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On solving periodic $\mathcal{H}_2$-optimal fault detection and isolation problems

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***Abstract*— A lifting-free computational method is proposed to solve approximate fault detection and isolation problems for periodic systems using an $\mathcal{H}_2$-optimal model matching approach. The synthesis procedure relies on two key computational procedures: a numerically reliable algorithm to determine least order annihilators of periodic systems to reduce the periodic $\mathcal{H}_2$-optimal model matching problem to a simpler standard form and a recently developed algorithm to compute inner-outer factorizations of periodic systems which allows a further reduction to a $\mathcal{H}_2$ minimal distance problem. If the resulting fault detection filter is not stable and/or not causal, then a final stabilization step is performed using periodic coprime factorization techniques. The overall computational algorithm has strongly coupled computational steps, where all available structural information at the end of each computational step are fully exploited in the subsequent computations.**

I. INTRODUCTION

The solution of the *periodic fault detection and isolation problem* (PFDIP) has its main application in solving fault isolation problems for multirate systems, which are frequently modelled as *linear time-periodic* (LTP) systems [1]. In the presence of unknown disturbance inputs, the PFDIP can be sometimes solved *exactly*, if these inputs can be fully decoupled from the residual signals used to decide on the presence or absence of faults. However, in most of cases, the PFDIP can be solved only *approximately*, because the unknown signals (e.g., sensor or process noise, or even parametric uncertainties recast as noise signals) can not be exactly decoupled from the residual signals. There are several possible approaches to solve both exact and approximate PFDIPs.

A straightforward approach for the solution of the PFDIP is to employ methods proposed for solving fault detection and isolation problems for *linear time-invariant* (LTI) systems. This approach is in principle simple and relies on building a lifted LTI model which is input-output equivalent to the given LTP system [2], [3]. Then, suitable linear synthesis methods can be applied to the lifted LTI model to solve exact or approximate model matching problems using numerically reliable computational methods as described in [4], [5], [6]. The final step consists in recovering a periodic realization of the fault detection filter which can be used for real-time implementations. Unfortunately, several intrinsic difficulties can impede the usage of lifting-based approaches, especially for systems with high orders or large periods. For example, building a lifted representation using the lifting technique of [2] involves explicitly forming many matrix products, thus this approach is completely unappropriate from numerical point of view. On the other hand, using the lifting technique proposed in [3] requires manipulating large sparse matrices of a descriptor system representation, which leads to computationally unacceptable costs. Even the final step of turning the designed lifted representation of the detector into a periodic state space representation (e.g., by using the algorithm of [7]) can lead to numerical difficulties in the case of high order systems. Therefore, lifting-free methods should be always preferred in solving computational problems for periodic systems [8].

The solution of the exact PFDIP (e.g., when ignoring some of unknown inputs) can be seen as a first step in solving the more involved approximate PFDIP and can be used to produce meaningful design specifications (e.g., in form of reference models). A suitable numerically reliable computational approach has been recently proposed in [9]. However, in the case when an exact solution of the model matching problem is not possible due to presence of many unknown inputs, approximate solutions can be computed by determining fault detection filters which minimize the effect of unknown inputs and achieve the best matching of a given reference model. A first attempt in this direction was the $\mathcal{H}_2$ approach proposed for fault estimation in [10] for the case of constant disturbance inputs. The $\mathcal{H}_2$ optimization based solution approach relies on two technical assumptions, both of which are not necessary for the problem solvability: the lack of infinite zeros and the lack of zeros on the unit circle. The resulting fault detection filter has a larger dynamical order as the original system.

In this paper we propose a new lifting-free approach to solve the approximate PFDIP in a general setting, by solving a $\mathcal{H}_2$ periodic model matching problem. The distinctive features of the proposed approach compared to [10] are: (1) the possibility to perform partial decoupling of unknown inputs in the residual, (2) a general problem setting based on a causal descriptor periodic model with time-varying state dimensions and an arbitrary reference model, (3) the possibility to determine suitable reference models which guarantee the solvability of the problem, and (4) the use of a numerically reliable integrated synthesis algorithm with closely tied computational steps.

The proposed approach can be seen as an extension to the periodic case of the $\mathcal{H}_2$ synthesis technique proposed in [5] for LTI systems. The synthesis procedure relies on two key computational procedures: a numerically reliable algorithm [11] to determine least order annihilators of periodic systems to reduce the periodic $\mathcal{H}_2$-optimal model matching problem to a simpler standard form and a recently developed algo-

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rithm [12] to compute inner-outer factorizations of periodic systems which allows a further reduction to a $\mathcal{H}_2$ minimal distance problem. If the resulting fault detection filter is not stable and/or not causal, then a final stabilization step is performed using periodic coprime factorization techniques. An important feature of the proposed method is its ability to determine or update a reference model used to solve the synthesis problem.

Notation. For an N-periodic matrix X_k we use alternatively the *script* notation $\mathcal{X} := \mathrm{diag}\,(X_1, X_2, \ldots, X_N)$, which associates the block-diagonal matrix $\mathcal{X}$ to the cyclic matrix sequence X_k, $k = 1, \ldots, N$.

II. The Approximate PFDIP

We consider periodic time-varying linear discrete-time descriptor systems of the form

$$\begin{aligned} E_k x(k+1) &= A_k x(k) + B_k^u u(k) + B_k^d d(k) \\ &\quad + B_k^w w(k) + B_k^f f(k) \\ y(k) &= C_k x(k) + D_k^u u(k) + D_k^d d(k) \\ &\quad + D_k^w w(k) + D_k^f f(k) \end{aligned} \tag{1}$$

where, for generality, the system state vector is assumed to have time-varying dimensions $x(k) \in \mathbf{R}^{n_k}$, $y(k) \in \mathbf{R}^p$ is the measured output vector, $u(k) \in \mathbf{R}^{m_u}$ is the plant control input vector, $d(k) \in \mathbf{R}^{m_d}$ is the disturbance vector, $w(k) \in \mathbf{R}^{m_w}$ is the noise vector and $f(k) \in \mathbf{R}^{m_f}$ is the fault signal vector. We assume that the system matrices E_k, A_k, B_k^u, $\ldots$ are periodic with period $N \geq 1$ and E_k is square for $k = 1, \ldots, N$. For invertible E_k the periodic system is *causal*, and often we can employ *standard* realizations with $E_k = I_{n_{k+1}}$ (i.e., the identity matrix of appropriate size). If E_k is singular, then the periodic system may be, in general, *non-causal*. Periodic descriptor systems as in (1) will be alternatively denoted by a quintuple of periodic matrices $(E_k, A_k, B_k, C_k, D_k)$ or by a quadruple (A_k, B_k, C_k, D_k) in the standard case, with appropriately defined B_k and D_k.

Consider a causal N-periodic linear residual generator having the general form

$$\begin{aligned} \widehat{x}(k+1) &= F_k \widehat{x}(k) + H_k \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \\ r(k) &= J_k \widehat{x}(k) + L_k \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \end{aligned} \tag{2}$$

which processes the measurable signals $y(k)$ and $u(k)$ and generates the residual signal $r(k) \in \mathbf{R}^q$. The role of the residual generator is to generate the residual signal vector $r(k)$ such that the relative sizes of its components $r_i(k)$, $i = 1, \ldots, m_f$, allow to discriminate between no fault and faulty situations in the corresponding components of the fault vector.

To address the fault isolation aspect, we will employ a reference model which specifies the desired behaviour of the residuals as functions of the fault signals. For the reference model we assume the following causal N-periodic system representation

$$\begin{aligned} x^r(k+1) &= A_k^r x^r(k) + B_k^r f(k) \\ y^r(k) &= C_k^r \widehat{x}^r(k) + D_k^r f(k) \end{aligned} \tag{3}$$

where $y^r(k) \in \mathbf{R}^{m_f}$ is the output of the reference model to be followed by the residual $r(t)$.

The *approximate periodic fault detection and isolation problem* (APFDIP) can be formulated as follows: Determine a stable N-periodic linear residual generator having the general form (2) such that for all control and disturbance inputs $u(k)$ and $d(k)$

(i) $r(k) = 0$ if $w(k) = 0$ and $f(k) = 0$, (decoupling condition)

(ii) $r_i(k) \approx y_i^r(k)$ if $f_i(k) \neq 0$, for $i = 1, \ldots, m_f$ (fault isolation condition)

where $k \geq 0$ and for (i) and (ii) we assume zero initial conditions for the state variables: $x(0) = 0$, $\widehat{x}(0) = 0$ and $x_r(0) = 0$.

The stability requirement can be expressed by the condition that all characteristic multipliers of the periodic matrix F_k (i.e., the eigenvalues of the monodromy matrix $\Psi_F := F_N \cdots F_2 F_1$) have moduli less than one [1]. More generally, the stability requirement can be formulated with respect to a certain *good* region $\mathbb{C}_g$ of the unit disk centered in the origin, by requiring that the spectrum of Ψ_F satisfies $\Lambda(\Psi_F) \subset \mathbb{C}_g$.

The above formulation of the APFDIP is quite general and has two important aspects to be mentioned. First of all, this formulation is independent of any potentially applicable solution method. This aspect is very important, because in the case when optimization-based synthesis methods are employed but no stable optimal solution exists (e.g., because of presence of some zeros on the unit circle), still an acceptable non-optimal solution can be determined starting from the optimal (but unstable) solution. The second aspect concerns the separation of unknown signals in two categories: the disturbances, which are aimed to be exactly decoupled, and the noise, whose effects need to be minimized in the residual signal. This allows to interprete the exact PFDIP as a particular case of the APFDIP (i.e., in the case of absence of noise). Therefore, it is straightforward to show that a sufficient condition for the solvability of the APFDIP is the solvability of the exact PFDIP.

III. Lifted reformulation of the APFDIP

An alternative theoretical insight to the APFDIP and guidance for its solution can be obtained if we reformulate the detector synthesis problem in terms of the *transfer-function matrices* (TFMs) corresponding to the associated *stacked* lifted representation of [3]. First we build a state-space representation which uses the input-state-output behavior of the system over time intervals of length N, rather than 1. The lifted input, output and state vectors are defined as

$$\begin{aligned} \widetilde{u}(h) &= [u^T(hN+1) \cdots u^T(hN+N)]^T, \\ \widetilde{d}(h) &= [d^T(hN+1) \cdots d^T(hN+N)]^T, \\ \widetilde{w}(h) &= [w^T(hN+1) \cdots w^T(hN+N)]^T, \\ \widetilde{f}(h) &= [f^T(hN+1) \cdots f^T(hN+N)]^T, \\ \widetilde{y}(h) &= [y^T(hN+1) \cdots y^T(hN+N)]^T, \\ \widetilde{x}(h) &= [x^T(hN+1) \cdots x^T(hN+N)]^T. \end{aligned}$$

and the corresponding lifted system can be represented by a LTI descriptor system of the form (notice the usage of script notation)

$$\begin{aligned} E^S\widetilde{x}(h+1) &= A^S\widetilde{x}(h) + \mathcal{B}^u\widetilde{u}(h) + \mathcal{B}^d\widetilde{d}(h) \\ &\quad + \mathcal{B}^w\widetilde{w}(h) + \mathcal{B}^f\widetilde{f}(h) \\ \widetilde{y}(h) &= \mathcal{C}\widetilde{x}(h) + \mathcal{D}^u\widetilde{u}(h) + \mathcal{D}^d\widetilde{d}(h) \\ &\quad + \mathcal{D}^w\widetilde{w}(h) + \mathcal{D}^f\widetilde{f}(h) \end{aligned}, \qquad (4)$$

where the *pole pencil* corresponding to the periodic pair (A_k, E_k)

$$A^S - zE^S = \left[\begin{array}{ccccc} A_1 & -E_1 & O & \cdots & O \\ O & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -E_{N-2} & O \\ O & & \ddots & A_{N-1} & -E_{N-1} \\ -zE_N & O & \cdots & O & A_N \end{array}\right] \qquad (5)$$

is regular. For the lifted system (4), the TFMs $G_u(z)$, $G_d(z)$, $G_w(z)$, $G_f(z)$ from the control, disturbance, noise and fault inputs to the system output are

$$G_\xi(z) = \mathcal{C}(zE^S - A^S)^{-1}\mathcal{B}^\xi + \mathcal{D}^\xi, \qquad (6)$$

where ξ stays for u, d, w, and f, respectively.

Assume that the residual generator (2) has a lifted representation with the corresponding TFM $Q(z)$. Let $R_u(z)$, $R_d(z)$, $R_w(z)$ and $R_f(z)$ be the corresponding TFMs from the control, disturbance, noise and fault inputs to the residual given by

$$\begin{aligned} &[\, R_u(z)\ R_d(z)\ R_w(z)\ R_f(z)\,] \\ &= Q(z)\left[\begin{array}{cccc} G_u(z) & G_d(z) & G_w(z) & G_f(z) \\ I_{Nm_u} & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

For the APFDIP, the decoupling condition (i) requires $R_u(z) = 0$ and $R_d(z) = 0$, or equivalently

$$Q(z)G(z) = 0 \qquad (7)$$

where

$$G(z) = \left[\begin{array}{cc} G_u(z) & G_d(z) \\ I_{Nm_u} & O \end{array}\right]. \qquad (8)$$

The fault isolation condition (ii) is equivalent to ask

$$\|R_f(z) - M_r(z)\| \approx 0, \qquad \|R_w(z)\| \approx 0, \qquad (9)$$

for a suitable TFM norm, where $M_r(z)$ is the lifted TFM of the reference model (3). When solving fault isolation problems, $M_r(z)$ usually corresponds to m_f diagonally stacked *single-input single-output* (SISO) stable periodic systems and thus is invertible. Therefore, for the solution of the APFDIP we ask that $R_f(z)$ is invertible as well. For the solution of the exact PFDIP we can freely ask $R_f(z) = M_r(z)$.

The conditions (7) and (9) complemented with the requirement of stability of $Q(z)$ lead naturally to a constrained *model-matching* formulation of the APFDIP, where we determine an "optimal" stable $Q(z)$ satisfying (7) which also minimizes the $\mathcal{H}_2$-norm error $\|R(z)\|_2$ with

$$R(z) := [\, M_r(z)\ 0\,] - Q(z)\left[\begin{array}{cc} G_f(z) & G_w(z) \\ 0 & 0 \end{array}\right] \qquad (10)$$

The actual choice of $M_r(z)$ may lead to a solution $Q(z)$ which corresponds to an unstable periodic system. Therefore, besides determining $Q(z)$, we also consider the determination of a suitable updating factor $M(z)$ of $M_r(z)$ to ensure the stability of the solution $Q(z)$ satisfying

$$\|R_f(z) - M(z)M_r(z)\| \approx 0.$$

Obviously, $M(z)$ must be chosen such that it corresponds to a causal, stable and invertible periodic system. Additionally, $M(z)$ must be chosen to have a block diagonal structure compatible with $M_r(z)$.

In the next section we propose a new synthesis algorithm of a residual generator (2) which solves the APFDIP by solving the above formulated constrained model matching problem. In Section V we present a lifting-free version of the proposed procedure based on explicit periodic state-space representations.

IV. Procedure for the Solution of the APFDIP

As basis for our synthesis procedure, we extend an algorithm proposed for LTI systems in [5] to LTP systems. The proposed *integrated* synthesis approach relies on repeated updating of an initial fault detection filter. The final filter results in a factored form with an explicitly determined periodic state space realization. For more clarity, we describe first the main steps of this algorithm in terms of the TFMs of the lifted representation. In the next section, we describe the computational variant of this algorithm in terms of the original periodic representation (1).

Step 1. Nullspace based reduction

We choose $Q(z)$ in a factored form

$$Q(z) = \overline{Q}_1(z)Q_1(z), \qquad (11)$$

where $Q_1(z)$ is a proper left rational nullspace basis satisfying $Q_1(z)G(z) = 0$ and $\overline{Q}_1(z)$ is a factor to be subsequently determined. With this choice, it follows that $Q(z)$ automatically fulfills the decoupling conditions in (i), namely, $R_u(z) = 0$ and $R_d(z) = 0$. The resulting $Q_1(z)$ has maximal row rank $Np - r_d$, where $r_d = \mathrm{rank}\, G_d(z)$. Thus, the existence condition of a nonempty rational nullspace basis $Q_1(z)$ in combination with the condition for the invertibility of $R_f(z)$ require that $Nm_f \le Np - r_d$. If this condition is not fulfilled, then part of disturbance inputs in $d(t)$ must be redefined as noise inputs to be included in $w(t)$.

With the above detector, $R(z)$ in (10) becomes

$$R(z) = [\, M_r(z)\ 0\,] - \overline{Q}_1(z)[\,\overline{G}_f(z)\ \overline{G}_w(z)\,] \qquad (12)$$

with

$$[\,\overline{G}_f(z)\ \overline{G}_w(z)\,] := Q_1(z)\left[\begin{array}{cc} G_f(z) & G_w(z) \\ 0 & 0 \end{array}\right] \qquad (13)$$

Note that we can always choose $Q_1(z)$ to correspond to a causal and stable periodic system and such that $[\,\overline{G}_f(z)\ \overline{G}_w(z)\,]$ defined in (13) corresponds to a causal and stable periodic system as well [11].

With this first preprocessing step, we reduced the original APFDIP formulated for the periodic system (1) to one formulated for a reduced periodic system with the lifted TFMs $\overline{G}_f(z)$ and $\overline{G}_w(z)$ and without control and disturbance inputs. For this system, we have to determine the TFM $\overline{Q}_1(z)$ which minimizes $\|R(z)\|_2$ with $R(z)$ redefined as in (12).

The requirement on the invertibility of $R_f(z)$ can be fulfilled only if $\overline{G}_f(z)$ is left invertible, that is

$$\operatorname{rank} \overline{G}_f(z) = Nm_f \tag{14}$$

Step 2. Row regularization

We can choose $\overline{Q}_1(z)$ in the form

$$\overline{Q}_1(z) = \overline{Q}_2(z)Q_2(z), \tag{15}$$

where $\overline{Q}_2(z)$ is still to be determined and $Q_2(z)$ is a prefilter chosen such that

$$[\,\widetilde{G}_f(z)\ \widetilde{G}_w(z)\,] := Q_2(z)[\overline{G}_f(z)\ \overline{G}_w(z)\,] \tag{16}$$

is full row rank. With this choice, $R(z)$ becomes

$$R(z) = [\,M_r(z)\ 0\,] - \overline{Q}_2(z)[\,\widetilde{G}_f(z)\ \widetilde{G}_w(z)\,] \tag{17}$$

The simplest choice of $Q_2(z)$ is a constant projection matrix which simply selects Nm_f linearly independent rows of $\overline{G}_f(z)$. A more involved choice is one leading to an invertible $\widetilde{G}_f(z)$ and simultaneously to $Q_2(z)Q_1(z)$ having least dynamical order. Such a choice is possible using periodic minimal dynamic cover techniques [13] (see Section V).

Step 3. Inner-outer factorization-based updating

This step is standard in solving $\mathcal{H}_2$-norm optimization problems and consists in compressing the full row rank TFM $[\,\widetilde{G}_f(z)\ \widetilde{G}_w(z)\,]$ to a full column rank (thus invertible) TFM. For this, we compute a quasi-co-outer–inner factorization

$$\begin{aligned}[\,\widetilde{G}_f(z)\ \widetilde{G}_w(z)\,] &= [\,G_{o,1}(z)\ 0\,]\left[\begin{array}{c} G_{i,1}(z) \\ G_{i,2}(z) \end{array}\right], \\ &:= G_o(z)G_i(z)\end{aligned} \tag{18}$$

where $G_i(z)$ is a $N(m_f+m_w)\times N(m_f+m_w)$ inner TFM and $G_{o,1}(z)$ is an $Nm_f \times Nm_f$ invertible TFM. Recall that a square TFM $G_i(z)$ is *inner* (and simultaneously *co-inner*) if it has only stable poles and satisfies $G_i(z)G_i^*(z) = I$ with $G^*(z) := G_i^T(1/z)$. The *quasi-co-outer* factor $G_o(z)$ may have besides stable zeros, also zeros which lie on the unit circle.

We can refine the parametrization of the detector by defining $Q_3(z) = G_{o,1}^{-1}(z)$ and choosing $\overline{Q}_2(z)$ of the form

$$\overline{Q}_2(z) = \overline{Q}_3(z)Q_3(z) \tag{19}$$

where $\overline{Q}_3(z)$ is to be determined. Using (18), (16), (15) and (19), we can express $R(z)$ in (17) as $R(z) = \overline{R}(z)G_i(z)$, with

$$\overline{R}(z) = \left[\ \overline{F}_1(z) - \overline{Q}_3(z)\ \middle|\ \overline{F}_2(z)\ \right], \tag{20}$$

where

$$[\,\overline{F}_1(z)\ \overline{F}_2(z)\,] = [\,M_r(z)\ 0\,]G_i^*(z) \tag{21}$$

Since $G_i(z)$ is an inner TFM, we have $\|R(z)\|_2 = \|\overline{R}(z)\|_2$. Thus, the determination of a stable and proper $\overline{Q}_3(z)$ which minimizes $\|\overline{R}(z)\|_2$ is an $\mathcal{L}_2$-norm least-distance problem.

Step 4: $\mathcal{H}_2$-optimal synthesis

The solution of the least distance problem in the case of $\mathcal{L}_2$-norm is straightforward. We determine $\overline{Q}_3(z)$ in the form $\overline{Q}_3(z) = Q_5(z)Q_4(z)$, where $Q_4(z)$ is the stable projection

$$Q_4(z) = \{\overline{F}_1(z)\}_+$$

and $Q_5(z) = M(z)$ is to be determined at the next step. Here, $\{\cdot\}_+$ denotes the stable part of the underlying TFM including the direct feedthrough term, while $\{\cdot\}_-$ denotes the unstable part. With the above choice, we achieved the minimum $\mathcal{H}_2$-norm of $R(z)$, which can be computed as

$$\|R(z)\|_2 = \|\overline{R}(z)\|_2 = \|[\,\{\overline{F}_1(z)\}_-\ \ \overline{F}_2(z)\,]\|_2 \tag{22}$$

Since the underlying TFMs are unstable, the $\mathcal{L}_2$-norm is used instead of the $\mathcal{H}_2$-norm in the last equation.

Step 5. Stabilization

The resulting final detector with the lifted TFM

$$Q(z) = Q_5(z)Q_4(z)Q_3(z)Q_2(z)Q_1(z) \tag{23}$$

must correspond to a stable and causal periodic system realization. If $M_r(z)$ has been appropriately chosen, then $Q_4(z)Q_3(z)Q_2(z)Q_1(z)$ is stable and we can choose $Q_5(z)$ simply the identity matrix of order Nm_f. With this choice, we achieved the minimum $\mathcal{H}_2$-norm of $R(z)$, which can be computed as in (22).

In the general case, we choose $Q_5(z) = M(z)$, where $M(z)$ ensures that the resulting final detector (23) corresponds to a stable and causal periodic system realization. $M(z)$ can be determined using stable and proper coprime factorization techniques (see Section V) and can be interpreted as an updating factor for $M_r(z)$. In this case, $Q(z)$ can be interpreted as approximation of the solution of the weighted minimization problem

$$\|M(z)R(z)\|_2 = \min$$

and $M(z)M_r(z)$ is the updated reference model.

The described synthesis method in Steps 1-5 extends the method proposed for LTI systems in [5], [6]. For numerical computations, an integrated synthesis algorithm is described in the next section.

V. Computational algorithms

In this sections, we present an equivalent synthesis approach for periodic systems without manipulating explicitly lifted representations. The proposed computational approach operates directly on the matrices of the original periodic state-space description (1) and computes left annihilators directly in periodic minimal state-space representations. All subsequent computations to determine a stable fault detection filter which solves the APFDIP can be interpreted as updates of the initial representation and can be done using reliable numerical techniques based on state-space computations.

Step 1. Computation of a maximal left annihilator

In this computational step we employ the computational approach of [11] to determine a maximal left annihilator with the lifted TFM $Q_1(z)$ for the periodic system

$$\begin{aligned} E_k x_G(k+1) &= A_k x_G(k) + B_k^u u(k) + B_k^d d(k) \\ \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} &= \begin{bmatrix} C_k \\ 0 \end{bmatrix} x_G(k) + \begin{bmatrix} D_k^u \\ I_{m_u} \end{bmatrix} u(k) + \begin{bmatrix} D_k^d \\ 0 \end{bmatrix} d(k), \end{aligned} \tag{24}$$

which corresponds to the lifted TFM $G(z)$ in (8). In terms of lifted representations, this amounts to determine a periodic system (e.g., of the form (2)), whose lifted TFM $Q_1(z)$ is a proper rational matrix whose rows form a rational basis for the left nullspace of $G(z)$ (i.e., $Q_1(z)G(z) = 0$) see, for example, [14].

According to [11], $Q_1(z)$ can be determined to have a causal periodic realization of the form

$$\begin{aligned} E_k^l \overline{x}(k+1) &= A_k^l \overline{x}(k) + B_k^l \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \\ \overline{y}(k) &= C_k^l \overline{x}(k) + D_k^l \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \end{aligned} \tag{25}$$

with nonsingular E_k^l. Note that, this detector realization is observable, and is obtained in general with time-varying dimensions of the state vector ($\overline{n}_k$) and output vector (q_k). The output vector dimensions sum up to $\sum_{k=1}^N q_k = Np - r_d$.

A realization of $[\, \overline{G}_f(z)\ \overline{G}_w(z) \,]$ in (13) can be obtained in the form

$$\begin{aligned} E_k^l x_f(k+1) &= A_k^l x_f(k) + \widetilde{B}_k^f f(k) + \widetilde{B}_k^w w(k) \\ r(k) &= C_k^l x_f(k) + \widetilde{D}_k^f f(k) + \widetilde{D}_k^w w(k) \end{aligned} \tag{26}$$

As it can be observed, the realizations of $Q_1(z)$, $\overline{G}_f(z)$ and $\overline{G}_w(z)$ share the same matrices E_k^l, A_k^l, and C_k^l.

To determine the left annihilator (25), a single reduction of a periodic pair to a periodic Kronecker-like form has to be performed using the algorithm of [15]. This algorithm performs exclusively orthogonal transformations on a pair of periodic matrices, and it is possible to easily prove that all computed matrices are exact for a slightly perturbed original system. It follows that the algorithm to compute the left annihilator (25) and the corresponding realization of $[\, \overline{G}_f(z)\ \overline{G}_w(z) \,]$ in (26) is *numerically stable*.

To check the left invertibility condition (14) we can apply the numerically stable algorithm of [15] to compute the periodic Kronecker-like form of the periodic pair

$$\left(\begin{bmatrix} A_k^l & \widetilde{B}_k^f \\ C_k^l & \widetilde{D}_k^f \end{bmatrix}, \begin{bmatrix} E_k^l & 0 \\ 0 & 0 \end{bmatrix} \right)$$

and check the absence of right structure. Alternatively, the so-called *fast* algorithms based on orthogonal reductions (see [16]) can be employed to check the full column rank of the system pencil of the lifted system associated to the periodic system $(E_k^l, A_k^l, \widetilde{B}_k^f, C_k^l, D_k^f)$.

Step 2. Row regularization

The computational details for this computation are presented in [13] (see also [17] for a shorter account). The resulting realizations of $Q_2(z)Q_1(z)$ of least McMillan degree can be obtained as a minimal state space realization

$$(\widehat{E}_k, \widehat{A}_k, \widehat{B}_k, \widehat{C}_k, \widehat{D}_k), \tag{27}$$

with $\widehat{E}_k$ invertible. The periodic realization of the resulting $[\, \widetilde{G}_f(z)\ \widetilde{G}_w(z) \,] := Q_2(z)[\, \overline{G}_f(z)\ \overline{G}_w(z) \,]$ can be computed in the form

$$(\widehat{E}_k, \widehat{A}_k, [\, \widehat{B}_k^f\ \widehat{B}_k^w \,], \widehat{C}_k, [\, \widehat{D}_k^f\ \widehat{D}_k^w \,]) \tag{28}$$

Once again, the realizations of $Q_2(z)Q_1(z)$ and $[\, \widetilde{G}_f(z)\ \ \widetilde{G}_w(z) \,]$ share the same matrices $\widehat{E}_k$, $\widehat{A}_k$, and $\widehat{C}_k$, a property which is instrumental in performing the next computational step.

Step 3. Inner-outer factorization-based updating

The inner-outer factorization (18) of $[\, \widetilde{G}_f(z)\ \widetilde{G}_w(z) \,]$ with the periodic state-space realization (28) can be computed using the recent algorithm of [12]. An important feature of this algorithm is that the periodic state-space realization of the outer factor $G_{o,1}(z)$ can be obtained in the form

$$(\widehat{E}_k, \widehat{A}_k, \overline{B}_k^o, \widehat{C}_k, \overline{D}_k^o) \tag{29}$$

This allows to compute an explicit realization of $Q_3(z)Q_2(z)Q_1(z) = \widetilde{G}_{o,1}^{-1}(z)Q_2(z)Q_1(z)$ as the periodic descriptor system realization (see [9])

$$\left(\begin{bmatrix} \widehat{E}_k & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \widehat{A}_k & \overline{B}_k^o \\ \widehat{C}_k & \overline{D}_k^o \end{bmatrix}, \begin{bmatrix} \widehat{B}_k \\ \widehat{D}_k \end{bmatrix}, [0\ \ -I_{m_f}], 0 \right) \tag{30}$$

This realization corresponds to a causal but possibly unstable realization (e.g., with some characteristic multipliers on the unit circle).

The stable inner factor $G_i(z)$ has a periodic state-space realization, whose order reflects the number of unstable zeros of $[\, \widetilde{G}_f(z)\ \widetilde{G}_w(z) \,]$. All unstable zeros at infinity are reflected into null characteristic multipliers (i.e., null poles) in the realization of $G_i(z)$. Because of possible null characteristic multipliers, the inverse $G_i^{-1}(z) = G_i^*(z)$ has in general a periodic descriptor realization $(E_k^i, A_k^i, B_k^i, C_k^i, D_k^i)$ with possibly singular E_k^i. This realization can be explicitly obtained using the know formulas for building conjugated or inverse periodic systems (see for example [18]). The periodic realization of $[\, \overline{F}_1(z)\ \overline{F}_2(z) \,]$ in (21) has the form

$$(E_k^F, A_k^F, B_k^F, C_k^F, D_k^F) = \left(\begin{bmatrix} I & 0 \\ 0 & E_k^i \end{bmatrix}, \begin{bmatrix} A_k^r & B_k^r C_k^i \\ 0 & A_k^i \end{bmatrix}, \begin{bmatrix} B_k^r D_k^i \\ B_k^i \end{bmatrix}, [\, C_k^r\ D_k^r C_k^i \,], D_k^r D_k^i \right)$$

Step 4: $\mathcal{H}_2$-optimal synthesis

To compute the additive spectral separation

$$\overline{F}_1(z) = \{\overline{F}_1(z)\}_+ \{\overline{F}_1(z)\}_- \tag{31}$$

we exploit the structure of the realization of $\overline{F}_1(z)$, where the leading block A_k^r of A_k^F has only stable characteristic multipliers, while the trailing pair (E_k^i, A_k^i), with A_k^i invertible, has only unstable characteristic multipliers. To achieve the additive decomposition (31), we apply a Lyapunov similarity transformation with the periodic matrices

$$W_k = \begin{bmatrix} I & X_k \\ 0 & I \end{bmatrix}, \quad Z_k = \begin{bmatrix} I & -X_{k-1}E^i_{k-1} \\ 0 & I \end{bmatrix}$$

to annihilate the off-diagonal $(1,2)$-blocks of the pair $(W_kE^F_kZ_{k+1}, W_kA^F_kZ_k)$, that is

$$X_kA^i_k - A^r_kX_{k-1}E^i_{k-1} + B^r_kC^i_k = 0$$

Thus, for the computation of $\{\overline{F}_1(z)\}_+$ we need only to solve this generalized periodic Sylvester equation. Suitable algorithms for this purpose have been proposed in [19].

The solution $Q_4(z) = \{\overline{F}_1(z)\}_+$ of the least distance problem we obtain with the explicit periodic realization

$$(A^r_k, B^r_kD^{i,1}_k + X_kB^{i,1}_k, C^r_k, D^r_kD^{i,1}_k)$$

where $B^{i,1}_k$ and $D^{i,1}_k$ are the first m_f columns of the matrices B^i_k and D^i_k, respectively

Step 5. Stabilization

We obtain the state-space realization of $\overline{Q}(z) = Q_4(z)Q_3(z)Q_2(z)Q_1(z)$ in a straightforward way using series coupling formulas. If this realization is not stable, we have to perform the additional stabilization step to determine the updating factor $M(z)$ of $M_r(z)$, which ensures the stability of the final fault detection filter. Assume that $(\overline{E}_k, \overline{A}_k, \overline{B}_k, \overline{C}_k, \overline{D}_k)$ is the periodic realization of $\overline{Q}(z)$ as resulted at the previous step (e.g., after applying the minimal realization algorithm of [20]). Since the updating factor $M(z)$ must correspond to m_f diagonally stacked SISO stable periodic systems, we have

$$M(z) = \text{diag}\,\{M_1(z), M_2(z), \ldots, M_{m_f}(z)\},$$

where $M_j(z)$ is the lifted TFM of the j-th SISO periodic subsystem. In what follows, we describe the computation of a minimal realization of $M_j(z)$ and of the corresponding block rows $Q_j(z) := M_j(z)\overline{Q}_j(z)$, where $\overline{Q}_j(z)$ is the block row of $\overline{Q}(z)$ corresponding to the j-th filter output.

Let $\overline{C}_{k,j}$ and $\overline{D}_{k,j}$ denote the j-th rows of $\overline{C}_k$ and $\overline{D}_k$, respectively. Thus, $(\overline{E}_k, \overline{A}_k, \overline{B}_k, \overline{C}_{k,j}, \overline{D}_{k,j})$ is a realization of $\overline{Q}_j(z)$ which may be non-minimal. A minimal realization can be computed using the algorithm of [20]. For simplicity, we reuse the same notation for the resulting minimal realization. The PRCF Algorithm of [18] can be employed to compute a stable *periodic right coprime factorization* (PRCF) of $\overline{Q}_j(z)$ as

$$\overline{Q}_j(z) = M_j^{-1}(z)Q_j(z),$$

where both $M_j(z)$ and $Q_j(z)$ correspond to stable periodic systems. The underlying computational algorithm is numerically reliable and rely on sound computational techniques, as the computation of coprime factorizations using recursive generalized periodic Schur technique for the assignment of characteristic multipliers.

VI. Conclusions

We proposed a general, lifting-free computational approaches to solve the APFDIP using a $\mathcal{H}_2$-norm minimization approach. The proposed numerical solution of the APFDIP is an example of a numerically reliable *integrated* computational algorithm, with closely connected computational steps, where the fault detection filter is implicitly determined in a factored form, where each factor corresponds to a typical computational step. The proposed procedure is able to determine a solution to the APFDIP in the most general setting, as for example, without constraints on the system zeros. For all computational steps numerically reliable or even numerically stable algorithms are available.

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