

Structural properties and graphical implications of Webster's delay formula

Thorsten Neumann^{1,*} and Robert Oertel¹

1. German Aerospace Center (DLR), Institute of Transportation Systems, 12489 Berlin, Germany

Abstract: Stochastic queuing models have been and still are an essential topic in the field of traffic flow theory at signalized intersections. The present article enhances the existing theoretical knowledge by a systematic review of the mutual relationship between traffic demand, green time split and average delay in context of a coupled system of Webster equations for a simple intersection scenario. Formally proved conditions for the existence and uniqueness of valid fixed time control strategies are derived such that these are able to simultaneously handle given traffic flows at the four approaches of the considered intersection. At that, consistency with measured or planned (maximum) average delays, for instance, is ensured. The strictly mathematical analysis finally leads to a new way of illustrating the dependencies of the three above named variables in terms of level curve diagrams that become an easy-to-understand graphical tool for answering a number of practical questions in context of traffic signal planning. Several theoretic examples are discussed.

Key words: Traffic flow theory, queuing model, average delay, signal control, fixed time control, level function.

1. Introduction

Traffic signals and waiting times have a significant effect on the quality of urban traffic. Thus, it is not surprising that delay models for signalized intersections have been playing an important role in transportation research and traffic engineering for many decades up to today and have become an indispensable tool for practical purposes in the field of traffic management and transportation planning (cf. [1]). In this context, Webster's delay formula (cf. [2]), as originally published in 1958 based on former theoretical studies by Wardrop [3] and Kendall [4], can be regarded as the basic steady-state model for non-deterministic (i.e. Poissonian) traffic demand.

It describes the average delay d [sec] per vehicle at a traffic signal in case of fixed-time control given cycle time c [sec], (effective) green time g [sec], traffic demand q [veh/sec] and saturation flow s [veh/sec]. Precisely,

$$d = \frac{c(1-\lambda)^2}{2(1-\lambda x)} + \frac{x^2}{2q(1-x)} - 0.65 \left(\frac{c}{q^2} \right)^{1/3} x^{(2+5\lambda)} \quad (1)$$

where $x = q/(\lambda s)$ is the degree of saturation and $\lambda = g/c$ the "proportion of the cycle which is effectively green for the phase under consideration" [2]. Thus, given fixed signal parameters, delay is usually interpreted as a function of demand (cf. Figure 1), i.e. $d = d(q)$.

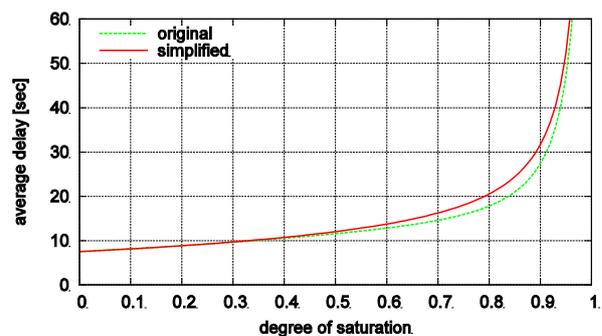


Fig. 1 Average delay based on Webster with (green dashed line) and without (red solid line) heuristic correction term ($c = 60$ sec, $g = 30$ sec, $s = 0.5$ veh/sec)

Based on that, Webster applied Equation (1) for deriving optimal cycle times and green time splits given the measured demand for all approaches of the

Corresponding author: Thorsten Neumann, PhD; research fields: traffic management, intelligent transportation systems and traffic flow theory. E-mail: thorsten.neumann@dlr.de.

Structural properties and graphical implications of Webster's delay formula

considered intersection (cf. [2]), for instance. Other researchers used the formula as the origin for the development of non-steady-state delay models that are valid also for oversaturated traffic (cf. [1, 5, 6]) and that have become an integral part of common guidelines in traffic engineering such as the famous Highway Capacity Manual [7].

Following that, Webster's results have always been and still are an important benchmark for other queuing models (cf. [8, 9]) and even more for innovative signal control strategies (cf. [10, 11]). Current research activities in context of Webster's theory include the validation and modification of the original terms for the optimal cycle length (cf. [12, 13]) or the deeper analysis of the "effect of green time on stochastic queues at traffic signals" [14]. Thus, although nearly 60 years old, there are still interesting features in the above formula that are worthy to be studied.

In this regard, the present contribution discusses the inversion of the delay formula from Equation (1). As modern sensor technologies for traffic state detection more and more facilitate measuring travel times and delay, the question arises what the corresponding demand is given the delay, i.e. $q = q(d)$. Moreover, it might be interesting to ask what the maximum demands are that can be handled by common (fixed-time) control strategies based on preset objectives regarding maximum delays for each individual intersection approach. This paper provides the answers in a strictly mathematical way in case of a standard two-phase intersection with coupled green times for the concurrent traffic streams based on Webster's theory. By that, it reveals new theoretical insights into the structural properties of the delay formula. In particular, it yields necessary and sufficient conditions for the existence and uniqueness of the solution of the equation system that arises from coupling the delay formulas for more than one intersection approach.

The paper is structured as follows: At first, Section 2 fleshes out the mathematical problem that will be

solved. This includes a detailed description of the considered two-phase intersection as well as the exact formulation of the resulting equation system for the delays. The strictly mathematical solution is then derived in Section 3. As the results can be interpreted graphically very well, this is part of Section 4 which continues the theoretical argumentation but also discusses practical applications of the findings. Section 5 finally is the conclusion.

2. Problem statement and specification

Consider the simple two-phase intersection as depicted in Figure 2 with the single-lane approaches $i = 1, 1', 2, 2'$ each of them having the same saturation flow s and an individual stochastic (i.e. Poissonian, cf. [2, 15]), but stationary traffic demand q_i .

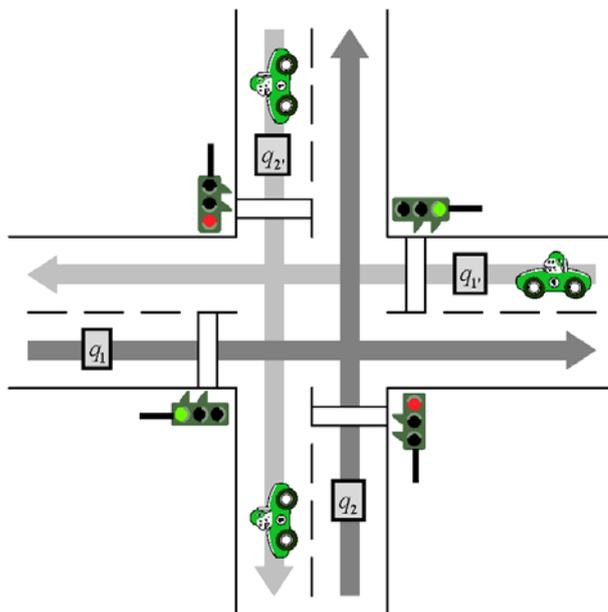


Fig. 2 Intersection layout with traffic streams and demand

Let g_i denote the (effective) green time of approach i for all i while the cycle time c is fixed. Amber times are ignored (cf. [2]). Two-phase signalization then means that $g_i = g_{i'}$ for $i = 1, 2$ and $g_1 + g_2 = c$. Hence, write $g_1 = g_{1'} = \kappa c$ and $g_2 = g_{2'} = (1-\kappa)c$ where $0 < \kappa < 1$. Consequently, with c and s fixed, the delay formula from Equation (1) reads

$$d_i = f_i(q_i, \kappa) \quad (2)$$

for all $i = 1, 1', 2, 2'$ where

$$f_i(q_i, \kappa) := \frac{c(1-\kappa)^2}{2(1-q_i/s)} + \frac{(q_i/(\kappa s))^2}{2q_i(1-q_i/(\kappa s))} \quad (3)$$

for $i = 1, 1'$ and

$$f_i(q_i, \kappa) := \frac{c\kappa^2}{2(1-q_i/s)} + \frac{(q_i/((1-\kappa)s))^2}{2q_i(1-q_i/((1-\kappa)s))} \quad (4)$$

for $i = 2, 2'$. Note that the heuristic correction term, i.e. the third addend in Equation (1) – as it was originally introduced by Webster in order to get a better fit to his empirical data (cf. [2]) – is omitted here for simplicity (cf. Figure 1). Needless to say, d_i denotes the average delay at the approach i for all i . Note that f_1 and $f_{1'}$ as well as f_2 and $f_{2'}$ yield identical values given the same demand q_i . Moreover, $f_1(q, \kappa) = f_2(q, 1-\kappa)$ for all q . That is, f_1 and f_2 are symmetrical with regard to the axis $\kappa = 1/2$.

As is well known (cf. [1]), the above delay formulas are practically valid for undersaturation only (i.e., $q_{1,1'} < \kappa s$ and $q_{2,2'} < (1-\kappa)s$, respectively) because of the relevant poles of f_i at $x = 1$ for all i (cf. Figure 1). But, when $q_{1,1'} \in [0, \kappa s)$ and $q_{2,2'} \in [0, (1-\kappa)s)$, they directly yield more or less reasonable estimates of the delay given κ . However, the contrary question is: Are there always such non-negative q_i for $i = 1, 1', 2, 2'$ together with $\kappa \in (0, 1)$ that solve the system of equations (2) given arbitrary (measured) delays d_i for $i = 1, 1', 2, 2'$? Moreover, is the solution unique or what are the conditions for its uniqueness?

3. Mathematical solution

The mathematical analysis of the delay functions f_i from Equation (2) can be reduced to studying f_1 only. Due to the identity of f_1 and $f_{1'}$, the results for $f_{1'}$ are exactly the same as for f_1 , and those for f_2 and $f_{2'}$ are directly obtained by replacing κ with $(1-\kappa)$ for symmetrical reasons.

Lemma 1. *Let*

$$f_1(q_1, \kappa) = \frac{c(1-\kappa)^2}{2(1-q_1/s)} + \frac{(q_1/(\kappa s))^2}{2q_1(1-q_1/(\kappa s))} \quad (5)$$

with $c > 0$ and $s > 0$ fixed. Given $\kappa \in (0, 1)$, f_1 is a strictly increasing function for $0 \leq q_1 < \kappa s$.

Proof. The function f_1 is differentiable for all $0 \leq q_1 < \kappa s$ with

$$\begin{aligned} \frac{\partial}{\partial q_1} f_1(q_1, \kappa) &= \frac{c(1-\kappa)^2}{2s(1-q_1/s)^2} \\ &+ \frac{1}{2\kappa^2 s^2 (1-q_1/(\kappa s))^2} \\ &+ \frac{q_1}{2\kappa^3 s^3 (1-q_1/(\kappa s))^2} \end{aligned} \quad (6)$$

Hence, $\partial/\partial q_1 f_1(q_1, \kappa) > 0$ for all $0 \leq q_1 < \kappa s$ because of $\kappa \in (0, 1)$, and the proof is completed. \square

The strict monotonicity of f_1 , together with the knowledge that $f_1(q_1, \kappa) \rightarrow \infty$ as $q_1 \uparrow \kappa s$, implies that there always is a unique demand value $q_1 \in [0, \kappa s)$ for the isolated intersection approach that solves $d_1 = f_1(q_1, \kappa)$ for given $\kappa \in (0, 1)$ whenever

$$d_1 \geq f_1(0, \kappa) = \frac{c}{2}(1-\kappa)^2. \quad (7)$$

In fact, q_1 is the solution of a quadratic equation as will be shown in Lemma 2. However, note before that there is no such positive $q_1 \in [0, \kappa s)$ if $d_1 < f_1(0, \kappa)$, of course. Hence, Equation (7) is a necessary and sufficient condition for the existence of a unique solution of $d_1 = f_1(q_1, \kappa)$ in the relevant interval $[0, \kappa s)$ for any given $\kappa \in (0, 1)$.

Lemma 2. *Let $c > 0$, $s > 0$ and $\kappa \in (0, 1)$ be fixed. Then, given $d_1 \geq 0$,*

$$q_1 = -\frac{\alpha_{d_1}^{(\kappa)}}{2} - \sqrt{\left(\frac{\alpha_{d_1}^{(\kappa)}}{2}\right)^2 - \beta_{d_1}^{(\kappa)}} \quad (8)$$

with

Structural properties and graphical implications of Webster's delay formula

$$\alpha_{d_1}^{(\kappa)} := -\frac{2d_1\kappa^2s^2 + s - (c(1-\kappa)^2 - 2d_1)\kappa s^2}{2d_1\kappa s + 1} \quad (9)$$

$$\beta_{d_1}^{(\kappa)} := -\frac{(c(1-\kappa)^2 - 2d_1)\kappa^2s^3}{2d_1\kappa s + 1}$$

is a solution of $d_1 = f_1(q_1, \kappa)$. Moreover, if the condition of Equation (7) holds for d_1 , this solution also satisfies $0 \leq q_1 < \kappa s$.

Proof. Let $d_1 \geq 0$. Elementary transformations show that $d_1 = f_1(q_1, \kappa)$ is equivalent to the simple quadratic equation

$$q_1^2 + \alpha_{d_1}^{(\kappa)} q_1 + \beta_{d_1}^{(\kappa)} = 0. \quad (10)$$

with the coefficients from Equation (9). Thus, given the occurring square root exists as a real number, the values

$$q_1^{+/-} = -\frac{\alpha_{d_1}^{(\kappa)}}{2} \pm \sqrt{\left(\frac{\alpha_{d_1}^{(\kappa)}}{2}\right)^2 - \beta_{d_1}^{(\kappa)}} \quad (11)$$

are the natural solutions of $d_1 = f_1(q_1, \kappa)$. In this context, the validity of

$$\left(\frac{\alpha_{d_1}^{(\kappa)}}{2}\right)^2 - \beta_{d_1}^{(\kappa)} \geq 0 \quad (12)$$

can be proved either directly by some elementary calculus or by the following arguments: For $0 < \kappa < 1$, the function f_1 has two poles, namely $q_1 = \kappa s$ and $q_1 = s$, with $f_1(q_1, \kappa) \rightarrow -\infty$ as $q_1 \downarrow \kappa s$ and $f_1(q_1, \kappa) \rightarrow \infty$ as $q_1 \uparrow s$. In between, f_1 is a continuous function. That is, for any $d_1 \geq 0$, there is a real solution of $d_1 = f_1(q_1, \kappa)$ that necessarily has the form as in Equation (11). Consequently, Equation (12) must hold for all $d_1 \geq 0$ and $\kappa \in (0, 1)$ because otherwise there were complex solutions of Equation (10) only.

It remains to show that

$$q_1^- = -\frac{\alpha_{d_1}^{(\kappa)}}{2} - \sqrt{\left(\frac{\alpha_{d_1}^{(\kappa)}}{2}\right)^2 - \beta_{d_1}^{(\kappa)}} \in [0, \kappa s] \quad (13)$$

given that d_1 satisfies Equation (7). For this purpose, note that previous arguments already showed that there

always are a unique solution with $q_1 \in [0, \kappa s)$ and a second one with $q_1 \in (\kappa s, s)$ in the considered case. Since now q_1^+ and q_1^- are the only possible solutions and $q_1^+ \geq q_1^-$, it directly follows that $q_1^- \in [0, \kappa s)$ as proposed. \square

So far, Equation (7) has been shown to be a necessary and sufficient condition for the existence and uniqueness of the solution of $d_1 = f_1(q_1, \kappa)$ given $\kappa \in (0, 1)$. Thus, in order to have a corresponding solution of the original system of equations (2), the above-named (symmetric) condition needs to hold simultaneously for all four intersection approaches, i.e.

$$d_1, d_{1'} \geq \frac{c}{2}(1-\kappa)^2 \quad \text{and} \quad d_2, d_{2'} \geq \frac{c}{2}\kappa^2. \quad (14)$$

By simple considerations, this is equivalent to

$$1 - \sqrt{\frac{2 \min\{d_1, d_{1'}\}}{c}} \leq \kappa \leq \sqrt{\frac{2 \min\{d_2, d_{2'}\}}{c}} \quad (15)$$

while still $0 < \kappa < 1$, of course. That means, there are some additional limitations concerning κ when searching for a solution of the system of equations (2). In particular, it turns out that

$$\sqrt{\frac{2 \min\{d_1, d_{1'}\}}{c}} + \sqrt{\frac{2 \min\{d_2, d_{2'}\}}{c}} \geq 1 \quad (16)$$

together with $d_i > 0$ for all $i = 1, 1', 2, 2'$ is a necessary and sufficient condition for the existence of a solution because only then there is a $\kappa \in (0, 1)$ that satisfies Equation (15). Note that usually such a κ is not uniquely defined. That is, given κ is a free variable, Equation (16) does not guarantee the uniqueness of the solution except for the specific case where

$$\sqrt{\frac{2 \min\{d_1, d_{1'}\}}{c}} + \sqrt{\frac{2 \min\{d_2, d_{2'}\}}{c}} = 1. \quad (17)$$

Needless to say, if $\kappa \in (0, 1)$ is fixed such that Equation (15) holds, the solution becomes unique in any case (cf. Lemma 2).

Remark 3. Let $q_i > 0$ for $i = 1, 1', 2, 2'$. Then, Equation (16) implies that it is impossible to make the

Structural properties and graphical implications of Webster's delay formula

corresponding delays d_i arbitrarily small for all intersection approaches simultaneously except if $c \rightarrow 0$ is allowed. However, there are practical limitations such as fixed loss times per signal phase that forbid to choose extremely short cycle lengths c .

4. Graphical interpretation and applications

As in Section 3, the following analysis concentrates on $i = 1$ first while the results for the other intersection approaches are finally obtained by symmetry arguments. Thus, let $d_1 \geq 0$. Lemma 2 then implies that $h_{d_1}: (0,1) \rightarrow \mathbb{R}$ with

$$h_{d_1}(\kappa) := -\frac{\alpha_{d_1}^{(\kappa)}}{2} - \sqrt{\left(\frac{\alpha_{d_1}^{(\kappa)}}{2}\right)^2 - \beta_{d_1}^{(\kappa)}} \quad (18)$$

is a well-defined function that solves

$$f_1(h_{d_1}(\kappa), \kappa) = d_1 \quad (19)$$

for all $\kappa \in (0,1)$. Hence, the term from Equation (18) can be interpreted as or determines the level curve of the delay function f_1 in the q_1 - κ -plane that belongs to the level d_1 . Note that d_1 does not need to satisfy the condition of Equation (7) at this point. Figure 3 gives a first graphical impression about the structure of the level curves for different $d_1 \geq 0$.

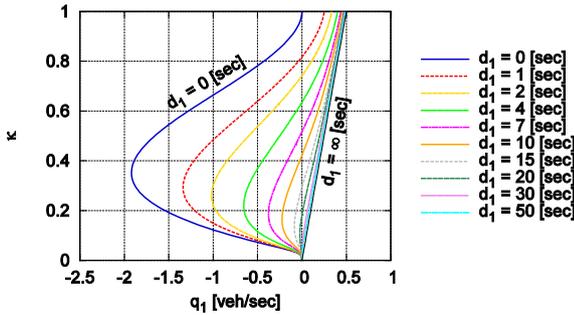


Fig. 3 Level curves of the delay function f_1 with $c = 60$ sec and $s = 0.5$ veh/sec

4.1 Specific properties of the level curves

A detailed summary of the relevant properties of the level curves – including the mathematical proofs – is given in the following.

Lemma 4. Let $c > 0$ and $s > 0$ be fixed. Then, given that $d_1 \geq 0$, the following propositions regarding the level function $h_{d_1}: (0,1) \rightarrow \mathbb{R}$ from Equation (18) hold:

- h_{d_1} is a continuous function for all $\kappa \in (0,1)$.
- If $2d_1 < c$,

$$\kappa_0 = 1 - \sqrt{\frac{2d_1}{c}} \quad (20)$$

is the only root of h_{d_1} with $h_{d_1}(\kappa_0) = 0$ in the open interval $(0,1)$. Otherwise, if $2d_1 \geq c$, there is no root of h_{d_1} in $(0,1)$.

- The following limits hold:

$$h_{d_1}(0) := \lim_{\kappa \downarrow 0} h_{d_1}(\kappa) = 0 \quad (21)$$

$$h_{d_1}(1) := \lim_{\kappa \uparrow 1} h_{d_1}(\kappa) = s \cdot \frac{2d_1 s}{1 + 2d_1 s} \quad (22)$$

- There is a natural upper boundary for h_{d_1} , namely $h_{d_1}(\kappa) < \kappa s$ for all $\kappa \in (0,1)$.
- Given $\kappa \in (0,1)$, one obtains $h_{d_1}(\kappa) > 0$ if and only if $\max\{0, \kappa_0\} < \kappa < 1$ with κ_0 as in Equation (20).
- h_{d_1} is a strictly monotone function when $\max\{0, \kappa_0\} < \kappa < 1$ with κ_0 as in Equation (20).

Proof. a) The continuity of h_{d_1} is directly obtained from its definition in Equation (18).

b) Let $\kappa \in (0,1)$. According to Equation (19), the condition $h_{d_1}(\kappa) = 0$ then implies

$$d_1 = f_1(h_{d_1}(\kappa), \kappa) = f_1(0, \kappa) = \frac{c}{2}(1 - \kappa)^2. \quad (23)$$

By simple transformations, one finds that κ_0 from Equation (20) is the only root of h_{d_1} in the open interval $(0,1)$ given $2d_1 < c$. Moreover, when $2d_1 \geq c$, there is no $\kappa \in (0,1)$ such that $h_{d_1}(\kappa) = 0$.

c) The proposed limits are easily obtained from Equation (18) by inserting $\kappa = 0$ and $\kappa = 1$, respectively.

d) Proposition d) has already been proved in Lemma 2 for the situation when the condition from Equation (7) holds, i.e. when $\kappa_0 \leq \kappa < 1$ with κ_0 as in Equation (20).

Structural properties and graphical implications of Webster's delay formula

In the general case, assume that there is a $\kappa \in (0,1)$ such that $h_{d_1}(\kappa) \geq \kappa s$. The observation that $h_{d_1}(1) < 1 \cdot s$ (see proposition c)), together with the continuity of h_{d_1} (see proposition a)), then implies that there is a $\kappa' \in [\kappa, 1)$ such that $h_{d_1}(\kappa') = \kappa' s$ (cf. Figure 4). Thus,

$$f_1(\kappa' s, \kappa') = f_1(h_{d_1}(\kappa'), \kappa') = d_1 \quad (24)$$

in contradiction to the fact that $f_1(\cdot, \kappa')$ has a pole at $q_1 = \kappa' s$. That means, the original assumption must be wrong and proposition d) holds.

e) Let $\max\{0, \kappa_0\} < \kappa < 1$ with κ_0 as in Equation (20). Then, Lemma 2 yields that $h_{d_1}(\kappa) \geq 0$. Since κ_0 is the only possible root of h_{d_1} in the open interval $(0,1)$, i.e. $h_{d_1}(\kappa) \neq 0$ for all $\kappa \in (0,1)$ where $\kappa \neq \kappa_0$, this directly proves the first direction of the equivalence in proposition e).

In order to show the opposite direction, let $2d_1 < c$ so that $\kappa_0 > 0$. (Otherwise, there is nothing to do.) Then, assume that $h_{d_1}(\kappa) > 0$ for any $\kappa \in (0,1)$ with $\kappa \leq \kappa_0$. Consequently, since $h_{d_1}(\kappa) < \kappa s$ (see proposition d)), the strict monotonicity of f_1 for $0 \leq q_1 < \kappa s$ (see Lemma 1) implies that

$$d_1 = f_1(h_{d_1}(\kappa), \kappa) > f_1(0, \kappa) = \frac{c}{2}(1 - \kappa)^2. \quad (25)$$

Thus, $\kappa > \kappa_0$ in contradiction to the assumption, and the proposition holds.

f) Assume that h_{d_1} is not strictly monotone for $\max\{0, \kappa_0\} < \kappa < 1$. The continuity of h_{d_1} (see proposition a)) then yields the existence of numbers κ_1 and κ_2 such that $\max\{0, \kappa_0\} < \kappa_1 < \kappa_2 < 1$ with $h_{d_1}(\kappa_1) = h_{d_1}(\kappa_2) =: q$. Moreover, the propositions d) and e) imply that $0 < q < \kappa_1 s$ and $0 < q < \kappa_2 s$. Thus,

$$\begin{aligned} d_1 &= f_1(h_{d_1}(\kappa_1), \kappa_1) \\ &= \frac{c(1 - \kappa_1)^2}{2(1 - q/s)} + \frac{q}{2\kappa_1^2 s^2 (1 - q/(\kappa_1 s))} \\ &> \frac{c(1 - \kappa_2)^2}{2(1 - q/s)} + \frac{q}{2\kappa_2^2 s^2 (1 - q/(\kappa_2 s))} \\ &= f_1(h_{d_1}(\kappa_2), \kappa_2) \\ &= d_1 \end{aligned} \quad (26)$$

which is obviously wrong. Consequently, h_{d_1} must be strictly monotone for $\max\{0, \kappa_0\} < \kappa < 1$. \square

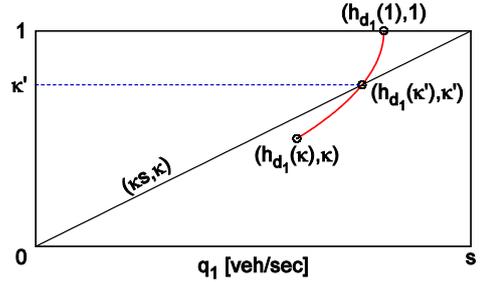


Fig. 4 Schematic illustration of the proof of Lemma 4d)

Lemma 4 contains all necessary information for a detailed drawing of the level function h_{d_1} with regard to positive q_1 (see Figure 5). Note that the same picture holds for $i = 1'$ as well, of course. As can be seen, the propositions b) and c) yield the exact location of the intersection points between the level curves and the two important axes $q_1 = 0$ and $\kappa = 1$.

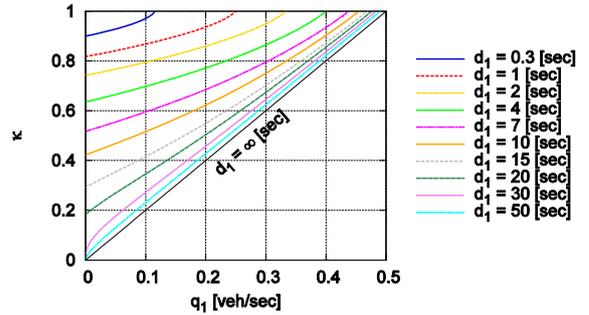


Fig. 5 Close-up view of the level curves of f_1 and f_1' with $c = 60$ sec and $s = 0.5$ veh/sec

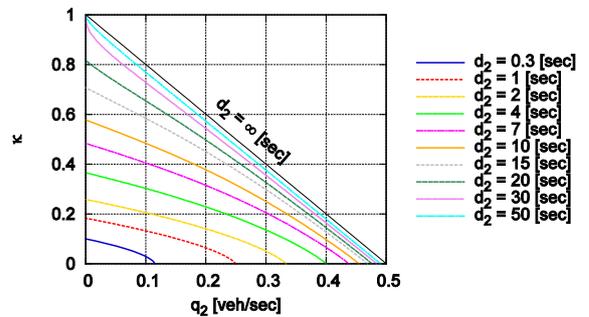


Fig. 6 Corresponding level curves of the delay functions curves of f_2 and f_2' with $c = 60$ sec and $s = 0.5$ veh/sec

Structural properties and graphical implications of Webster's delay formula

The corresponding plots for $i = 2, 2'$ are obtained by simple symmetry arguments as already discussed (see Figure 6).

Finally, Lemma 5 reveals some further mathematical properties of the above level functions mainly based on the results of Lemma 4.

Lemma 5. *Let $c > 0$ and $s > 0$ be fixed as well as $d_1 > 0$.*

a) *Then, given $q_1 > 0$ with*

$$q_1 < s \cdot \frac{2d_1s}{1 + 2d_1s}, \quad (27)$$

there is a unique $\kappa^ \in (\max\{0, \kappa_0\}, 1)$ such that $h_{d_1}(\kappa^*) = q_1$ where κ_0 as in Equation (20). On the contrary, there is no such κ^* if*

$$q_1 \geq s \cdot \frac{2d_1s}{1 + 2d_1s}. \quad (28)$$

b) *Let $q_1 > 0$ according to Equation (27), and $\kappa > \kappa^*$ where κ^* as in proposition a). Then, there is $d_1^* > 0$ such that $h_{d_1^*}(\kappa) = h_{d_1}(\kappa^*)$. Moreover, $d_1^* < d_1$ holds.*

Proof. a) Let $d_1 > 0$ and $q_1 > 0$ such that Equation (27) holds. Lemma 4b) and c) then imply

$$\lim_{\kappa \downarrow \max\{0, \kappa_0\}} h_{d_1}(\kappa) = 0 < q_1 \quad (29)$$

$$\lim_{\kappa \uparrow 1} h_{d_1}(\kappa) = s \cdot \frac{2d_1s}{1 + 2d_1s} > q_1 \quad (30)$$

where κ_0 is defined as in Equation (20). Consequently, since h_{d_1} is a continuous function (see Lemma 4a)), there is a $\kappa^* \in (\max\{0, \kappa_0\}, 1)$ such that $h_{d_1}(\kappa^*) = q_1$. Moreover, the strict monotonicity of h_{d_1} (see Lemma 4f)) for $\kappa \in (\max\{0, \kappa_0\}, 1)$ also proves the uniqueness of κ^* .

On the contrary, the same monotonicity together with Equation (28) yields

$$h_{d_1}(\kappa^*) < \lim_{\kappa \uparrow 1} h_{d_1}(\kappa) = s \cdot \frac{2d_1s}{1 + 2d_1s} \leq q_1 \quad (31)$$

for all $\kappa^* \in (\max\{0, \kappa_0\}, 1)$. Thus, the proof of proposition a) is completed.

b) Let $q_1 > 0$ such that Equation (27) holds, and κ^* as in proposition a). For any $\kappa \in (\kappa^*, 1)$, Lemma 4d) then implies

$$0 < h_{d_1}(\kappa^*) < \kappa^* s < \kappa s. \quad (32)$$

That means – as proposed –

$$d_1^* := f_1(h_{d_1}(\kappa^*), \kappa) > f_1(0, \kappa) > 0 \quad (33)$$

is a well-defined value such that $d_1^* = f_1(h_{d_1^*}(\kappa), \kappa)$, and thus $h_{d_1^*}(\kappa) = h_{d_1}(\kappa^*)$ because of the uniqueness of the solution of $d_1^* = f_1(q_1, \kappa)$.

In the following, note that

$$\frac{\partial}{\partial \kappa'} f_1(q_1, \kappa') = -\frac{c(1 - \kappa')}{1 - q_1/s} - \frac{q_1}{(\kappa')^3(1 - q_1/(\kappa's))^2}. \quad (34)$$

Consequently, f_1 is a strictly monotone function regarding $\kappa' \in (0, 1)$ as long as $0 < q_1 < \kappa's$ because then $\partial/\partial \kappa' f_1(q_1, \kappa') < 0$. Hence, due to Equation (32), one obtains

$$d_1^* = f_1(h_{d_1}(\kappa^*), \kappa) < f_1(h_{d_1}(\kappa^*), \kappa^*) = d_1, \quad (35)$$

and the proof is completed. \square

Remark 6. *Let, as a kind of external requirement, $d_1 > 0$ be the maximum acceptable average delay for the considered intersection approach and $q_1 > 0$ such that Equation (27) holds. Given $\kappa^* \in (0, 1)$ where $h_{d_1}(\kappa^*) = q_1$, Lemma 5 then implies that the same traffic demand q_1 can be served by the traffic signal with even less delay in case of any green time split where $\kappa > \kappa^*$. Thus, Lemma 5 is the formal proof of the intuitive statement that more green time for a given traffic stream with fixed demand necessarily reduces the corresponding delay.*

On the contrary, given q_1 is fixed, it can be shown very similar as in Lemma 5b) that a reduction of the green time ($\kappa < \kappa^$) always leads to increased delays or*

Structural properties and graphical implications of Webster's delay formula

even results in oversaturation when κ becomes too small.

4.2 Graphical illustration

The previous results from Section 4.1 – including the (symmetric) versions of Lemma 4 and 5 in case of $i = 1, 1', 2, 2'$ – have several graphical implications in context of the existence and uniqueness of the solution of the original equation system (2). For, let $d_i > 0$ for all $i = 1, 1', 2, 2'$. The roots of h_{d_i} (cf. Lemma 4b) and c) together with the monotonicity of the level functions (cf. Lemma 4f) then show that the necessary and sufficient conditions from Equations (15) and (16) are equivalent to the fact that there is an area as in Figure 7 where for each fixed κ within the depicted “ κ -band” there are intersection points with all four level curves (cf. Figure 8). Obviously, the corresponding q_i are uniquely defined due to the strict monotonicity of h_{d_i} for all i whenever κ is fixed.

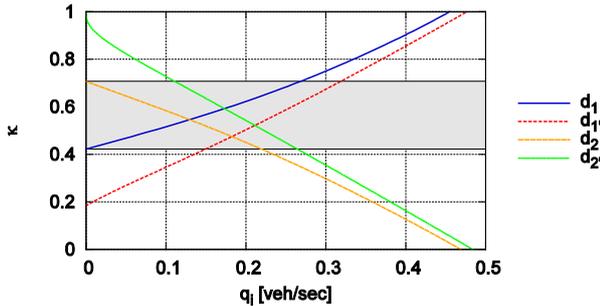


Fig. 7 Graphical illustration of the “ κ -band” with $c = 60$ sec and $s = 0.5$ veh/sec

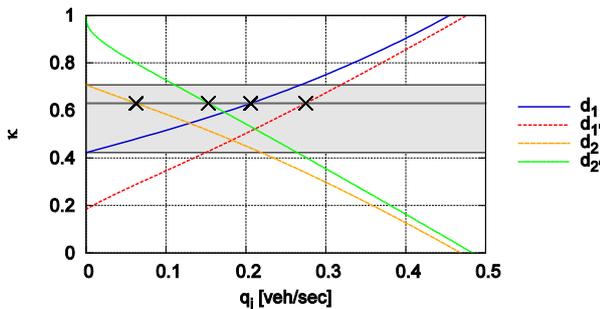


Fig. 8 Graphical illustration of the solution given $\kappa = 0.63$ with $c = 60$ sec and $s = 0.5$ veh/sec

Interestingly, Figure 8 also implies some further conditions or scenarios for the uniqueness of the solution of the system of equations (2) in the case where κ is unknown. Namely, let

$$q_i \geq h_{d_i} \left(1 - \sqrt{\frac{2 \min\{d_1, d_{1'}\}}{c}} \right) \quad (36)$$

and

$$q_i \leq h_{d_i} \left(\sqrt{\frac{2 \min\{d_2, d_{2'}\}}{c}} \right) \quad (37)$$

for any $i \in \{1, 1', 2, 2'\}$. Then, there is a unique κ located within the depicted “ κ -band” such that $h_{d_i}(\kappa) = q_i$. And thus, the overall solution comprising the traffic demands for all four intersection approaches is unique again. But, note that there is no such solution if q_i does not satisfy the Equations (36) and (37). In general, that means the knowledge about the actual green time split as represented by κ is mathematically equivalent to the knowledge about the traffic demand q_i for any of the four intersection approaches.

4.3 Applications

From a practical point of view, variants of Figure 7 can also be seen as a simple graphical tool for traffic signal planning. For, let $d_i > 0$ be the maximum acceptable delays for the considered intersection approaches $i = 1, 1', 2, 2'$ (cf. Figure 2) in the sense of external planning requirements. Then, the values

$$q_i^{\max} := h_{d_i} \left(\min \left\{ 1, \sqrt{\frac{2 \min\{d_2, d_{2'}\}}{c}} \right\} \right) \quad (38)$$

for $i = 1, 1'$ and

$$q_i^{\max} := h_{d_i} \left(\max \left\{ 0, 1 - \sqrt{\frac{2 \min\{d_1, d_{1'}\}}{c}} \right\} \right) \quad (39)$$

for $i = 2, 2'$ are easily derived from Figure 7 as can be seen in Figure 9. Obviously, for all $i \in \{1, 1', 2, 2'\}$, the defined q_i^{\max} represents the maximum traffic demand for the intersection approach i that can be served by a simple fixed time control with two phases while

Structural properties and graphical implications of Webster's delay formula

ensuring that the complete set of given delay requirements d_i is satisfied. Needless to say, the maximum flow q_i^{\max} for the approach i can only be realized in case of suitable flows at the other three intersection approaches. Of course, higher manageable flows q_1 and $q_{1'}$ are always at the cost of lower manageable flows q_2 and $q_{2'}$, and vice versa. By varying κ within the depicted “ κ -band”, it is graphically possible to find a valid trade-off between the conflicting traffic streams.

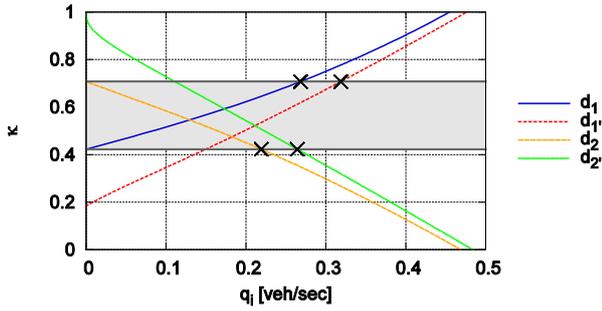


Fig. 9 Maximum flow per intersection approach given delay requirements with $c = 60$ sec and $s = 0.5$ veh/sec

On the contrary, the diagrams may also be used to decide whether some observed or planned traffic demands q_i for $i = 1, 1', 2, 2'$ can be handled at all by a corresponding two-phase signal control given maximum acceptable delays d_i for all intersection approaches as before. For this purpose, note that the delay requirement for the intersection approach 1, for instance, is met if and only if κ lies above the (potential) intersection point of the level curve h_{d_1} and the vertical line as defined by q_1 (cf. Remark 6). Of course, the analogous (eventually symmetric) statements hold for $i = 1', 2, 2'$ as well. That means the delay requirements are satisfied for all four intersection approaches simultaneously if and only if κ is chosen within the reduced “ κ -band” as depicted in Figure 10. Clearly, this “ κ -band” may even vanish completely depending on the concrete values d_i and q_i for $i = 1, 1', 2, 2'$ so that there is no solution of the described specific problem in that case.

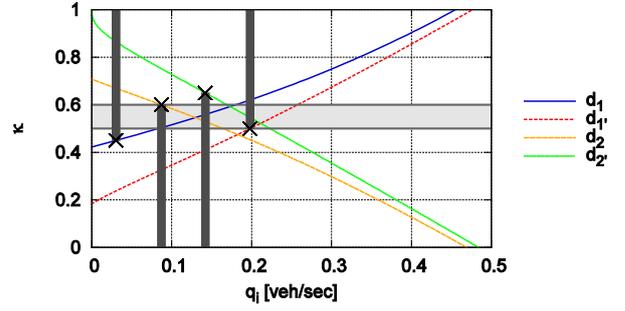


Fig. 10 Reduced “ κ -band” given traffic flows and delay requirements with $c = 60$ sec and $s = 0.5$ veh/sec

Finally, the discussed theory concerning the inversion of Webster's delay formula could, for instance, be applied for generating a simple adaptive signal control scheme with temporarily fixed green time splits and static cycle times. For, assume that there are periodical measurements of the average delay (i.e. $d_{i,t} > 0$) at all four intersection approaches (cf. Figure 2) based on floating car data or other detection techniques. Then, the above theory would allow to derive the corresponding traffic demand $q_{i,t}$ for all $i = 1, 1', 2, 2'$ at the regular time intervals t based on a known green time split κ_t . Consequently, a suitable green time split κ_{t+1} for the next time interval is defined by

$$\kappa_{t+1} := \frac{\max\{h_{d_{1,t}}(\kappa_t), h_{d_{1',t}}(\kappa_t)\}}{\Sigma}. \quad (40)$$

where

$$\begin{aligned} \Sigma := & \max\{h_{d_{1,t}}(\kappa_t), h_{d_{1',t}}(\kappa_t)\} \\ & + \max\{h_{d_{2,t}}(1-\kappa_t), h_{d_{2',t}}(1-\kappa_t)\}. \end{aligned} \quad (41)$$

That is, green times are allocated more or less proportionally to the computed traffic demand as in standard fixed time traffic signal planning (Note that the saturation flow s and the number of lanes were assumed to be identical for all four intersection approaches!). Some first prototypical simulations of this simple adaptive control scheme however showed that it becomes unstable very fast and requires a deeper

analysis first which is out of the scope of this present article.

5. Conclusion

Webster's delay formula belongs to the fundamentals of traffic flow theory at signalized intersections and is still of interest for the research community as discussed in the introduction. Based on strictly mathematical considerations, a number of properties in context of its inversion were derived in the present article for a simple intersection scenario which includes explicit proofs of conditions for the existence and uniqueness of the solutions of the system of equations (2). In particular, the analysis of the implicit level functions in Section 4 showed up a new and highly informative way of illustrating the relationship between traffic flow, average delay and green time split (cf. Figure 7). By that, a number of practical questions in context of traffic signal planning can be answered directly by graphical arguments only.

At the moment, of course, the described theory is valid only for simple two-phase intersections with fixed time control as depicted in Figure 2. Thus, further studies should extend the proved propositions and lemmas in order to cover also more complex intersection scenarios. In this regard, the idea of using level curve diagrams for graphical traffic signal planning is not necessarily limited to Webster's delay formula, but may be adapted to other delay models including those for adaptive control strategies as well. Consequently, the presented work is not only interesting for theoreticians, but may also evolve into helpful tools for practitioners in the field of signalized traffic flow.

Acknowledgments

The authors would like to thank Dr. Peter Wagner for his readiness to discuss the presented theory at several stages of its development and for his strong encouragement to write this publication.

References

- [1] V.F. Hurdle, Signalized Intersection Delay Models – A Primer for the Uninitiated, Transportation Research Record 971 (1984) 96-105.
- [2] F.V. Webster, Traffic Signal Settings, Road Research Technical Paper No. 39, Her Majesty's Stationery Office, London, UK, 1958.
- [3] J.G. Wardrop, Some Theoretical Aspects of Road Traffic Research, ICE Proceedings 1 (1952) 325-362.
- [4] D.G. Kendall, Some Problems in the Theory of Queues, Journal of the Royal Statistical Society, Series B (Methodological) 13 (1951) 151-185.
- [5] R.M. Kimber, E.M. Hollis, Traffic queues and delays at road junctions, TRRL Laboratory Report 909, Crowthorne, Berkshire, 1979.
- [6] R. Akcelik, The Highway Capacity Manual Delay Formula for Signalized Intersections, ITE Journal 58 (1988) 23-27.
- [7] Transportation Research Board (TRB), HCM 2010 Highway Capacity Manual, Washington D.C., USA, 2010
- [8] G.F. Newell, Queues for a fixed-cycle traffic light, The Annals of Mathematical Statistics 31 (1960) 589-597.
- [9] W. Brilon, N. Wu, Delays at fixed-time traffic signals under time-dependent traffic conditions, Traffic engineering & control (TEC), December 1990, pp. 623-631.
- [10] A. Kouvelas, K. Aboudolas, M. Papageorgiou, E.B. Kosmatopoulos, Hybrid Strategy for Real-Time Traffic Signal Control of Urban Road Networks, IEEE Transactions on Intelligent Transportation Systems 12 (2011) 884-894.
- [11] K. Pandit, D. Ghosal, H.M. Zhang, C.N. Chuah, Adaptive Traffic Signal Control With Vehicular Ad Hoc Networks, IEEE Transactions on Vehicular Technology 62 (2013) 1459-1471.
- [12] D.X. Cheng, C.J. Messer, Z.Z. Tian, J. Liu, Modification of Webster's Minimum Delay Cycle Length Equation Based on HCM 2000, in: TRB 82th Annual Meeting, Washington D.C., 2003.
- [13] P. Wagner, N.H. Gartner, T. Lu, R. Oertel, Webster's Delay Formula – revisited, in: TRB 93rd Annual Meeting, Washington D.C., 2014.
- [14] N.B. Taylor, B.G. Heydecker, The effect of green time on stochastic queues at traffic signals, Transportation Planning and Technology 37 (2014) 3-19.
- [15] W.F. Adams, Road Traffic Considered as a Random Series, Journal of the Institution of Civil Engineers 4 (1936) 121-130.