

TRANSPORT USER BENEFITS CALCULATION WITH THE “RULE OF A HALF” FOR TRAVEL DEMAND MODELS WITH CONSTRAINTS

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Christian Winkler

Institute of Transport Research, German Aerospace Center (DLR), Rutherfordstraße 2, 12489 Berlin, Germany

Email for correspondence: christian.winkler@dlr.de

ABSTRACT

The importance of user benefits in transport projects assessments is well-known by transport planners and economists. Generally they have the greatest impact on the result of cost-benefit analysis. It is common practice to adopt the consumer surplus measure for calculating transport user benefits. Normally the well-known “Rule of a Half”, as a practical approximation for the integral of the demand curve, is used to determine the change of consumer surplus. In this paper we enter into the question of whether the Rule of a Half is valid in the case of travel demand models with multiple constraints. Such models are often used for travel demand modeling of large-scale areas. The most discussed and well-known model in transport modeling field is the doubly constrained gravity model. Beside this model with inelastic constraints there are also more flexible models with elastic constraints. The theoretical analysis in this paper provides a mathematical proof for the validity of the concept of the Rule of a Half for travel demand models with multiple elastic and inelastic constraints. In this case the Rule of a Half is also a correct approximation of the change of consumer surplus.

Keywords: consumer surplus, Rule of a Half, travel demand model, multinomial logit model, constraints

1 INTRODUCTION

Investments in the transport sector are hallmarked by a long-term useful life and high investment costs. Due to restricted public financial resources not all transport projects are realizable and they have to be economically appraised. For this purpose different methodical approaches exist. The most important concept is the cost-benefit analysis, which is preferentially used worldwide.

Generally in transport project appraisals, transport user benefits have the most impact. The user benefits result from the changes of user costs like travel time and travel cost between the "Do-Minimum" situation (the most likely transport situation over the course of the appraisal period if no intervention were to occur) and the "Do-Something" situation (the Do-Minimum with transport intervention). The economic theory – as the theoretical basis of the cost-benefit analysis – provides the concept of the consumer surplus for the calculation of the transport user benefits. This measure can be derived by the mathematical integral of the demand function (See, for example, Varian, 1992).

The importance of user benefits in transport projects assessments is well-known by transport planners and economists. But the calculation of the exact integral of multiple travel demand functions is quite challenging. Hence, normally the "Rule of a Half", as a practical approximation of the integral of the demand function, is used to estimate the change of consumer surplus. This approach is proven and correct for travel demand models without multiple sets of constraints.

But for travel demand modeling of large-scale areas, models dealing with two sets of constraints are often used. The most discussed and well-known model in the transport modeling field is the doubly constrained gravity model. Beside this model with inelastic constraints, there are also more flexible models with elastic constraints. For doubly (or more) constrained models (regardless whether inelastic or elastic) the correctness of the Rule of a Half has not yet been proved. Because of the impact and importance of the correct transport user benefits measure for cost-benefit analysis, the investigation of this aspect is highly relevant.

2 TRAVEL DEMAND MODELS

For travel demand modeling many different theories and approaches exist (for a good overview see, for example, Ortúzar and Willumsen, 2011). Beside the main goal of realistic travel demand modeling the theoretical base of the models is also highly important for the context here discussed. Thereby one can differentiate between doubly constrained models, which satisfy the given total numbers of trips originating in a travel zone **and** trips attracted to a travel zone respectively. Such constraints are powerful instruments and enable more realistic results especially for travel demand modeling of large-scale areas. The total number of trips is a result of the trip generation (first step of the 4-step-algorithm) and an input

variable of the trip distribution. But the economic consistency of such models is unclear, because of their theoretical derivations from analogies of natural laws (e.g. gravitation, entropy maximization).

Beside traditional doubly constrained models, discrete choice models are highly relevant for the context considered here. The most famous representative of them is the multinomial logit model. But the model is only able to satisfy one set of constraints:

1. the total number of produced trips in each zone or
2. the total number of attracted trips in each zone or
3. the total number over all zones.

However, the multinomial logit model is derived from the economic concept of utility maximization and is completely economically consistent. Both approaches – doubly constrained models and the multinomial logit model – have advantages and disadvantages in the considered context and their strengths should be combined. In particular, the constraints should be combined with the economical interpretability of the multinomial logit model.

2.1 The Doubly Constrained Trip Distribution Model

Constraints are used to satisfy the given number of trips in all travel zones. They are a very useful and pragmatic method to achieve good results for trip distribution for large-scale areas. The most well-known model is the doubly constrained gravity model, using inelastic constraints solely (see Wilson, 1967). Another, more flexible, model is the “EVA model” (from the German terms for production (**E**rzeugung), distribution (**V**erteilung) and mode choice (**A**ufteilung)). It is a simultaneous model for production, distribution and mode choice but here we only consider the distribution part for the sake of simplicity (for a more detailed view see Vrtic et al., 2007). Differently to the “Wilson model”, the EVA model deals with inelastic and/or elastic constraints.

The content of this paper is based on the EVA model. Since it is not possible to show all facets, in the following the model specification with a set of inelastic constraints for the origins and a set of elastic constraints for the destinations will be used. The EVA model is then:

$$\begin{aligned}
 T_{ij} &= f(g_{ij}) \cdot a_i \cdot b_j \\
 \left. \begin{aligned}
 O_i^{min} &\leq \sum_j T_{ij} = O_i \leq O_i^{max} && \text{with } O_i^{min} = O_i^{max} \\
 D_j^{min} &\leq \sum_i T_{ij} = D_j \leq D_j^{max} && \text{with } D_j^{min} < D_j^{max}
 \end{aligned} \right\} \text{constraints}
 \end{aligned} \tag{1}$$

with

a_i, b_j balancing factors to satisfy the constraints

f deterrence function with regard to the generalized costs g_{ij} from zone i to zone j

O_i total number of trips originating at zone i
 D_j total number of trips attracted to zone j
 T_{ij} number of trips from zone i to zone j

Thereby the number of trips is calculated taking into account generalized costs and the two sets of constraints. The constraints are satisfied by the balancing factors, which are solved iteratively. For solving this optimization problem different methods can be used, e.g. the Furness algorithm (Furness, 1965). The constraints are calculated in the trip generation a priori, but in contrast to the fixed origin constraints ($O_i^{min} = O_i^{max}$) only minimal and maximal values are defined for the destination constraints. In this intermediate range the resulting number of trips attracted to the different zones is calculated in the trip distribution. Thus, the number of attracted trips does not depend only on the number of attractors (e.g. number of persons) but also on the accessibility of the zones. In particular, this is important for substitutable trips like shopping.

There are many different versions of the above-mentioned deterrence function, although the exponential function has gained a very high importance and will be applied below. The integrated generalized costs g_{ij} are composed by travel time and travel cost from origin i to destination j and the value of travel time for monetizing (Goodwin, 1974). Hence, the function can be written:

$$f(g_{ij}) = e^{-g_{ij}} \quad \text{with } g_{ij} = c_{ij} + \text{vot} \cdot t_{ij} \quad (2)$$

with

c_{ij} travel cost from zone i to zone j
 t travel time from zone i to zone j
vot value of time

The value of time can be derived either from special travel behavioral surveys (e.g. stated response surveys) or from the model itself (as a result of model calibration).

2.2 The Multinomial Logit Model

Discrete choice modeling and in particular the (here considered) multinomial logit model (MNL model) is probably the most used decision concept in the travel modeling context currently. These models are grounded on the economic concept of utility maximization and the individuals are assumed choosing the most beneficial (optimal) alternative (trip from i to j). Unlike the constrained models, the MNL model calculates probabilities for the discrete alternatives and not the trips directly. These are definable by multiplying the probabilities with the total number of trips.

The comparison of alternatives is on the basis of the alternative specific utilities. Therefore utility functions including decision-relevant variables like travel costs and travel time have to be defined. Due to travel modelers' not being able to observe all decision relevant

components, as well as travelers’ inaccurate “perception” of the objective costs, the utility function additionally includes, beside objective components, a stochastic portion of utility standing for the unobservable part. The utility function can then be described as (Ben-Akiva and Lerman, 1985):

$$u_{ij} = \bar{u}_{ij} + \varepsilon_{ij} \quad (3)$$

with

u_{ij} utility of an alternative (trip from zone i to zone j)

\bar{u}_{ij} observable part of utility (deterministic)

ε_{ij} unobservable part of utility (stochastic)

The chosen distribution of the stochastic part of utility determines different discrete choice models. The MNL model results from the use of Gumbel distribution and the calculation of the probabilities of trips from zone i to zone j can be written:

$$P_{ij} = \frac{e^{\bar{u}_{ij}}}{\sum_{i'} \sum_{j'} e^{\bar{u}_{i'j'}}} \quad (4)$$

with

P_{ij} probability of choosing alternative i-j

e exponential function

The wanted number of trips from zone i to zone j result from a globally fixed MNL model (satisfies only the total number over all zones):

$$T_{ij} = P_{ij} \cdot T \quad (5)$$

with

T total number of trips over all zones

The observable part of utility, which is here defined by the negative generalized costs, is highly relevant. Thus, it is:

$$\bar{u}_{ij} = -g_{ij} = -c_{ij} - \text{vot} \cdot t_{ij} \quad (6)$$

The simplicity of calculating travel demand on the ground on the MNL model is one important model advantage but there is another good point in addition. In several publications and especially in a ground-breaking theoretical analysis by McFadden, the embedding of discrete choice models in the economic theory could be shown. This is of great value for integrated demand modeling and benefit calculating. Hence, travel demand thus derived is economically consistent and the integral of the travel demand function provides the

consumer surplus. The great practical and theoretical advantages of the MNL model are accompanied by the disadvantage of the absence of two sets of constraints.

3 THE DOUBLY CONSTRAINED MULTINOMIAL LOGIT MODEL

The combination of economic consistence on the one hand and the simplicity of travel demand estimation on the other hand is a big plus of the MNL model. However the limitation of considering multiple constraints is a serious drawback for several transport planning problems. Hence, a doubly constraint MNL model has to be formulated enabling the satisfaction of production **and** attraction constraints. Thus, the objective is the formulation of the EVA model in terms of the MNL model. For the sake of simplicity, the derivation is carried out only for the case with a set of inelastic production and a set of elastic attraction constraints. Of course other combinations are possible as well.

The utility function is the central element to account for decision-relevant characteristics in the MNL model. In order to consider two sets of constraints the utility function in (6) origin and destination “shadow prices” has to be added to the generalized costs (Neuburger and Wilcox, 1976). The term “shadow” is used, since these “prices” are not definable a priori but derivable from the balancing factors a_i and b_j of the EVA model a posteriori. The given number of originating and attracting trips contain additional information about the underlying decision process of the transport users. This information is implemented in the utility function by the shadow prices which serve the emendation of the “a priori utility”. Then, by means of the shadow prices, the utility maximization process provides the probabilities of alternatives producing a result with satisfied constraints. The utility function can then be written:

$$\begin{aligned}\bar{u}_{ij} &= -c_{ij} - \text{vot} \cdot t_{ij} + \theta_i + \tau_j \\ &= -g_{ij} + \theta_i + \tau_j\end{aligned}\quad (7)$$

with

θ_i, τ_j shadow prices of origin i an destination j

The shadow prices in this function – as well as the generalized costs – are monetary factors. By implementing this function in the MNL model we get the doubly constrained MNL model for estimating trips T_{ij} :

$$\begin{aligned}T_{ij} &= P_{ij} \cdot T = \frac{e^{(-g_{ij} + \theta_i + \tau_j)}}{\sum_{i'} \sum_{j'} e^{(-g_{i'j'} + \theta_{i'} + \tau_{j'})}} \cdot T \\ O_i^{\min} &\leq \sum_j T_{ij} = O_i \leq O_i^{\max} \quad \text{with } O_i^{\min} = O_i^{\max} \\ D_j^{\min} &\leq \sum_i T_{ij} = D_j \leq D_j^{\max} \quad \text{with } D_j^{\min} < D_j^{\max}\end{aligned}\quad (8)$$

4 THE RULE OF A HALF

The concept of the Rule of a Half (RoH) as an approximation of calculation of the change of consumer surplus arising from specific transport intervention was used for the first time at the end of the 1960s (Williams, 1977). In all likelihood it is the most-commonly applied instrument in practice for estimating user benefits. For example, the RoH is recommended by the Economic Commission for Europe of the United Nations (United Nations, 2003). Its popularity is based on the easy calculability and the universal applicability.

The change in consumer surplus over all transport users calculated by the RoH is:

$$\begin{aligned}\Delta CS &= \sum_i \sum_j \Delta CS_{ij} \\ &= \sum_i \sum_j \left(\frac{1}{2} \cdot (g_{ij}^0 - g_{ij}^1) (T_{ij}^0 + T_{ij}^1) \right)\end{aligned}\quad (9)$$

with

ΔCS change in consumer surpluses of all transport users and trip relations i-j

ΔCS_{ij} change in consumer surpluses of all transport users T_{ij}

The RoH is a pragmatic and comprehensible method to estimate the change in consumer surplus. Williams (1976) contributed a mathematical analysis of the concept, which proved the correctness of the RoH as an approximation of the true benefit measure in economic sense. Hence, the methodical approach is theoretically consistent. But the main question here is whether this is true for the case of travel demand models with multiple constraints.

5 THE RULE OF A HALF FOR TRAVEL DEMAND MODELS WITH CONSTRAINS

The RoH method aims to measure the user benefits by an approximate integration of the travel demand function between the initial and final generalized costs. In the case of the doubly constrained MNL model, however, the demand is influenced by shadow prices as well. Now, the question is, which consequences do these additional (model) factors have on the change of consumer surplus quantified by the RoH? At first, of course, the RoH can be used for the doubly constrained MNL model. Then, all deterministic components of the utility function – including shadow prices – have to be considered by integrating the demand function. So we obtain:

$$\begin{aligned}\Delta CS^* &= \sum_i \sum_j \Delta CS_{ij}^* \\ &= \sum_i \sum_j \left(\frac{1}{2} \cdot \left((g_{ij}^0 - \theta_i^0 - \tau_j^0) - (g_{ij}^1 - \theta_i^1 - \tau_j^1) \right) \cdot (T_{ij}^0 + T_{ij}^1) \right)\end{aligned}\quad (10)$$

ΔCS^* stands for the approximate integral of the doubly constrained MNL model, in which the influence of shadow prices is included. However, the desired change in consumer surplus is only a function of the real quantities of generalized costs (Williams, 1976). So ΔCS^* has to be adjusted for the contribution of the constraints (by shadow prices), since (Winkler, 2011):

$$\Delta CS^* \neq \Delta CS. \quad (11)$$

It now needs clarifying, whether the RoH, taking only generalized costs into consideration (and neglecting the shadow prices), gives a correct (approximate) benefit measure in the case of a doubly constrained MNL model. The mathematical decomposition of the integral of the doubly constrained MNL model is needed to prove this.

It can be shown that we can write (Winkler, 2011):

$$\Delta CS^* = - \int_{g^0}^{g^1} \sum_i \sum_j T_{ij} dg_{ij} + \int_{\theta^0}^{\theta^1} \sum_i \sum_j T_{ij} d\theta_i + \int_{\tau^0}^{\tau^1} \sum_i \sum_j T_{ij} d\tau_j. \quad (12)$$

The desired component ΔCS , which encompasses the change of consumer surplus due to the change in real costs (generalized costs), is represented only by the first term on the right side. In contrast, the effect of the shadow prices has to be extracted. Initially, for finding of ΔCS equation (12) can be expressed by:

$$\Delta CS = \Delta CS^* - \int_{\theta^0}^{\theta^1} \sum_i \sum_j T_{ij} d\theta_i - \int_{\tau^0}^{\tau^1} \sum_i \sum_j T_{ij} d\tau_j. \quad (13)$$

The term ΔCS^* can be approximated by the RoH (as shown above) and is here revised to the portion of the shadow prices. Unfortunately the two shadow price integrals are not readily solvable, but further simplifications are possible. Both integrals are one-dimensional and cover the dimension of the production or attraction side only. Since a change in θ_i does not have any impact on the total number of attracted trips D_j (the same applies to τ_j and O_i), it can be written (with $\sum_j T_{ij} = O_i$ and $\sum_i T_{ij} = D_j$):

$$\Delta CS = \Delta CS^* - \int_{PA} \sum_i O_i d\theta_i - \int_{PA} \sum_j D_j d\tau_j. \quad (14)$$

After these simplifications and mathematical redrafts, the integration of functions of the total number of originated and attracted trips over the shadow prices is now necessary. But there are no (explicit) closed formulas for the calculus of the O_i s and D_j s and integration cannot easily be achieved.

It is of great importance for the solution of the integrals, whether the initial and final total number of trips O_i and D_j , respectively, are equal or not. Generally, in the case of inelastic constraints, the numbers are equal. On the other hand with elastic constraints changes

almost always occur. As hitherto, for solving the integrals, we consider inelastic constraints on the total number of originating trips and elastic constraints on the total number of attracting trips in each zone.

For the inelastic case there are no formal interdependences of the numbers of trips of the zones and the O_i s can be taken outside the integrals. Hence, it can be shown that the solution is:

$$\int_{PA} \sum_i O_i d\theta_i = \sum_i O_i \cdot (\theta_i^1 - \theta_i^0). \quad (15)$$

All required factors are available – the O_i s from the trip generation and the θ_i s from the trip distribution. So there is no problem calculating the first portion of benefits.

For the elastic case the D_j s are not constant and, in principal, it is necessary to integrate the (unknown) functions of the resulting D_j s. But we only have information about the D_j s for the two equilibrium (initial and final) states. Hence, we have to make an assumption about the shape of the function. According to the RoH we assume a linear form and we get:

$$\int_{PA} \sum_j D_j d\tau_j = \sum_j \left(\frac{D_j^0 + D_j^1}{2} \right) \cdot (\tau_j^1 - \tau_j^0). \quad (16)$$

In this case all required factors are given by the trip distribution.

In order to solve equation (14) we can insert a simplified version of equation (10):

$$\begin{aligned} \Delta CS^* &= \sum_i \sum_j \frac{1}{2} \cdot (g_{ij}^0 - g_{ij}^1) \cdot (T_{ij}^0 + T_{ij}^1) \\ &+ \sum_i \frac{1}{2} \cdot (\theta_i^1 - \theta_i^0) \cdot (O_i^0 + O_i^1) + \sum_j \frac{1}{2} \cdot (\tau_j^1 - \tau_j^0) \cdot (D_j^0 + D_j^1) \end{aligned} \quad (17)$$

The insertion of the solved integrals (15) and (16) as well as the simplification (17) in equation (14) provide:

$$\begin{aligned} \Delta CS &= \sum_i \sum_j \frac{1}{2} \cdot (g_{ij}^0 - g_{ij}^1) \cdot (T_{ij}^0 + T_{ij}^1) \\ &+ \sum_i \frac{1}{2} \cdot (\theta_i^1 - \theta_i^0) \cdot (O_i^0 + O_i^1) + \sum_j \frac{1}{2} \cdot (\tau_j^1 - \tau_j^0) \cdot (D_j^0 + D_j^1) \\ &- \sum_i O_i \cdot (\theta_i^1 - \theta_i^0) - \sum_j \left(\frac{D_j^0 + D_j^1}{2} \right) \cdot (\tau_j^1 - \tau_j^0) \end{aligned} \quad (18)$$

Considering, that is $O_i^0 = O_i^1 = O_i$ for the inelastic case, we obtain:

$$\sum_i \frac{1}{2} \cdot (\theta_i^1 - \theta_i^0) \cdot (O_i^0 + O_i^1) = \sum_i O_i \cdot (\theta_i^1 - \theta_i^0). \quad (19)$$

Furthermore, it is:

$$\sum_j \frac{1}{2} \cdot (\tau_j^1 - \tau_j^0) \cdot (D_j^0 + D_j^1) = \sum_j \left(\frac{D_j^0 + D_j^1}{2} \right) \cdot (\tau_j^1 - \tau_j^0). \quad (20)$$

Consequently, the last four terms in equation (18) cancel out. So then we can see, that the RoH is a correct approximation of the change of consumer surplus for travel demand models with multiple constraints as well. We have derived the approaches for inelastic constraints on the total number of originating trips and elastic constraints on the total number of attracting trips. Of course, the approaches are applicable in any combination (See Winkler, 2011).

6 CONCLUSION

On the one hand we have been able to show that doubly constrained travel demand models can be expressed in terms of the MNL model. On the other hand we have presented a proof for the correctness of the widely used RoH on the basis of doubly constrained travel demand models. There is no need for correction terms. The shadow prices do not have any effect on real user benefits and can be disregarded in benefits calculation.

The findings are of theoretical interest, since they show the possibility of an economic formulation of travel demand models with multiple constraints and the applicability of the RoH. For practical use the results are very important too, since there is a verification of all previous user benefits evaluations by the RoH independent of the used travel demand model – with or without multiple constraints. The same applies to all following evaluations which use this concept.

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