AN ATMOSPHERIC MESOSCALE MODEL: TREATMENT OF HYDROSTATIC FLOWS AND APPLICATION TO FLOWS WITH HYDRAULIC JUMPS

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SUMMARY

For numerical simulation of three-dimensional atmospheric flows at micro- and mesoscales a finite-difference method has been developed (programme MESOSCOP). In its non-hydrostatic variant it requires the inversion of an elliptic (Poisson) equation. Here we describe an alternative which applies the hydrostatic approximation and thus avoids the elliptic equation. It has been found, however, that the hydrostatic version requires smaller time steps for stability. The method is applied to investigate the formation of hydraulic jumps in shallow nonlinear fluid flow over a mountain ridge on a rotating plane. By scale analysis and numerical experiments it is shown that the effect of rotation is negligble essentially if F^2/Ro is small, where F is the Froude number and Ro the Rossby number.

INTRODUCTION

A three-dimensional numerical model (MESOSCOP) has been developed to simulate microscale and mesoscale atmospheric flows (horizontal scales less than a few hundred kilometres). The model is implemented in three versions in order to study clouds and precipitation processes over flat terrain, airflow over mountains and turbulent boundary layers. Mesoscale models use either the (complete) non-hydrostatic equations or the hydrostatic approximation [5]. The hydrostatic approximation is valid if the horizontal scales of the flow are much larger than the vertical ones. The approximation results in a possibly more efficient coding since it does not require the solution of an elliptic equation for the pressure as it is necessary for non-hydrostatic incompressible flows. Often, however, the validity of this approximation is not easy to prove. Therefore, it is desirable to have both, the hydrostatic and non-hydrostatic version available, so that the degree of hydrostasy can be determined in concrete flow situations. In a recent paper [8], the numerical version of MESOSCOP for non-hydrostatic flows is described together with five validation examples. Here, we introduce a hydrostatic variant and apply it to investigate the generation of hydraulic jumps in rotating and non-rotating flows. The behaviour of such hydraulic jumps is of interest with respect to cold fronts which are deformed or blocked by high mountains [1, 6].

THE NON-HYDROSTATIC ALGORITHM

MESOSCOP is based on the conservation laws for density ρ, volume specific momentum $\rho\vec{v}$, and several scalars $\rho\psi_k$, $k=1, \ldots, K$. The number K and the meaning of the scalars depend on the specific applications. The complete model applies for compressible as well as for incompressible fluids. Since the hydrostatic version is restricted to incompressible Boussinesq fluids, we give the equations for this case only; $\overline{\rho}$ is the time-independent reference density:

$$\mathrm{div}(\overline{\rho}\vec{v}) \quad = \quad 0\,, \tag{1}$$

$$\frac{\partial \overline{\rho}\vec{v}}{\partial t} + \mathrm{div}(\overline{\rho}\vec{v}\vec{v}) + 2\vec{\Omega}\times(\overline{\rho}\vec{v}) + \mathrm{div}(\mathbf{F}) = -\,\vec{\mathrm{grad}}(p) - \rho\,\vec{g}\,, \tag{2}$$

$$\frac{\partial \overline{\rho}\psi_k}{\partial t} + \mathrm{div}(\overline{\rho}\vec{v}\,\psi_k) + \mathrm{div}(\vec{f}_k) = \overline{\rho}\,q_k\,; \qquad k = 1,..,K\,. \tag{3}$$

$$\rho = \rho(p,\,\psi_k)\,. \tag{4}$$

Equation (1) is the continuity equation. Equation (2) describes the nonhydrostatic momentum balance and includes Coriolis forces due to earth's rotation $\vec{\Omega}$, friction by diffusive momentum fluxes $\mathbf{F}$, pressure p, and buoyancy forces by gravity $\vec{g}$. Equation (3) expresses the budget of a scalar ψ_k with nonadvective fluxes f_k , and mass specific sources q_k . Equation (4) represents the equation of state.

For numerical integration, we use a finite difference method based on a staggered grid in space and a combination of the Adams-Bashforth method for momentum with the Smolarkiewicz scheme for scalars. The integration algorithm starts from suitable initial values $\vec{v}^n$, ψ_k^n, and p^n at time t^n, $n = 0$. For brevity we assume that all sources q_k can be evaluated explicitly. Then the algorithm computes in sequence for $n=0,1,\ldots$:

$$\overline{\rho}\psi_k^{n+1} = \overline{\rho}\psi_k^n - \Delta t\,[\mathrm{div}((\overline{\rho}\vec{v})^n\,\psi_k^n) + \mathrm{div}(\vec{f}_k^n)] + \Delta t\,\overline{\rho}q_k^n \tag{5}$$

$$\rho^{\sim} = \rho(p^n, \psi_k^{n+1})\,, \tag{6}$$

$$\vec{b}^n = \mathrm{div}((\overline{\rho}\vec{v})^n\,\vec{v}^n) + \mathrm{div}(\mathbf{F}^n) + 2\vec{\Omega}\times(\overline{\rho}\vec{v})^n\,, \tag{7}$$

$$(\overline{\rho}\vec{v})^{\sim} = (\overline{\rho}\vec{v})^n - \Delta t\,[\gamma_0\,\vec{b}^n + \gamma_1\,\vec{b}^{n-1} + \vec{\mathrm{grad}}(p)^n + \rho^{\sim}\,\vec{g}\,]\,, \tag{8}$$

($\gamma_0 = 1, \gamma_1 = 0$ for $n = 0$; $\gamma_0 = 1.5, \gamma_1 = -0.5$ for $n > 0$.)

$$r = \frac{1}{\Delta t}\,\mathrm{div}\,(\overline{\rho}\vec{v})^{\sim}\,. \tag{9}$$

Now we invert the Poisson equation

$$\mathrm{div}\,\vec{\mathrm{grad}}\,(\Delta p) = r \tag{10}$$

to obtain the mass conserving pressure increments Δp . Finally, pressure and momentum are updated, and time is incremented to the next level:

$$p^{n+1} = p^n + \Delta p\,, \tag{11}$$

$$(\overline{\rho}\vec{v})^{n+1} = (\overline{\rho}\vec{v})^{\sim} - \Delta t\,\vec{\mathrm{grad}}\,(\Delta p)\,,\ \vec{v}^{n+1} = \frac{(\overline{\rho}\vec{v})^{n+1}}{\overline{\rho}}\,, \tag{12}$$

$$t^{n+1} = t^n + \Delta t\,. \tag{13}$$

Boundary conditions are described in [8]. The hydrostatic version will be restricted to cases where the boundary condition at the top surface of the computational domain (at $z = z_t$) prescribes the pressure either explicitly or as a function of the vertical velocity at this plane. For flows over mountains with height $z_s(x, y)$, we apply terrain following coordinates [7]. In this case the Poisson equation corresponds to a 25-point operator which is inverted in a block iteration using direct Poisson solvers for constant coefficients. This part of the computation may well require up to 50 % of the total time.

THE HYDROSTATIC ALGORITHM

In the hydrostatic limit, the vertical component of Equation (2) reduces to

$$\frac{\partial p}{\partial z} = -\rho g . \tag{14}$$

Therefore, the hydrostatic version of the algorithm proceeds as follows:

$\bar{\rho}\psi_k^{n+1}$ and $\rho^{\sim}$ are computed as in Equations (5,6).

The horizontal components of $\vec{b}$ and $(\bar{\rho}\vec{v})^{n+1} = (\bar{\rho}\vec{v})^{\sim}$ follow from Equations (7, 8).

Instead of Equation (9) we compute $r = -\rho^{\sim}g$, determine the top pressure from the boundary condition and integrate

$$\frac{\partial p^{n+1}}{\partial z} = r . \tag{15}$$

downwards to optain the new pressure p^{n+1}.

Then the vertical velocity at the surface is determined from the kinematic boundary condition

$$w = u\frac{\partial z_s}{\partial x} + v\frac{\partial z_s}{\partial y} , \tag{16}$$

and finally the vertical component of the momentum $\bar{\rho}w$ follows by integrating the continuity Equation (1) vertically upwards.

In discrete curvilinear coordinates, using the notation defined in [7], the relevant equations read:

$$\delta_3 VG^{33}p = -\overline{V\rho^{\sim}}^3 g, \tag{17}$$

$$G^{33}\bar{\rho}w = \overline{G^{31}\bar{\rho}u}^1 + \overline{G^{32}\bar{\rho}v}^2 , \tag{18}$$

at the ground, and

$$\delta_3 G^{33}\bar{\rho}w = -\delta_1 a - \delta_2 b - \delta_3[\overline{G^{31}\bar{\rho}u}^1 + \overline{G^{32}\bar{\rho}v}^2] \tag{19}$$

in the interior.

Tests have shown, that this algorithm requires smaller time steps than the non-hydrostatic version, in particular if large vertical velocities arise as near

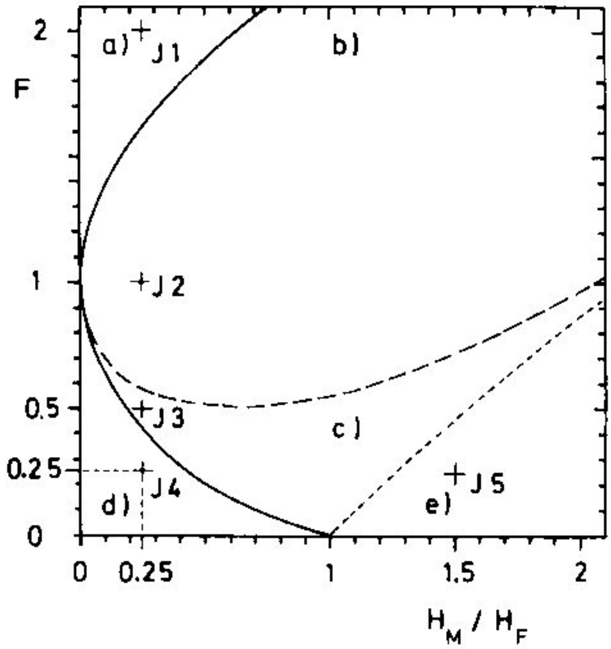

Fig. 1 Left: Flow regime map adapted from [3] as a function of Froude number F and height aspect ratio H. The crosses mark the parameters used in cases J1 to J5. The regime-labels a) to e) refer to Fig. 2.

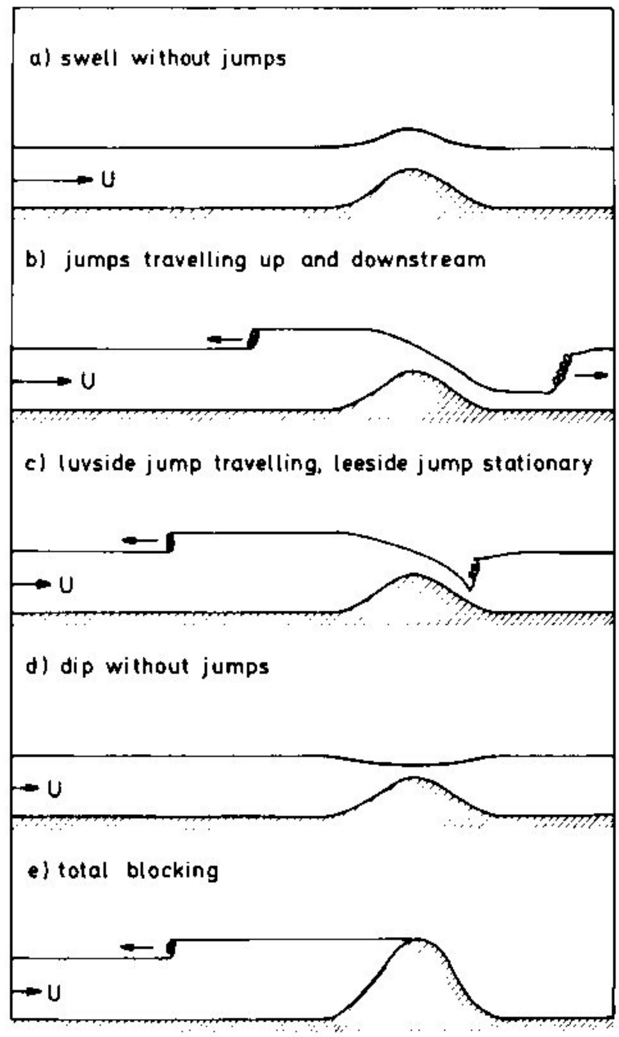

Fig. 2 Right: Illustration of flow regimes referring to the parameter domains a) to e) indicated in Fig. 1.

hydraulic jumps. Presumably, this stems from the fact that the hydrostatic equations neglect the vertical inertia. Thus, the gain in computational efficiency by avoiding the Poisson equation is at least partly compensated by the larger number of time steps.

INFLUENCE OF ROTATION ON THE FORMATION OF HYDRAULIC JUMPS

We consider a layer of incompressible fluid flowing on a plane rotating at frequency $|\vec{\Omega}| = f/2$ around the vertical axis. The fluid approaches an isolated ridge with uniform speed U. The ridge of height H_M and width L is assumed to have the profile $h_M = H_M \sin^2(\pi x/L)$, for $0 \leq x \leq L$, $h_M = 0$ elsewhere. The fluid is layered in that we have heavy fluid below lighter fluid. The height of the interface between the two layers far upstream the ridge is H_F. The density difference is $\Delta\rho$. Thus the effective gravity amounts to $g' = g\Delta\rho/\bar{\rho}$. Small disturbances in the height of the interface move with the speed of gravity waves $C = (g'H_F)^{1/2}$ in the absence of rotational forces. We assume that the flow can be approximated by shallow water equations, i.e. $H_F \ll L$, $H_M \ll L$. This implies hydrostatic flow. Then the problem is determined by three characteristic numbers:

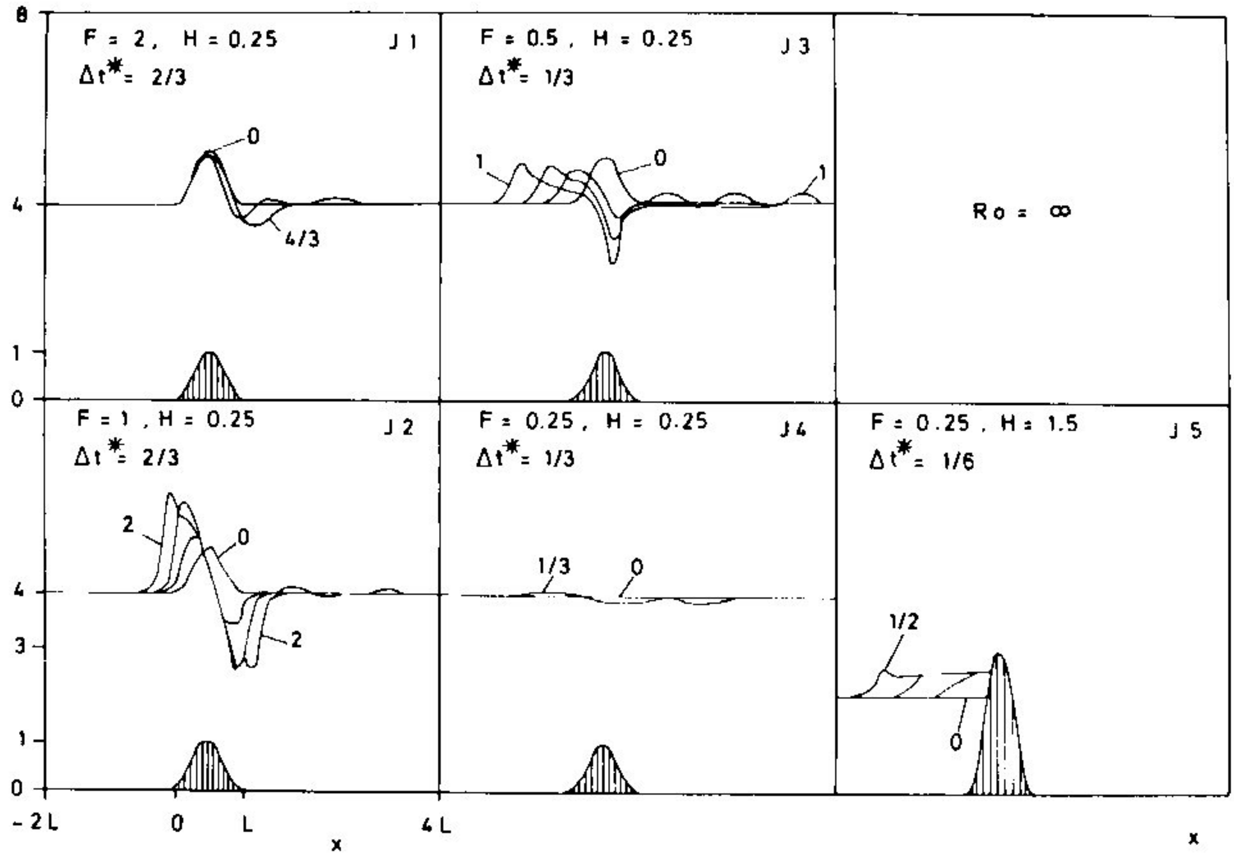

Fig. 3 Development of the top interface of the heavy fluid layer. Each panel shows the results for one of the cases J1 to J5 at a sequence of times with increments Δt^* for $Ro = \infty$ (without rotation). The numbers at the curves denote the respective value of the the normalized time t^*.

$F = U/C$, the Froude number,
$H = H_M/H_F$, the height aspect ratio,
$Ro = U/(fL)$, the Rossby number.

For irrotational flows ($f = 0$, $Ro = \infty$), Houghton and Kasahara [3] and Long [4] have shown that a shallow layer of inviscid fluid flowing at constant speed over a mountain ridge can create hydraulic jumps. Figure 1, maps regimes with and without jumps as a function of the Froude number F and the height ratio H. The various types of flows are illustrated in Fig. 2. The stationary theory predicts the formation of jumps in pairs, one upstream and one downstream of the ridge.

We have applied MESOSCOP to simulate the formation of hydraulic jumps. Five cases are considered (J1 to J5) with Froude-numbers F and height aspect ratios H as indicated by the crosses in Fig. 1. The mountain width is $L = 100$. The computational domain extends over $-2L < x \leq 4L$, $0 < z \leq z_t = 8$, the height of the interface is $H_F = 4$ (except for case J5 where $H_F = 2$), the velocity of the undisturbed flow is $U = 1$. $t^* = tU/L$ defines a non-dimensional time.

Figure 3 shows the development of the flows at a sequence of times for the five cases. As can be seen, the simulation results in the formation of hydraulic jumps in fair agreement with the expectations from Figs. 1 and 2, although the transient solutions are not quite so clear as the idealized picture given in Fig. 2. For J1, the disturbance induced by the mountain cannot propagate upstream because this flow is super-critical in the hydraulic sense. Case J2 shows jumps at the upstream and downstream sides both travelling away from the ridge while for case J3 the downstream jump is stationary. For case J4 the disturbance is small so that the picture is not so clear. Case J5 clearly shows the blocking effect as expected from Fig. 1. The numerical solutions of the hydrostatic and non-hydrostatic algorithms show very small differences. We conclude that both methods are well suited to investigate the formation of hydraulic jumps.

No equivalent theory exists for flows over mountains in a rotating system. Houghton [2] and Williams and Hori [9] consider the transient motion of a shallow water layer on an f-plane without mountains starting from an initial velocity disturbance of magnitude U over a length L. They find that Coriolis forces tend to reduce the amplitude of hydraulic jumps and to delay its formation but apparently do not totally suppress its formation. The reduction of hydraulic jumps by Coriolis forces can be explained as follows: Behind the upstream hydraulic jump the velocity u is less than U. As a consequence, the flow downstream the jump experiences positive acceleration in y-direction due to the Coriolis forces. This in turn causes Coriolis-acceleration in x-direction which tends to reduce the hydraulic jump. The time scale of Coriolis forces is f^{-1}, that of jump creating nonlinear forces is L/U. Hence the delay is large if the Rossby number $Ro = U/(fL)$ is less than a certain limit. No precise information exists on the value of this limiting number.

For the purpose of scale analysis we refer to the shallow water equations where $u(x, t)$ and $v(x,t)$ are the vertically constant velocities in the fluid layer and $h(x,t)$ is the height of the free surface above ground:

$$\frac{du}{dt} = - g'\frac{\partial}{\partial x}(h + h_M) + f v , \tag{20}$$

$$\frac{dv}{dt} = - f u , \tag{21}$$

We introduce non-dimensional variables (denoted by an over-bar) which are of order unity:

$$u = U\bar{u},\ v = U\bar{v},\ x = L\bar{x},\ t = \bar{t}L/U,\ h = H_F\bar{h},\ h_M = H_M\bar{h}_M . \tag{22}$$

This yields the following system:

$$\frac{d\bar{u}}{d\bar{t}} = - F^{-2}\frac{\partial}{\partial\bar{x}}(\bar{h} + H\bar{h}_M) + Ro^{-1}\bar{v} , \tag{23}$$

$$\frac{d\bar{v}}{d\bar{t}} = - Ro^{-1}\bar{u} . \tag{24}$$

From these equations we see that the Coriolis forces (last terms) are small in comparison to the inertia forces if

$$Ro^{-1} \ll 1. \tag{25}$$

In order to have negligible effects of rotation in comparison to the smaller of the two gravity force terms, the following conditions have to be satisfied:

$$F^2/Ro \ll 1,\ \text{if } H > 1, \tag{26}$$

$$F^2/(RoH) \ll 1,\ \text{if } H < 1. \tag{27}$$

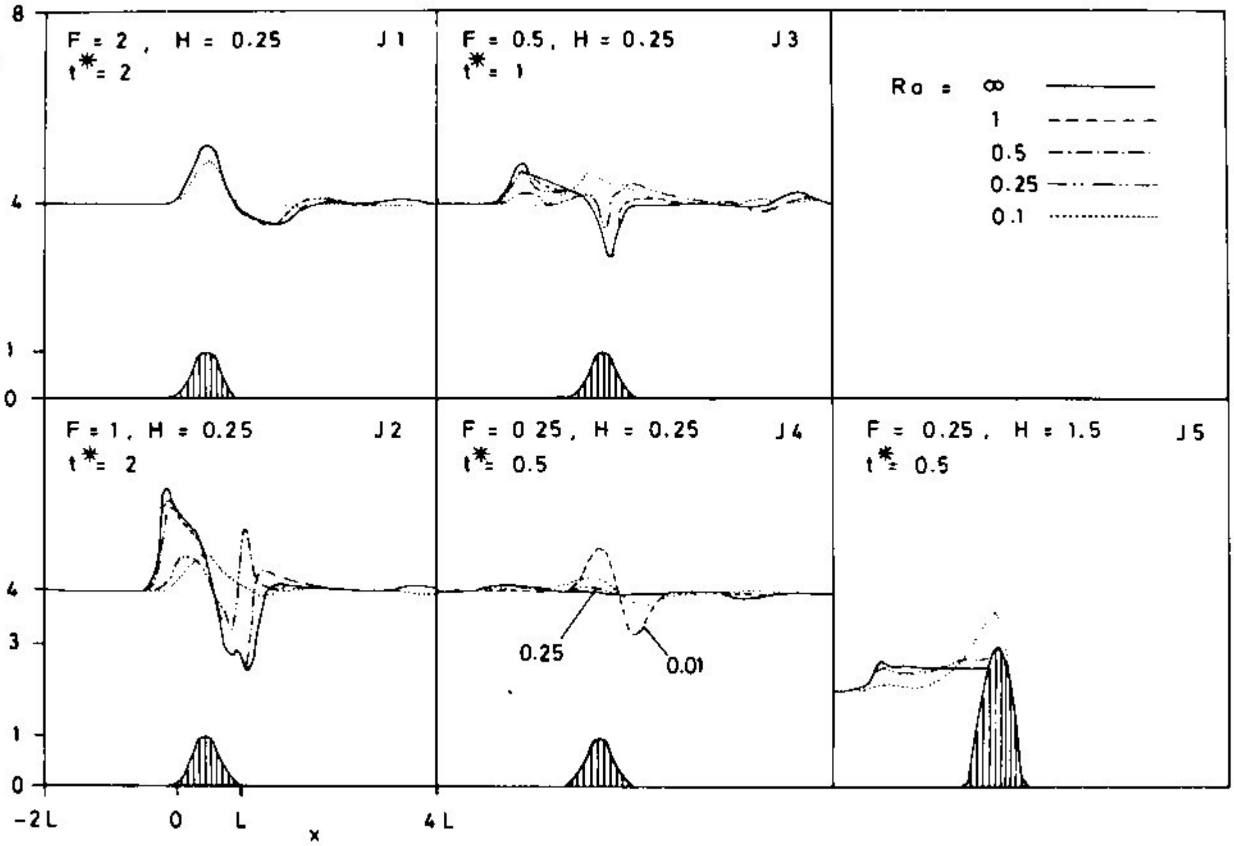

Fig. 4 Influence of rotation on the formation of hydraulic jumps. Each panel shows the results for one of the cases J1 to J5 at a fixed normalized instant of time t^* for various values of the Rossby number *Ro* . For case J4, the labelling of the curves differs and is given explicitly.

In simplified terms, the scale analysis suggests that Coriolis forces are small in comparison to gravity forces if F^2/Ro is small.

Figure 4 shows the solutions for the five cases at fixed times for various values of the Rossby number. For Rossby numbers larger than unity the effect of rotation is negligble throughout all five cases. For very small Rossby numbers, the Coriolis forces cause steady flow over the ridge at constant depth of the fluid layer. We see that the effect of rotation in case J1 with the largest Froude number is negligble for Rossby numbers greater than 1 ($F^2/Ro = 4$). For smaller Froude numbers, e.g. for cases J4 with F = 0.25, the effect of rotation is negligble for the even smaller Rossby number 0.1 ($F^2/Ro = 1.25$). Therefore this case was run also for $Ro = 0.01$. This shows, that F^2/Ro is the controlling characteristic number and that the effect of rotation on the formation of hydraulic jumps is negligble for F^2/Ro less than unity at least if H is of order unity and F is not too large. Otherwise the more complicated conditions given in Eqs. (25-27) apply.

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