Chapter 6

Descriptor system techniques in solving $\mathcal{H}_{2/\infty}$-optimal fault detection and isolation problems

A. Varga*

Abstract. This chapter illustrates the effectiveness of descriptor systems based algorithms in solving $\mathcal{H}_{2/\infty}$-optimal fault detection and isolation problems. The descriptor system based formulation allows the solution of these problems in the most general setting by eliminating all technical assumptions required when using standard approaches. The underlying numerical algorithms to compute rational nullspace bases, inner-outer factorizations or proper coprime factorizations are based on descriptor system representations and rely on orthogonal matrix pencil reductions. The developed integrated computational approaches fully exploit the structural aspects at each solution step and produce fault detectors of least orders.

6.1 Introduction

We consider time-invariant linear *descriptor* system representations of the form

$$\begin{array}{rcl} E\lambda x(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \qquad (6.1)$$

where $x(t)$ is the system state vector, $u(t)$ is the system input vector, $y(t)$ is the system output vectors, and where $\lambda x(t) = \dot{x}(t)$ for a continuous-time system and $\lambda x(t) = x(t+1)$ for a discrete-time system. All intervening matrices in (6.1) are real, and we assume E is a square matrix, which can be in general singular. A *standard* system corresponds to $E = I$. Continuous-time descriptor systems with singular E arise frequently from modelling interconnected systems with standard tools like Simulink or object oriented modeling with Modelica. For example, descriptor models are commonly employed to model constrained mechanical systems (e.g., contact problems) [6]. Discrete-time descriptor representations are frequently used to model economic processes.

For our study, the importance of descriptor system representations lies primarily in solving computational problems involving the manipulation of rational matrices in a numerically reliable way. Note that polynomial matrices can be considered a particular case of rational matrices.

*A. Varga is with German Aerospace Center, DLR-Oberpfaffenhofen, Institute of Robotics and Mechatronics, D-82234 Wessling, Germany, andreas.varga@dlr.de

Each real rational matrix $R(\lambda)$ can be interpreted as the *transfer-function matrix* (TFM) of a continuous- or discrete-time descriptor system of the form (6.1) from input u to output y, where the associated quadruple $(A - \lambda E, B, C, D)$ with $A - \lambda E$ assumed *regular*, satisfies

$$R(\lambda) = C(\lambda E - A)^{-1}B + D$$

Here, λ stays for the *frequency variable*, which is either the complex variable s in the Laplace-transform or the complex variable z in the Z-transform for a continuous- or discrete-time realization, respectively. $R(\lambda)$ is a *proper* TFM if $R(\infty)$ is finite, otherwise is called *improper*. $R(\lambda)$ is called *strictly proper* if $R(\infty) = 0$. A proper $R(\lambda)$ has a standard system realization with $E = I$. Many basic operations with improper and proper rational matrices (e.g., sum, difference, product, transposition, etc.) can be immediately transcribed in equivalent descriptor or standard system realizations, respectively. However, there are some operations on proper TFMs like conjugation or inversion, which can not be generally expressed as standard system realizations, but can always be expressed as descriptor system realizations.

Many numerically reliable computational methods for descriptor systems are based on matrix pencil manipulation using orthogonal transformation matrices. Examples where pencil reduction techniques play a crucial role are the computation of controllability/observability staircase forms [20, 23], determination of infinite poles/zeros structure and minimal indices [14], performing additive spectral separations like finite-infinite or stable-unstable splitting [12], conversions between descriptor representations and rational matrices [22], determination of various inverses [27]. We must say, that virtually, matrix pencil techniques can be used for every linear system analysis and design computation! Interestingly, the underlying pencil reduction tools are useful even in solving several difficult control problems for standard systems, as for example, the solution of discrete-time Riccati equations [21], computation of infinite zeros [14], determination of the Kronecker structure of the system pencil [19, 2], computation of coprime factorizations [25, 17], computation of inner-outer and spectral factorizations [18, 16].

Based on the above mentioned algorithms, a DESCRIPTOR SYSTEMS toolbox for MATLAB has been developed by the author [26, 11]. Initially, the toolbox was mainly intended to provide an extended functionality for the CONTROL TOOLBOX of MATLAB, which formally supported descriptor systems, but only with nonsingular E. Consequently, some functions in the DESCRIPTOR SYSTEMS toolbox simply represent extensions of functions already present in the CONTROL TOOLBOX. Many functions are merely interfaces to powerful structure exploiting computational routines implemented as MEX-functions which call Fortran routines from the high performance linear algebra numerical library LAPACK [1] and control library SLICOT [11]. The implementations of all functions exploit the best of MATLAB and Fortran programming, by trying to balance the matrix manipulation power of MATLAB with the intrinsic high efficiency of carefully implemented structure-exploiting Fortran codes. For the contents of the current version of the toolbox (presently 1.06), see the associated web site[1].

The existence of the powerful descriptor systems algorithms and of the associated software tools in the DESCRIPTOR SYSTEMS toolbox allowed to formulate and implement a new generation of numerically reliable algorithms to solve *fault detection* (FD) and *fault detection and isolation* (FDI) problems [40, 29, 30, 32, 33, 34, 36, 37, 39]. The new computational methods have several distinctive features. The solution methods are *general*, being applicable to both continuous- and discrete-time systems in standard or descriptor system forms. Therefore, a solution can be always computed whenever one exists and no technical assumptions are necessary for the computation of the solution. The new synthesis algorithms belong to the so-called family of *integrated algorithms*, where the successive computational steps are strongly connected such that

[1] `http://www.robotic.de/fileadmin/control/varga/Desc_contents_01.m`

all structural information at the end of a computational step is fully exploited in the next step. For this, state-space representation based updating techniques of the intermediary solutions are employed, where the underlying descriptor system based formulation is of primordial importance. A collection of software tools for the synthesis of residual generators for fault detection has been implemented into a FAULT DETECTION toolbox by the author [31, 38].

In this chapter, we present computational procedures for the optimal synthesis of fault detection filters (or residual generators) for solving approximately FDI problems and illustrate how descriptor system techniques are instrumental for solving these problems in the most general setting and in a numerically reliable way. For illustration purpose, we describe two recently proposed algorithms for the synthesis of residual generators to solve $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal FDI problems [36, 37]. After formulating the approximate synthesis problem, we provide a high level description of the synthesis algorithms based on rational matrix manipulation techniques. Then we discuss an integrated approach based entirely on state-space computational techniques. Two academic examples illustrates the applicability of the algorithms to solve robust FDI problems.

6.2 The fault detection and isolation problem

Consider additive fault models described by input-output representations of the form

$$\mathbf{y}(\lambda) = G_u(\lambda)\mathbf{u}(\lambda) + G_d(\lambda)\mathbf{d}(\lambda) + G_w(\lambda)\mathbf{w}(\lambda) + G_f(\lambda)\mathbf{f}(\lambda), \tag{6.2}$$

where $\mathbf{y}(\lambda)$, $\mathbf{u}(\lambda)$, $\mathbf{d}(\lambda)$, $\mathbf{w}(\lambda)$, and $\mathbf{f}(\lambda)$ are Laplace- or Z-transformed vectors of the p-dimensional system output vector $y(t)$, m_u-dimensional control input vector $u(t)$, m_d-dimensional disturbance vector $d(t)$, m_w-dimensional noise vector $w(t)$ and m_f-dimensional fault vector $f(t)$, respectively, and where $G_u(\lambda)$, $G_d(\lambda)$, $G_w(\lambda)$ and $G_f(\lambda)$ are the TFMs from the control inputs to outputs, disturbance inputs to outputs, noise inputs to outputs, and fault inputs to outputs, respectively. For complete generality of our problem setting, we will allow that these TFMs are general rational matrices (proper or improper) for which we will not *a priori* assume any further properties.

A linear residual generator (or fault detection filter, or simply fault detector) processes the measurable system outputs $y(t)$ and control inputs $u(t)$ and generates the residual signals $r(t)$ which serve for decision making on the presence or absence of faults. The input-output form of this filter is

$$\mathbf{r}(\lambda) = Q(\lambda)\begin{bmatrix} \mathbf{y}(\lambda) \\ \mathbf{u}(\lambda) \end{bmatrix} \tag{6.3}$$

where $Q(\lambda)$ is the TFM of the filter. For a physically realizable filter, $Q(\lambda)$ must be *proper* (i.e., only with finite poles) and *stable* (i.e., only with poles having negative real parts for a continuous-time system or magnitudes less than one for a discrete-time system). The (dynamic) *order* of $Q(\lambda)$ (also known as *McMillan degree*) is the dimension of the state vector of a minimal state-space realization of $Q(\lambda)$. The dimension q of the residual vector $r(t)$ depends on the fault detection problem to be solved.

The residual signal $r(t)$ in (6.3) generally depends via the system outputs $y(t)$ of all system inputs $u(t)$, $d(t)$, $w(t)$ and $f(t)$. The residual generation system, obtained by replacing in (6.3) $\mathbf{y}(\lambda)$ by its expression in (6.2), is given by

$$r(\lambda) = R_u(\lambda)\mathbf{u}(\lambda) + R_d(\lambda)\mathbf{d}(\lambda) + R_w(\lambda)\mathbf{w}(\lambda) + R_f(\lambda)\mathbf{f}(\lambda) \tag{6.4}$$

where

$$[\, R_u(\lambda) \mid R_d(\lambda) \mid R_w(\lambda) \mid R_f(\lambda)\,] := Q(\lambda)\left[\begin{array}{c|c|c|c} G_u(\lambda) & G_d(\lambda) & G_w(\lambda) & G_f(\lambda) \\ I_{m_u} & 0 & 0 & 0 \end{array}\right] \tag{6.5}$$

For a successfully designed filter $Q(\lambda)$, the corresponding residual generation system is proper and stable and achieves specific fault detection requirements.

For the solution of fault detection problems it is always possible to completely decouple the control input $u(t)$ from the residuals $r(t)$ by requiring $R_u(\lambda) = 0$. Regarding the disturbance input $d(t)$ and noise input $w(t)$ we aim to impose a similar condition on the disturbances input $d(t)$ by requiring $R_d(\lambda) = 0$, while minimizing simultaneously the effect of noise input $w(t)$ on the residual (e.g., by minimizing the norm of $R_w(\lambda)$). Thus, from a practical synthesis point of view, the distinction between $d(t)$ and $w(t)$ lies solely in the way these signals are treated when solving the residual generator synthesis problem.

Let $M_r(\lambda)$ be a suitably chosen reference model (i.e., stable, proper, diagonal and invertible) representing the desired TFM from the faults to residuals. We want to achieve that $\mathbf{r}(\lambda) \approx M_r(\lambda)\mathbf{f}(\lambda)$, that is, each residual $r_i(t)$ is influenced mainly by fault $f_i(t)$. Our formulation of the *approximate fault detection and isolation problem* (AFDIP) extends the formulation of the model-matching approach of [5, 3] by requiring to determine a stable and proper filter $Q(\lambda)$ such that the following conditions are fulfilled:

$$\begin{array}{ll} (i) & R_u(\lambda) = 0, \\ (ii) & R_d(\lambda) = 0, \\ (iii) & R_f(\lambda) \approx M_r(\lambda), \ \text{ with } \ R_f(\lambda) \ \text{ stable;} \\ (iv) & R_w(\lambda) \approx 0, \ \text{ with } \ R_w(\lambda) \ \text{ stable.} \end{array} \tag{6.6}$$

The *exact fault detection and isolation problem* (EFDIP) requiring $R_f(\lambda) = M_r(\lambda)$ is included in this formulation and corresponds to $m_w = 0$, while the formulation of the AFDIP in [5, 3] corresponds to $m_d = 0$.

It is straightforward to show that for the solution of the AFDIP, the solvability conditions are those for the solvability of the EFDIP stated in [9] (see [36] for a proof).

Theorem 6.1. *For the system (6.2) there exists a stable, diagonal, proper, and invertible $M_r(\lambda)$ such that the AFIDP is solvable if and only if*

$$\mathrm{rank}\,[\,G_f(\lambda)\ G_d(\lambda)\,] = m_f + \mathrm{rank}\,G_d(\lambda) \tag{6.7}$$

Generically, the condition (6.7) is fulfilled if $p \geq m_f + m_d$, which implies that the system must have a sufficiently large number of measurements. For the case $m_d = 0$ considered in [5, 3], this condition reduces to the simple left invertibility condition:

$$\mathrm{rank}\,G_f(\lambda) = m_f \tag{6.8}$$

In the next section we describe the solution of the AFDIP by solving an approximate model-matching problem using $\mathcal{H}_{2/\infty}$-norm minimization techniques.

6.3 The optimal model-matching approach

Assume $m_u + m_d > 0$ and consider $Q(\lambda)$ in a factored form

$$Q(\lambda) = Q_1(\lambda)N_l(\lambda), \tag{6.9}$$

where $N_l(\lambda)$ is a proper left rational nullspace basis satisfying $N_l(\lambda)G(\lambda) = 0$, where $G(\lambda)$ is defined as

$$G(\lambda) = \left[\begin{array}{cc} G_d(\lambda) & G_u(\lambda) \\ 0 & I_{m_u} \end{array} \right] \tag{6.10}$$

and $Q_1(\lambda)$ is a factor to be further determined. With this choice it follows that $Q(\lambda)$ automatically fulfills the first two conditions in (6.6). The existence of $N_l(\lambda)$ is guaranteed provided condition (6.7) is fulfilled. The resulting $N_l(\lambda)$ has maximal row rank $p-r_d$, where $r_d = rank\, G_d(\lambda)$. Moreover, we can choose $N_l(\lambda)$ stable and such that both $N_f(\lambda)$ and $N_w(\lambda)$ defined as

$$[\, N_f(\lambda)\; N_w(\lambda)\,] := N_l(\lambda) \left[\begin{array}{cc} G_f(\lambda) & G_w(\lambda) \\ 0 & 0 \end{array} \right] \tag{6.11}$$

are proper and stable TFMs [33]. If $m_u = m_d = 0$, we can determine $N_l(\lambda)$ in (6.9) simply from a proper and stable left coprime factorization

$$[\, G_f(\lambda)\; G_w(\lambda)\,] = N_l^{-1}(\lambda)[\, N_f(\lambda)\; N_w(\lambda)\,]$$

To fulfill the last two conditions in (6.6) we can solve a $\mathcal{H}_{2/\infty}$-norm minimization problem to determine a stable and proper $Q_1(\lambda)$ such that

$$\|[\, Q_1(\lambda)N_f(\lambda) - M_r(\lambda) \;\; Q_1(\lambda)N_w(\lambda)\,]\|_{2/\infty} = \min$$

This $\mathcal{H}_{2/\infty}$ model matching problem can be easily reformulated as a standard $\mathcal{H}_{2/\infty}$-norm minimization based "controller" synthesis problem [41] as shown in Fig. 6.1. Here, the underlying equations are

$$\begin{aligned} \mathbf{e}(\lambda) &= \mathbf{r}(\lambda) - M_r(\lambda)\mathbf{f}(\lambda) \\ \overline{\mathbf{y}}(\lambda) &= N_f(\lambda)\mathbf{f}(\lambda) + N_w(\lambda)\mathbf{w}(\lambda) \\ \mathbf{r}(\lambda) &= Q_1(\lambda)\overline{\mathbf{y}}(\lambda) \end{aligned}$$

and lead to the following definition of the generalized plant

$$P(\lambda) = \left[\begin{array}{c|c} P_{11}(\lambda) & P_{12}(\lambda) \\ \hline P_{21}(\lambda) & P_{22}(\lambda) \end{array} \right] := \left[\begin{array}{cc|c} -M_r(\lambda) & 0 & I \\ \hline N_f(\lambda) & N_w(\lambda) & 0 \end{array} \right]$$

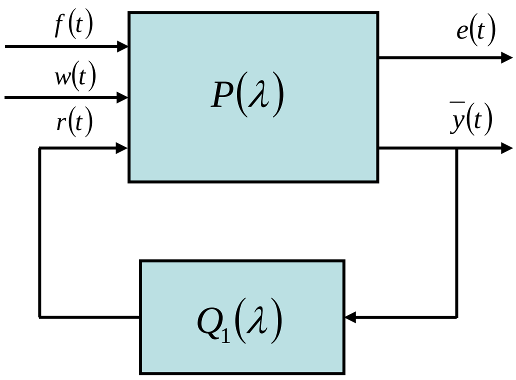

Figure 6.1. *Standard $\mathcal{H}_{2/\infty}$ synthesis setting.*

The minimization of the $\mathcal{H}_{2/\infty}$-norm of the TFM from $[\, f^T(t)\; w^T(t)\,]^T$ to $e(t)$) via an optimal $Q_1(\lambda)$ is thus formally a standard $\mathcal{H}_{2/\infty}$-synthesis problem for which software tools exist, as for example, the functions `h2syn`/`hinfsyn` available in the Matlab Robust Control Toolbox. The main problem when employing standard tools like `h2syn`/`hinfsyn`, is that, although a stable and proper solution of the AFDIP may exist, this solution can not be computed because of an inappropriate choice of $M_r(\lambda)$ or of the presence of technical assumptions which must be fulfilled.

The first aspect, related to the choice of $M_r(\lambda)$, has been already discussed in [15]. The difficulty is that the determination of the solution $Q_1(\lambda)$ can not be done simultaneously with the choice of $M_r(\lambda)$, because this would lead to the trivial (optimal) solution $M_r(\lambda) = 0$ and $Q_1(\lambda) = 0$. Therefore, the procedure suggested in [15] is to choose first $M_r(\lambda)$ and compute the corresponding $Q_1(\lambda)$, then solve the optimization problem for a new $M_r(\lambda)$ with fixed $Q_1(\lambda)$, and continue in this way until convergence to a satisfactory solution is achieved. However, it is not clear how to choose the initial $M_r(\lambda)$ nor how to keep the dynamical order of $M_r(\lambda)$ as low as possible. Furthermore, there is no even guarantee for the convergence of this process.

The second aspect, related to the presence of technical assumptions, is the need to fulfill by the underlying generalized plant model certain requirements when solving the $\mathcal{H}_{2/\infty}$-synthesis problem using available standard tools. For example, to obtain a proper solution, the TFM $P_{21}(\lambda)$ must not have zeros on the extended imaginary axis in the continuous-time case or on the unit circle. If this condition is not fulfilled, the synthesis fails, although an appropriate choice of $M_r(\lambda)$ would lead to the cancelation of these zeros in the final detector.

To face the above limitations, it is necessary to develop general synthesis procedures for which no such limitations exist. The key parameter to guarantee the stability and properness of the detector is $M_r(\lambda)$, the desired TFM relating the faults to the residuals. As already mentioned, the choice of $M_r(\lambda)$ is not obvious and can be even the object of an optimization based choice [15]. Often good candidates for $M_r(\lambda)$ result from an exact synthesis (for $m_w = 0$) [29]. However, in [30] a procedure has been proposed, where the choice of suitable $M_r(\lambda)$ is part of the solution. This procedure has been refined in [36, 37], by developing an integrated approach to the detector synthesis. An important feature of these computational approaches is that they rely on repeated updating of an initial fault detector. The underlying state-space representation based computations employ explicit least order realizations of the successive detectors, thus a least order of the final detector is guaranteed. Since the successive steps are strongly connected, all structural features of the computed intermediary results can be exploited in the next steps. This leads to an integrated computational procedure based on highly efficient structure exploiting computations.

6.4 Enhanced optimal model-matching procedure

In this section we present enhanced versions of the algorithms of [30], where we exploit the additional structure in the model (6.2) owing to the separation of the unknown inputs in two components $d(t)$ and $w(t)$. Moreover, by using a new parametrization of the detector, we derive integrated computational procedures based on detector updating techniques. We describe in what follows the main stages of the overall computational procedures, pointing out the commonalities and differences between the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm based optimization approaches. The enhanced synthesis procedures will be first presented as high-level computations in terms of TFMs. However, the actual computations have to be performed employing reliable numerical methods relying on descriptor system state-space representations. The corresponding computational methods are described in the next section.

Stage 1:

This stage employs the nullspace-based parametrization of the detector introduced in the previous section, which allows to reduce the initial synthesis problem to a standard $\mathcal{H}_{2/\infty}$-norm optimization problem. The importance of this step is two-fold. First, the control inputs u and disturbance inputs d are decoupled from the residual via this parametrization. This decoupling can be performed whether the system is proper or not. Second, this step is common to both exact and approximate synthesis based approaches, thus the exact synthesis procedure can be easily embedded into the approximate solution.

We employ the parametrization (6.9) of the detector in the factored form $Q(\lambda) = Q_1(\lambda)N_l(\lambda)$, where $N_l(\lambda)$ is a left proper rational nullspace basis satisfying (6.10), and $Q_1(\lambda)$ is a factor to be further determined. As already mentioned, the resulting $N_l(\lambda)$ has maximal row rank $p - r_d$, where $r_d = rank\, G_d(\lambda)$. Moreover, we can choose $N_l(\lambda)$ stable and such that both proper TFMs $N_f(\lambda)$ and $N_w(\lambda)$ defined in (6.11) are also stable [33]. However, as it will be later apparent, enforcing the stability and even properness conditions are not necessary at this stage. We can easily check now the solvability of the AFDIP by verifying that

$$rank\, N_f(\lambda) = m_f \tag{6.12}$$

To fulfill the last two conditions in (6.6) we can solve a $\mathcal{H}_{2/\infty}$-norm minimization problem for $\|\widetilde{\mathcal{R}}(\lambda)\|_{2/\infty}$ to determine $Q_1(\lambda)$, where

$$\widetilde{\mathcal{R}}(\lambda) := \widetilde{F}(\lambda) - Q_1(\lambda)\widetilde{G}(\lambda), \tag{6.13}$$

with $\widetilde{G}(\lambda) = [\, N_f(\lambda)\;\; N_w(\lambda) \,]$ and $\widetilde{F}(\lambda) = [\, M_r(\lambda)\;\; O \,]$. Here, $M_r(\lambda)$ is the TFM of a given reference model (i.e., stable, proper, diagonal, invertible).

Let ℓ be the rank of the $(p - r_d) \times (m_f + m_w)$ TFM $\widetilde{G}(\lambda)$. If $\ell < p - r_d$ (i.e., $\widetilde{G}(\lambda)$ has no full row rank), we can take instead $N_l(\lambda)$, ℓ linear combinations of basis vectors of the form $W(\lambda)N_l(\lambda)$, which ensures that $W(\lambda)\widetilde{G}(\lambda)$ has full row rank ℓ. A suitable choice of the $\ell \times (p - r_d)$ TFM $W(\lambda)$ which also minimizes the McMillan degree of $W(\lambda)N_l(\lambda)$ is described in [33, 39].

Stage 2:

The second stage is standard in solving $\mathcal{H}_{2/\infty}$-norm optimization problems and consists in compressing the full row rank TFM $\widetilde{G}(\lambda)$ to a full column rank (thus invertible) TFM. For this, we compute a quasi-co-outer–inner factorization

$$\widetilde{G}(\lambda) = [\, G_{o,1}(\lambda)\; 0 \,] \begin{bmatrix} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{bmatrix} := G_o(\lambda)G_i(\lambda), \tag{6.14}$$

where $G_i(\lambda)$ is a $(m_f + m_w) \times (m_f + m_w)$ inner TFM and $G_{o,1}(\lambda)$ is an $\ell \times \ell$ invertible TFM. Recall that a square TFM $G_i(\lambda)$ is *inner* (and simultaneously *co-inner*) if it has only stable poles and satisfies $G_i(\lambda)G_i^*(\lambda) = I$, where $G_i^*(s) := G_i^T(-s)$ in a continuous-time setting and $G_i^*(z) := G_i^T(1/z)$ in a discrete-time setting. The *quasi-co-outer* factor $G_o(\lambda)$ may have besides stable zeros, also zeros which lie on the boundary of the stability domain.

We can refine the parametrization (6.9) of the detector by choosing $Q_1(\lambda)$ of the form

$$Q_1(\lambda) = Q_2(\lambda)G_{o,1}^{-1}(\lambda) \tag{6.15}$$

where $Q_2(\lambda)$ is to be determined. Using (6.14) and (6.15), we can express $\widetilde{\mathcal{R}}(\lambda)$ in (6.13) as $\widetilde{\mathcal{R}}(\lambda) = \overline{\mathcal{R}}(\lambda)G_i(\lambda)$, with

$$\overline{\mathcal{R}}(\lambda) = \left[\; \overline{F}_1(\lambda) - Q_2(\lambda) \;\middle|\; \overline{F}_2(\lambda) \;\right], \tag{6.16}$$

where $\overline{F}_1(\lambda) := \widetilde{F}(\lambda)G_{i,1}^*(\lambda)$ and $\overline{F}_2(\lambda) := \widetilde{F}(\lambda)G_{i,2}^*(\lambda)$. Since $G_i(\lambda)$ is an inner TFM, we have $\|\widetilde{\mathcal{R}}(\lambda)\|_{2/\infty} = \|\overline{\mathcal{R}}(\lambda)\|_{2/\infty}$. The determination of a stable and proper $Q_2(\lambda)$ which minimizes

$\|\overline{\mathcal{R}}(\lambda)\|_{2/\infty}$ is a least-distance problem. Depending on the employed norm, different solutions have to be computed.

From (6.9) and (6.15), the overall detection filter $Q(\lambda)$ has the product form

$$Q(\lambda) = Q_2(\lambda)G_{o,1}^{-1}(\lambda)N_l(\lambda) := Q_2(\lambda)\overline{Q}(\lambda),$$

where $\overline{Q}(\lambda)$ can be interpreted as an updated partial detector, to which correspond the updated quantities

$$[\,\overline{N}_f(\lambda)\;\overline{N}_w(\lambda)\,] := G_{o,1}^{-1}(\lambda)[\,N_f(\lambda)\;N_w(\lambda)\,] = [\,I_\ell\;\;0\,]\left[\begin{array}{c} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{array}\right] = G_{i,1}(\lambda) \tag{6.17}$$

In the next section we show that pole-zero cancelations occur in forming $G_{o,1}^{-1}(\lambda)N_l(\lambda)$, thus $\overline{Q}(\lambda)$ has as only poles the zeros of $G_{o,1}(\lambda)$. These poles are stable, excepting possible poles corresponding to zeros of $G_{o,1}(\lambda)$ on the boundary of the stability domain (i.e., on the extended imaginary axis in the continuous-time case, or on the unit circle in discrete-time case).

Note: The solution of the EFDIP (for $m_w = 0$) corresponds to choosing $G_{o,1}(\lambda) = N_f(\lambda)$ and $G_i(\lambda) = I_{m_f}$. It follows that $Q_2(\lambda) = M_r(\lambda)$ and the next stage can be skipped.

Stage 3: $\mathcal{H}_2$-optimal synthesis

The solution of the least distance problem in the case of $\mathcal{H}_2$-norm is straightforward and involves computing the stable projection

$$Q_2(\lambda) = \{\overline{F}_1(\lambda)\}_+$$

Here, $\{\cdot\}_+$ denotes the stable part of the underlying TFM including the direct feedthrough term, while $\{\cdot\}_-$ denotes the unstable part. With the above choice, we achieved the minimum $\mathcal{H}_2$-norm of $\mathcal{R}(\lambda)$, which can be computed as

$$\|\widetilde{\mathcal{R}}(\lambda)\|_2 = \|\overline{\mathcal{R}}(\lambda)\|_2 = \|[\,\{\overline{F}_1(\lambda)\}_-\;\;\overline{F}_2(\lambda)\,]\|_2$$

Since the underlying TFMs are unstable, the $\mathcal{L}_2$-norm is used in the last equation.

Stage 3: $\mathcal{H}_\infty$-optimal synthesis

We determine a stable $Q_2(\lambda)$ as the solution of the suboptimal two-blocks least distance problem

$$\|[\,\overline{F}_1(\lambda) - Q_2(\lambda)\;\;\overline{F}_2(\lambda)\,]\|_\infty < \gamma, \tag{6.18}$$

where $\gamma_{opt} < \gamma \leq \gamma_{opt} + \varepsilon$, with ε an arbitrary user specified (accuracy) tolerance for the least achievable value γ_{opt} of γ. With the following lower and upper bounds for γ_{opt}

$$\gamma_l = \|\overline{F}_2(\lambda)\|_\infty, \quad \gamma_u = \left\|\left[\;\overline{F}_1(\lambda)\quad \overline{F}_2(\lambda)\;\right]\right\|_\infty \tag{6.19}$$

such a γ-suboptimal solution $Q_2(\lambda)$ can be computed using the bisection-based γ-iteration approach of [8], whose main steps are presented in what follows.

For a given $\gamma > \gamma_l$, we compute first the spectral factorization [41]

$$\gamma^2 I - \overline{F}_2(\lambda)\overline{F}_2^*(\lambda) = V(\lambda)V^*(\lambda), \tag{6.20}$$

where $V(\lambda)$ is biproper, stable and minimum-phase. Further, we compute the additive decomposition

$$L_s(\lambda) + L_u(\lambda) = V^{-1}(\lambda)\overline{F}_1(\lambda), \tag{6.21}$$

where $L_s(\lambda)$ is the stable part and $L_u(\lambda)$ is the unstable part. If $\gamma > \gamma_{opt}$, the two-blocks problem (6.18) is equivalent to the one-block problem [8]

$$\left\| V^{-1}(\lambda)\left(\overline{F}_1(\lambda) - Q_2(\lambda)\right)\right\|_\infty \leq 1 \tag{6.22}$$

and $\gamma_H := \|L_u^*(\lambda)\|_H < 1$ ($\|\cdot\|_H$ denotes the Hankel norm of a stable TFM). In this case we readjust $\gamma_u = \gamma$. If $\gamma_H \geq 1$, we readjust $\gamma_l = \gamma$. Then, for $\gamma = (\gamma_l + \gamma_u)/2$ we redo the factorization (6.20) and decomposition (6.21). This process is repeated until $\gamma_u - \gamma_l \leq \varepsilon$.

If $\gamma_u \geq \gamma > \gamma_{opt}$, the stable solution of (6.22) is

$$Q_2(\lambda) = V(\lambda)(L_s(\lambda) + Q_{2,s}(\lambda)), \tag{6.23}$$

where, for any γ_1 satisfying $1 \geq \gamma_1 > \gamma_H$, $Q_{2,s}(\lambda)$ is the stable solution of the γ_1-suboptimal Nehari problem

$$\|L_u(\lambda) - Q_{2,s}(\lambda)\|_\infty < \gamma_1. \tag{6.24}$$

Stage 4:

Given the stable solution $Q_2(\lambda)$ computed at previous stage, the resulting $\mathcal{H}_{2/\infty}$ optimal detector can be expressed as

$$\widehat{Q}(\lambda) = Q_2(\lambda)G_{o,1}^{-1}(\lambda)N_l(\lambda) \tag{6.25}$$

Taking into account the pole-zero cancelations when forming $G_{o,1}^{-1}(\lambda)N_l(\lambda)$, the optimal detector $\widehat{Q}(\lambda)$ is stable (and also proper) only if $G_{o,1}$ is minimum-phase. However, since $G_{o,1}(\lambda)$ is in general a quasi co-outer factor, it can still have unstable zeros on the boundary of the stability domain. Thus, these zeros may appear as poles of $\widehat{Q}(\lambda)$, if they are not canceled via the zeros of $Q_2(\lambda)$. To ensure that the final detector is stable (and thus also proper), the resulting $Q(\lambda)$ can be determined as

$$Q(\lambda) = M(\lambda)\widehat{Q}(\lambda),$$

where $M(\lambda)$ is chosen such that $Q(\lambda)$ is proper and stable. Since $\widehat{Q}_1(\lambda) := M(\lambda)Q_2(\lambda)G_{o,1}^{-1}(\lambda)$ can be seen as an approximation of the solution of the weighted minimization problem

$$\|M(\lambda)\widetilde{\mathcal{R}}(\lambda)\|_{2/\infty} = \min$$

it follows that $M(\lambda)M_r(\lambda)$ can be interpreted as an updated reference model.

The computation of appropriate $M(\lambda)$ can be done using the stable and proper coprime factorization algorithm of [25]. The choice of $M(\lambda)$ can be done such that $\|M(\lambda)\widetilde{\mathcal{R}}(\lambda)\|_{2/\infty} \approx \|\widetilde{\mathcal{R}}(\lambda)\|_{2/\infty}$ and $M(\lambda)$ has the least possible McMillan degree. For example, to ensure properness or strict properness, $M(\lambda)$ can be chosen diagonal with the diagonal terms $m_j(\lambda)$, $j = 1, \ldots, m_f$ having the form

$$m_j(\lambda) = \frac{1}{(\tau s + 1)^{k_j}} \quad \text{or} \quad m_j(z) = \frac{1}{z^{k_j}}$$

for continuous- or discrete-time settings, respectively. Both above factors have unit $\mathcal{H}_\infty$-norm. In the case of solving the EFDIP, the choice of $M(\lambda)$ can enforce arbitrary dynamics of the detector. In this case, the choice of a diagonal $M(\lambda)$ allows to obtain an updated diagonal specification $M(\lambda)M_r(\lambda)$ for $R_f(\lambda)$.

6.5 Computational issues

For computations we employ an equivalent *descriptor* state-space realization of the input-output model (6.2)

$$\begin{aligned} E\lambda x(t) &= Ax(t)+B_u u(t)+B_d d(t)+B_w w(t)+B_f f(t) \\ y(t) &= Cx(t)+D_u u(t)+D_d d(t)+D_w w(t)+D_f f(t) \end{aligned} \tag{6.26}$$

with the n-dimensional state vector $x(t)$. Recall that $\lambda x(t) = \dot{x}(t)$ or $\lambda x(t) = x(t+1)$ depending on the type of the system, continuous or discrete, respectively. In general, we can assume that the representation (6.26) is *minimal*, that is, the descriptor pair $(A - \lambda E, C)$ is *observable* and the pair $(A - \lambda E, [\, B_u \;\; B_d \;\; B_w \;\; B_f \,])$ is *controllable*. The corresponding TFMs of the model in (6.2) are

$$\begin{aligned} G_u(\lambda) &= C(\lambda E - A)^{-1}B_u + D_u \\ G_d(\lambda) &= C(\lambda E - A)^{-1}B_d + D_d \\ G_w(\lambda) &= C(\lambda E - A)^{-1}B_w + D_w \\ G_f(\lambda) &= C(\lambda E - A)^{-1}B_f + D_f \end{aligned} \tag{6.27}$$

or in an equivalent notation

$$\left[\; G_u(\lambda) \quad G_d(\lambda) \quad G_w(\lambda) \quad G_f(\lambda) \;\right] := \left[\begin{array}{c|cccc} A-\lambda E & B_u & B_d & B_w & B_f \\ \hline C & D_u & D_d & D_w & D_f \end{array}\right]$$

In what follows, we present the state-space representation based algorithms to be used to perform the successive stages of the synthesis procedures.

Stage 1:

If $m_u + m_d > 0$, we employ the orthogonal pencil reduction based algorithms described in [40, 33, 39] to compute a least order state-space representation of the $(p - r_d) \times (m_u + p)$ left proper rational nullspace $N_l(\lambda)$ satisfying $N_l(\lambda)G(\lambda) = 0$, with $G(\lambda)$ defined in (6.10). The corresponding $N_f(\lambda)$ and $N_w(\lambda)$ can be simultaneously obtained with realizations of the form

$$\left[\; N_l(\lambda) \quad N_f(\lambda) \quad N_w(\lambda) \;\right] = \left[\begin{array}{c|ccc} \widetilde{A} - \lambda\widetilde{E} & \widetilde{B}_{yu} & \widetilde{B}_f & \widetilde{B}_w \\ \hline \widetilde{C} & \widetilde{D}_{yu} & \widetilde{D}_f & \widetilde{D}_w \end{array}\right], \tag{6.28}$$

where the $n_l \times n_l$ matrix $\widetilde{E}$ is invertible (thus all TFMs are proper) and the pair $(\widetilde{A}, \widetilde{E})$ has only finite generalized eigenvalues which can be arbitrarily placed. The computation of the matrices of the state space realizations in (6.28) is detailed in the above references and involves only an orthogonal reduction of the system matrix of the state-space realization of $G(\lambda)$ defined in (6.10) to a suitable Kronecker-like form which reveals the left Kronecker structure of $G(\lambda)$. The resulting Kronecker-like form allows to read out, without further computations, a realization of a proper left nullspace basis $N_l(\lambda)$. The most important aspect, as shown in [33, 39], is that the realizations of $N_f(\lambda)$ and $N_w(\lambda)$ can be determined such that they share the same observable pair $(\widetilde{A} - \lambda\widetilde{E}, \widetilde{C})$ with the realization of $N_l(\lambda)$. This property can be preserved in the subsequent computation (see below) and exploited at the next computational stage. If $m_u = m_d = 0$, a realization of the form (6.28) can be still employed with the trivial choice $n_l = n$, $N_l(\lambda) = I_p$, $N_f(\lambda) = G_f(\lambda)$ and $N_w(\lambda) = G_w(\lambda)$. As it will be apparent at the next stage, the presence of possibly singular $\widetilde{E}$ can be easily accommodated by the employed factorization algorithm.

The checking of condition (6.12) can be done by verifying that there exists a λ_0 (e.g., a randomly generated value) such that

$$rank \begin{bmatrix} \widetilde{A} - \lambda_0 \widetilde{E} & \widetilde{B}_f \\ \widetilde{C} & \widetilde{D}_f \end{bmatrix} = n_l + m_f \tag{6.29}$$

If $\ell := rank[\, N_f(\lambda)\ N_w(\lambda)\,] < p - r_d$ we can determine, by using minimum dynamic cover techniques, a suitable $\ell \times (p - r_d)$ prefilter $W(\lambda)$ such that $W(\lambda)N_l(\lambda)$ has the least possible McMillan degree and $W(\lambda)[\, N_f(\lambda)\ N_w(\lambda)\,]$ has full row rank ℓ. The state-space realization of $W(\lambda)[\, N_l(\lambda)\ N_f(\lambda)\ N_w(\lambda)\,]$ has still the form (6.28) and can be obtained by using updating techniques described in [33, 39].

Stage 2:

For the computation of the quasi-co-outer–inner factorization of $\widetilde{G}(\lambda) := [\, N_f(\lambda)\ N_w(\lambda)]$ in (6.14), we employ the dual of the algorithm of [18] for the continuous-time case and the dual of the algorithm of [16] for the discrete-time case. In both cases, the quasi-co-outer factor $G_{o,1}(\lambda)$ is obtained in the form

$$G_{o,1}(\lambda) = \left[\begin{array}{c|c} \widetilde{A} - \lambda \widetilde{E} & \overline{B}_o \\ \hline \widetilde{C} & \overline{D}_o \end{array}\right], \tag{6.30}$$

where $\overline{B}_o$ and $\overline{D}_o$ are matrices with ℓ columns. The system with the TFM $G_{o,1}(\lambda)$ may have besides the stable zeros (partly resulted from the column compression), also zeros on the imaginary axis (including infinity) in the continuous-time case or on the unit circle in the discrete-time case. The $(m_f + m_w) \times (m_f + m_w)$ TFM of the inner factor is proper and stable and assume that its inverse (i.e., its conjugated TFM) has a state-space realization of the form

$$G_i^*(\lambda) = [\, G_{i,1}^*(\lambda)\ G_{i,2}^*(\lambda)\,] = \left[\begin{array}{c|cc} A_i - \lambda E_i & B_{i,1} & B_{i,2} \\ \hline C_i & D_{i,1} & D_{i,2} \end{array}\right] \tag{6.31}$$

where all generalized eigenvalues of the pair (A_i, E_i) are unstable. For a continuous-time system $E_i = I$, but this can not be assumed, in general, for a discrete-time system with $\lambda = z$, unless $G_i(z)$ *does not have* poles in the origin. If $G_i(z)$ has poles in the origin, the resulting E_i is singular and thus the pair (A_i, E_i) has also infinite (unstable) generalized eigenvalues.

To compute the updated partial detector $\overline{Q}(\lambda) := G_{o,1}^{-1}(\lambda)N_l(\lambda)$ as well as $\overline{N}_f(\lambda) := G_{o,1}^{-1}(\lambda)R_f(\lambda)$ and $\overline{N}_w(\lambda) := G_{o,1}^{-1}(\lambda)R_w(\lambda)$, we can solve the linear rational system of equations

$$G_{o,1}(\lambda)[\ \overline{Q}(\lambda)\ \ \overline{N}_f(\lambda)\ \ \overline{N}_w(\lambda)\ ] = [\ N_l(\lambda)\ \ N_f(\lambda)\ \ N_w(\lambda)\ ] \tag{6.32}$$

Observe that $G_{o,1}(\lambda)$, $N_l(\lambda)$, $N_f(\lambda)$ and $N_w(\lambda)$ have descriptor realizations which share the same state, descriptor and output matrices. Using these state-space realizations, the linear rational equation (6.32) can be equivalently solved (see [28]) by computing first the solution $X(\lambda)$ of

$$\begin{bmatrix} \widetilde{A} - \lambda \widetilde{E} & \overline{B}_o \\ \widetilde{C} & \overline{D}_o \end{bmatrix} X(\lambda) = \begin{bmatrix} \widetilde{B}_{yu} & \widetilde{B}_f & \widetilde{B}_w \\ \widetilde{D}_{yu} & \widetilde{D}_f & \widetilde{D}_w \end{bmatrix}$$

and then

$$[\ \overline{Q}(\lambda)\ \ \overline{N}_f(\lambda)\ \ \overline{N}_w(\lambda)\ ] = [\ 0\ \ \ I_\ell\ ]\, X(\lambda)$$

With the invertible system matrix

$$S_o(\lambda) = \begin{bmatrix} \widetilde{A} - \lambda \widetilde{E} & \overline{B}_o \\ \widetilde{C} & \overline{D}_o \end{bmatrix} \tag{6.33}$$

we obtain

$$\left[\begin{array}{ccc}\overline{Q}(\lambda) & \overline{N}_f(\lambda) & \overline{N}_w(\lambda)\end{array}\right] = \left[\begin{array}{cc}0 & I_\ell\end{array}\right] S_o^{-1}(\lambda)\left[\begin{array}{ccc}\widetilde{B}_{yu} & \widetilde{B}_f & \widetilde{B}_w \\ \widetilde{D}_{yu} & \widetilde{D}_f & \widetilde{D}_w\end{array}\right]$$

Alternatively, explicit realizations of $\overline{N}_f(\lambda)$ and $\overline{N}_w(\lambda)$ can be determined using (6.17).

Let the quadruple (A_r, B_r, C_r, D_r) describe the state-space realization of $[\,M_r(\lambda)\ 0\,]$. Then, using (6.31), the state-space realization of $[\,\overline{F}_1(\lambda)\ \overline{F}_2(\lambda)\,]$ has the form

$$[\,\overline{F}_1(\lambda)\ \overline{F}_2(\lambda)\,] = \left[\begin{array}{cc|cc} A_r-\lambda I & B_rC_i & B_rD_{i,1} & B_rD_{i,2} \\ 0 & A_i-\lambda E_i & B_{i,1} & B_{i,2} \\ \hline C_r & D_rC_i & D_rD_{i,1} & D_rD_{i,2}\end{array}\right], \tag{6.34}$$

where A_r has only stable eigenvalues, while the pair (A_i, E_i) has only unstable generalized eigenvalues. The partial detector $\overline{Q}(\lambda)$ has an explicit descriptor realization of the form

$$\overline{Q}(\lambda) = \left[\begin{array}{cc|c} \widetilde{A}-\lambda\widetilde{E} & \overline{B}_o & \widetilde{B}_{yu} \\ \widetilde{C} & \overline{D}_o & \widetilde{D}_{yu} \\ \hline 0 & -I_\ell & 0\end{array}\right] \tag{6.35}$$

Stage 3: $\mathcal{H}_2$-optimal synthesis

To compute the additive spectral separation

$$\overline{F}_1(\lambda) = \{\overline{F}_1(\lambda)\}_+ + \{\overline{F}_1(\lambda)\}_-$$

the general algorithm of [12] for descriptor systems can be employed. However, the structure of the realization of $\overline{F}_1(\lambda)$ in (6.34) can be easily exploited, taking into account that the first block $A_r-\lambda I$ has only stable eigenvalues, while the second block $A_i-\lambda E_i$ has only unstable eigenvalues. Thus, the computation of $\{\overline{F}_1(\lambda)\}_+$ involves only the solution of a Sylvester equation (or system) (see [12] for details).

Stage 3: $\mathcal{H}_\infty$-optimal synthesis

At this stage we need to perform the γ-iteration to solve the suboptimal two-block minimum distance problem (6.18). To start, we have to compute the $\mathcal{L}_\infty$-norms in (6.19) to obtain γ_l and γ_u. For this purpose, efficient algorithms can be employed based on extensions of the method of [4] (for which standard numerical tools are available in MATLAB). Note that for computing γ_u we can exploit that $\gamma_u = \|M_r(\lambda)\|_\infty$ as a consequence of the all-pass property of $G_i^*(\lambda)$.

The computation of the spectral factorization (6.20) involves two steps. Firstly, we compute a right coprime factorization of $\overline{F}_2(\lambda)$ with inner denominator such that $\overline{F}_2(\lambda) = \overline{N}_2(\lambda)\overline{M}_2^{-1}(\lambda)$, where $\overline{M}_2(\lambda)$ is inner. It follows that $\overline{F}_2(\lambda)\overline{F}_2^*(\lambda) = \overline{N}_2(\lambda)\overline{N}_2^*(\lambda)$. This computation needs to be performed only once and suitable algorithms for this purpose have been proposed in [25] for standard systems, or in [17] for discrete-time descriptor systems. In both cases, the resulting $\overline{N}_2(\lambda)$ has a standard system realization of the form

$$\overline{N}_2(\lambda) = \left[\begin{array}{cc|c} A_r-\lambda I & B_r\overline{C}_i & B_r\overline{D}_{i,2} \\ 0 & \overline{A}_i-\lambda I & \overline{B}_{i,2} \\ \hline C_r & D_r\overline{C}_i & D_r\overline{D}_{i,2}\end{array}\right]$$

Secondly, we solve the spectral factorization problem

$$\gamma^2 I - \overline{N}_2(\lambda)\overline{N}_2^*(\lambda) = V(\lambda)V^*(\lambda)$$

using methods described in [41] to obtain a realization of $V^{-1}(\lambda)\overline{F}_1(\lambda)$ of the form

$$V^{-1}(\lambda)\overline{F}_1(\lambda) = \left[\begin{array}{cc|c} \overline{A}_{11} - \lambda I & \overline{A}_{12} & \overline{B}_1 \\ 0 & A_i - \lambda E_i & \overline{B}_2 \\ \hline \overline{C}_1 & \overline{C}_2 & \overline{D} \end{array}\right],$$

where $\overline{A}_{11}$ has only stable eigenvalues, while the pair (A_i, E_i) has only unstable eigenvalues. This computation involves the solution of an algebraic Riccati equation at each iteration. For details see [41].

To compute the spectral separation (6.21) we perform a similarity transformation to obtain the transformed pole pencil

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{A}_{11} - \lambda I & \overline{A}_{12} \\ 0 & A_i - \lambda E_i \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$$

in a block diagonal form by annihilating its (1,2) block. This comes down to solve the Sylvester system of matrix equations

$$\begin{array}{rcl} 0 & = & XA_i + \overline{A}_{11}Y + \overline{A}_{12} \\ 0 & = & XE_i + Y \end{array}$$

After applying the transformations to the input and output matrices we obtain

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} = \begin{bmatrix} \overline{B}_1 + X\overline{B}_2 \\ \overline{B}_2 \end{bmatrix}, \qquad [\,\overline{C}_1\;\overline{C}_2\,] \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = [\,\overline{C}_1\;\overline{C}_1 Y + \overline{C}_2\,]$$

The stable and unstable terms are given by

$$L_s(\lambda) = \left[\begin{array}{c|c} \overline{A}_{11} - \lambda I & \overline{B}_1 + X\overline{B}_2 \\ \hline \overline{C}_1 & \overline{D} \end{array}\right], \quad L_u(\lambda) = \left[\begin{array}{c|c} A_i - \lambda E_i & \overline{B}_2 \\ \hline \overline{C}_1 Y + \overline{C}_2 & 0 \end{array}\right]$$

The computation of the Hankel-norm $\gamma_H = \|L_u^*(\lambda)\|_H$ can be performed using standard algorithms for proper and stable systems.

The computation of the Nehari approximation can be done using the algorithm of [10] for continuous-time systems. For discrete-time systems, the same algorithm is applicable after performing a bilinear transformation to map the exterior of the unit circle to the open right half plane. A suitable transformation and its inverse transformation are $z = \frac{1+s}{1-s}$ and $s = \frac{z-1}{z+1}$, respectively. Note that in the case of an improper $L_u(z)$, all infinite poles go to $s = 1$ and therefore the "equivalent" continuous-time system will be proper.

In the light of the cancelation theory for continuous-time two-block problems of [13], pole-zero cancelations occur when forming $Q_2(s)$ in (6.23). In accordance with this theory, the expected order of $Q_2(s)$ is $n_r + n_i - 1$, where n_r and n_i are the McMillan degrees of $M_r(s)$ and $G_i(s)$, respectively. It is conjectured that similar cancelations will occur also for discrete-time systems, where a cancelation theory for two-block problems is still missing. Although we were not able to derive an explicit minimal state-space realization of $Q_2(\lambda)$, we can safely employ minimal realization procedures which exploits that the resulting $Q_2(\lambda)$ is stable. Balancing related methods are especially well suited for this computation, as for example, the square-root balancing-free method of [24, Algorithm MR6].

Stage 4:

The resulting detector $\widehat{Q}(\lambda)$ in (6.25) may be improper and/or unstable, and therefore we need to determine a stable $M(\lambda)$ having the least McMillan degree such that $M(\lambda)\widehat{Q}(\lambda)$ is proper and stable. Suitable state-space based factorization algorithms are described in [25]. To choose $M(\lambda)$ diagonal, the same algorithms can be applied to the individual rows of $\widehat{Q}(\lambda)$, for which we can build minimal descriptor state-space realizations using the algorithm of [23].

6.6 Illustrative example

We consider the robust actuator fault detection and isolation example of [7]. The fault system (6.2) has a standard state-space realization (6.26) with $E = I$ and

$$A(\delta_1,\delta_2) = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.5(1+\delta_1) & 0.6(1+\delta_2) \\ 0 & -0.6(1+\delta_2) & -0.5(1+\delta_1) \end{bmatrix}$$

$$B_u = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_d = 0, \quad B_f = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D_u = 0, \quad D_d = 0, \quad D_f = 0$$

In the expression of $A(\delta_1,\delta_2)$, δ_1 and δ_2 are uncertainties in the real and imaginary parts of the two complex conjugated eigenvalues $\lambda_{1,2} = -0.5 \pm j0.6$ of the nominal value $A(0,0)$. The fault detection filter is aimed to provide robust fault detection and isolation of actuator faults in the presence of these parametric uncertainties.

We reformulate the problem by assimilating δ_1 and δ_2 with fictitious noise inputs. We take A in (6.26) simply as the nominal value $A(0,0)$ and additionally define

$$B_w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D_w = 0.$$

At **Stage 1** we choose as nullspace basis

$$N_l(s) = [\, I \;\; -G_u(s)\,] = \left[\begin{array}{c|cc} A - sI & 0 & -B_u \\ \hline C & I & -D_u \end{array}\right]$$

and the corresponding $N_f(s)$ and $N_w(s)$ are simply $N_f(s) = G_f(s)$ and $N_w(s) = G_w(s)$. The solvability condition (6.29) is fulfilled, thus the AFDIP is solvable. Note that $N_f(s)$ is invertible.

At **Stage 2** we compute the quasi-co-outer-inner factorization of $\widetilde{G}(s) = [\, N_f(s)\; N_w(s)\,]$ in (6.14). The resulting realization of $G_{o,1}(s)$ in (6.30) has the matrices

$$\overline{B}_0 = \begin{bmatrix} -1.2181 & 0.3638 \\ -0.9828 & 0.7115 \\ -0.9913 & -1.0321 \end{bmatrix}, \quad \overline{D}_0 = 0$$

Since $\widetilde{G}(s)$ has two zeros at infinity, $G_{o,1}(s)$ will also have these two zeros at infinity and an additional stable zero at -1.1336. This stable zero is also the only pole of the 4×4 inner factor $G_i(s)$.

The descriptor realization of the updated $\overline{Q}(s)$ in (6.35) is

$$\overline{Q}(s) = \left[\begin{array}{cc|cc} A - sI & \overline{B}_o & 0 & -B_u \\ C & \overline{D}_o & I & -D_u \\ \hline 0 & -I & 0 & 0 \end{array}\right]$$

While the updated detector $\overline{Q}(s)$ is improper (having two infinite poles), the updated $\overline{N}_f(s)$ and $\overline{N}_w(s)$ can alternatively be expressed as in (6.17) and therefore have minimal realizations which are stable standard systems (as parts of the inner factor).

With $M_r(s) = I_2$, we compute $\overline{F}_1(s)$ and $\overline{F}_2(s)$ as

$$[\, \overline{F}_1(s) \;\; \overline{F}_2(s) \,] = [\, I \;\; 0 \,][\, G^*_{i,1}(s) \;\; G^*_{i,2}(s) \,] = \left[\begin{array}{c|cc} \overline{A} - sI & \overline{B}_1 & \overline{B}_2 \\ \hline \overline{C} & \overline{D}_1 & \overline{D}_2 \end{array}\right]$$

where

$$\overline{A} = 1.134, \qquad \overline{B}_1 = \left[\begin{array}{cc} -0.2830 & -0.41822 \end{array}\right], \quad \overline{B}_2 = \left[\begin{array}{cc} 0.8756 & 1.4586 \end{array}\right],$$

$$\overline{C} = \left[\begin{array}{c} -0.0623 \\ -0.7413 \end{array}\right], \quad \overline{D}_1 = \left[\begin{array}{cc} -0.8314 & 0.2423 \\ -0.3914 & -0.3112 \end{array}\right], \quad \overline{D}_2 = \left[\begin{array}{cc} 0.4477 & -0.2226 \\ -0.7625 & -0.4105 \end{array}\right]$$

Both $\overline{F}_1(s)$ and $\overline{F}_2(s)$ are first order systems with an unstable eigenvalue at 1.1336.

$\mathcal{H}_2$-optimal synthesis

With $\{\overline{F}_1(s)\}_+ = \overline{D}_1$ at **Stage 3**, we determine at **Stage 4** the final detector as

$$Q(s) = M(s)\overline{D}_1 G^{-1}_{o,1}(s)[\, I \;\; -G_u(s) \,]$$

with $M(s) = \frac{10}{s+10}K$ chosen to ensure the properness and stability of the detector. The 2×2 scaling matrix K was determined to ensure that the resulting DC-gain of the TFM from faults to residuals is the identity matrix. The detector has a standard system realization of order 3. Note that the orders of the realizations of the individual factors are respectively 2, 0, 5, and 3, which sum together to 10. The resulting low order (in fact the least possible order) clearly illustrates the advantage of the integrated algorithm, which allows, via explicitly computable realizations, to obtain at each step least order representations of the detector.

For completeness, we give the matrices of the state-space representation of the resulting detector

$$Q(s) = \left[\begin{array}{c|c} A_Q - sI & B_Q \\ \hline C_Q & D_Q \end{array}\right] \tag{6.36}$$

with

$$A_Q = \left[\begin{array}{ccc} -10.0147 & -0.4346 & -3.0643 \\ -0.0057 & -10.1691 & -1.1925 \\ 0.0433 & 1.2836 & -0.9498 \end{array}\right],$$

$$B_Q = \left[\begin{array}{cccc} -2.4464 & 0.4409 & -0.1912 & -0.3260 \\ -1.6712 & 3.6086 & 0.5794 & 0.1533 \\ -0.1336 & -0.3443 & -0.0512 & 0.3378 \end{array}\right],$$

$$C_Q = \left[\begin{array}{ccc} -9.6340 & -21.1930 & -5.3805 \\ 36.3688 & 13.7607 & 12.5546 \end{array}\right],$$

$$D_Q = \left[\begin{array}{cccc} -6.3099 & 8.2359 & 0 & 0 \\ 11.3898 & -5.8839 & 0 & 0 \end{array}\right]$$

The corresponding residual norm is $\|M(s)\widetilde{\mathcal{R}}(s)\|_2 = 7.9203$.

In Figure 6.2 we present the results of a Monte Carlo analysis of step responses of the parameter dependent residual generation system (of the form (6.4)) from the fault and control inputs for 20 random samples of δ_1 and δ_2 in the range $[-0.25, 0.25]$. The simulations have been performed using the original parameter uncertain state-space model. The step responses corresponding to a randomly generated parameter combination have the same color. As it can be observed, with an appropriate choice of the detection threshold, the detection and isolation of constant faults can be reliably performed in the presence of parametric uncertainties.

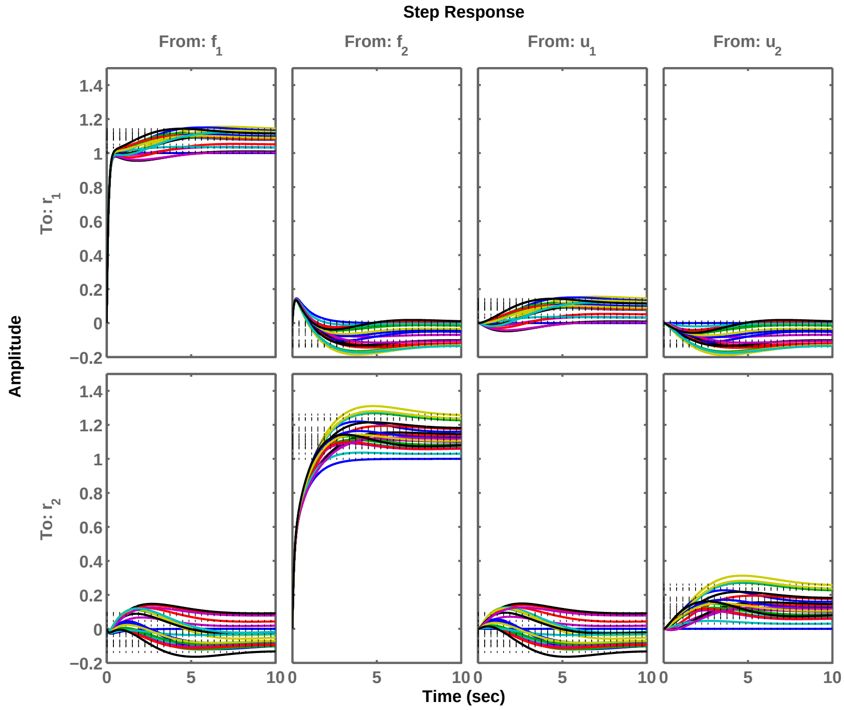

Figure 6.2. *Parametric step response analysis for $\mathcal{H}_2$-synthesis*

$\mathcal{H}_\infty$-optimal synthesis

At **Stage 3**, the γ-iteration starts with $\gamma_l = 0.9239$ and $\gamma_u = 1$ and ends with $\gamma = 0.9239 \ (= \gamma_l)$ for which the corresponding $\gamma_H = 0.5233$. The optimal Nehari approximation of the unstable part $L_u(s)$ has order zero, and the corresponding norm $\left\|V^{-1}(s)\left(\overline{F}_1(s) - Y(s)\right)\right\|_\infty = \gamma_H$. Full cancelation takes place when forming $Q_2(s)$ in (6.23), which thus results as a constant gain

$$Q_2(s) = \begin{bmatrix} -0.6389 & 0.5705 \\ -0.5081 & -0.1017 \end{bmatrix}$$

The zero McMillan degree of $Q_2(s)$ fully agrees with the degree theory of [13].

Finally, at **Stage 4** we choose $M(s) = \frac{10}{s+10}K$ to make $Q(s)$ proper and stable. The 2×2 scaling matrix K was determined to ensure that the resulting DC-gain of the TFM from faults to residuals is the identity matrix. The expression of the detector $Q(s)$ can be written down explicitly as

$$Q(s) = M(s)Q_2(s)G_{o,1}^{-1}(s)[\,I \ \ -G_u(s)\,]$$

which has a standard system realization of order 3. Note that the orders of the realizations of the individual factors are respectively 2, 0, 5, and 3, which, once again, sum together to 10.

The resulting state-space representation of the detector (6.36) has the matrices

$$A_Q = \begin{bmatrix} -10.0284 & -0.4410 & -3.2395 \\ -0.0070 & -10.1089 & -0.7999 \\ 0.0790 & 1.2256 & -0.9963 \end{bmatrix},$$

$$B_Q = \begin{bmatrix} -2.6035 & 0.5571 & -0.1846 & -0.3369 \\ -1.3316 & 3.4294 & 0.5806 & 0.1852 \\ -0.1780 & -0.3022 & -0.0473 & 0.3354 \end{bmatrix},$$

$$C_Q = \begin{bmatrix} -11.5159 & -21.6025 & -5.3846 \\ 36.0717 & 13.1418 & 12.5640 \end{bmatrix},$$

$$D_Q = \begin{bmatrix} -6.3099 & 8.2359 & 0 & 0 \\ 11.3898 & -5.8839 & 0 & 0 \end{bmatrix}$$

The corresponding residual norm is $\|M(s)\widetilde{\mathcal{R}}(s)\|_\infty = 3.3973$.

We evaluated the step responses of the parameter dependent residual generation system (of the form (6.4)) from the faults and control inputs on a 5×5 uniform grid for δ_1 and δ_2 in the range $[-0.25, 0.25]$. The resulting parametric step responses can be seen in Fig. 6.3. As it can be observed, with an appropriate choice of the detection threshold, the detection and isolation of constant faults can be reliably performed in the presence of parametric uncertainties.

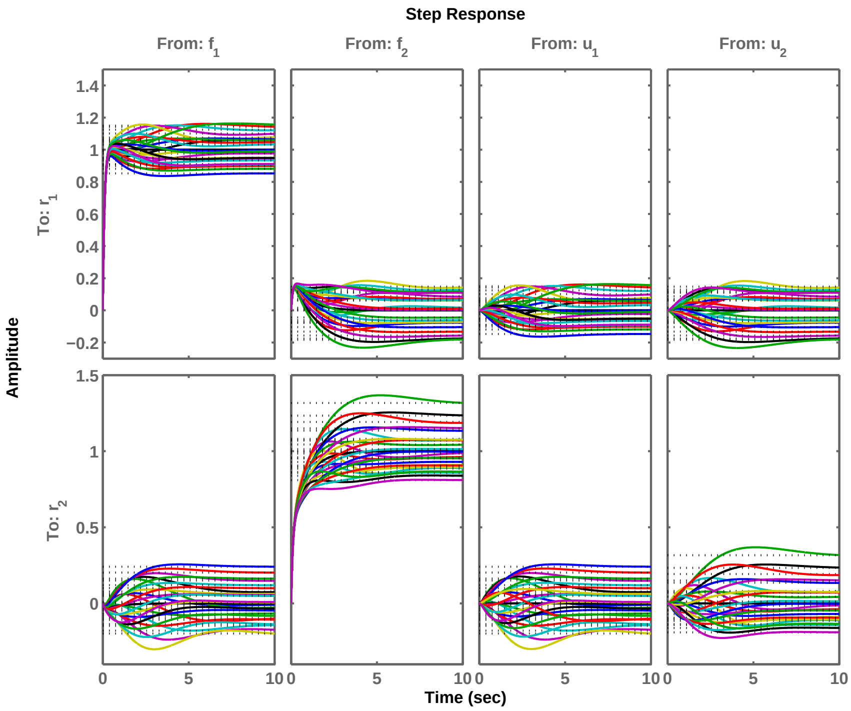

Figure 6.3. *Parametric step response analysis for $\mathcal{H}_\infty$-synthesis*

6.7 Conclusions

We presented general computational approaches to solve the $\mathcal{H}_{2/\infty}$-norm optimal FDI filter design problems. The new approaches reformulate the filter design problems as an equivalent model matching problems for which an integrated algorithm is proposed which is able to solve this problem in the most general setting. In this way, the technical difficulties often encountered by the existing methods when trying to reduce the approximation problems to standard $\mathcal{H}_{2/\infty}$-norm synthesis problems are completely avoided. For example, the presence of zeros or poles on the boundary of stability domains or problems with non-full rank and even improper transfer-function matrices can be easily handled. The underlying main computational algorithms are based on descriptor system representations and rely on orthogonal matrix pencil reductions. For all basic computations, reliable numerical software tools are available for Matlab in the Descriptor Systems Toolbox [26] and in the current version of the Fault Detection Toolbox [31, 35]. The described algorithms represent integrated alternative approaches to the exact synthesis method proposed in [29].

Bibliography

[1] E. Anderson, Z. Bai, J. Bishop, J. Demmel, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen, *LAPACK User's Guide, Second Edition*, SIAM, Philadelphia, 1995.

[2] T. Beelen and P. Van Dooren, *An improved algorithm for the computation of Kronecker's canonical form of a singular pencil*, Lin. Alg. & Appl., 105 (1988), pp. 9–65.

[3] M. Blanke, M. Kinnaert, J. Lunze, and M. Staroswiecki, *Diagnosis and Fault-Tolerant Control*, Springer-Verlag, Berlin, 2003.

[4] N.A. Bruinsma and M. Steinbuch, *A fast algorithm to compute the H_∞-norm of a transfer function*, Systems & Control Letters, 14 (1990), pp. 287–293.

[5] J. Chen and R. J. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Kluwer Academic Publishers, London, 1999.

[6] G.-R. Duan, *Analysis and Design of Descriptor Linear Systems*, Springer, 2010.

[7] A. Edelmayer and J. Bokor, *Optimal $\mathcal{H}_\infty$ scaling for sensitivity optimization of detection filters*, Int. J. Robust Nonlinear Control, 12 (2002), pp. 749–760.

[8] B. A. Francis, *A Course in H^∞ Theory*, Springer-Verlag, New York, 1987.

[9] P. M. Frank and X. Ding, *Frequency domain approach to optimally robust residual generation and evaluation for model-based fault diagnosis*, Automatica, 30 (1994), pp. 789–804.

[10] K. Glover, *All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds*, Int. J. Control, 39 (1984), pp. 1115–1193.

[11] S. Van Huffel, V. Sima, A. Varga, S. Hammarling, and F. Delebecque, *High-performance numerical software for control*, Control Systems Magazine, 24 (2004), pp. 60–76.

[12] B. Kågström and P. Van Dooren, *Additive decomposition of a transfer function with respect to a specified region*, in Proc. MTNS Symp., Brussels, 1989.

[13] D.J.N. Limebeer and G.D. Halikias, *A controller degree bound for $\mathcal{H}^\infty$-optimal control problems of the second kind*, SIAM J. Control Optim., 26 (1988), pp. 646–677.

[14] P. Misra, P. Van Dooren, and A. Varga, *Computation of structural invariants of generalized state-space systems*, Automatica, 30 (1994), pp. 1921–1936.

[15] H. Niemann and J. Stoustrup, *Design of fault detectors using H_∞ optimization*, in Proc. of CDC'00, Sydney, Australia, 2000, pp. 4237–4238.

[16] C. Oară, *Constructive solutions to spectral and inner-outer factorizations with respect to the disk*, Automatica, 41 (2005), pp. 1855 – 1866.

[17] C. Oară and A. Varga, *Minimal degree coprime factorization of rational matrices*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 245–278.

[18] ——, *Computation of general inner-outer and spectral factorizations*, IEEE Trans. Automat. Control, 45 (2000), pp. 2307–2325.

[19] P. Van Dooren, *The computation of Kronecker's canonical form of a singular pencil*, Lin. Alg. & Appl., 27 (1979), pp. 103–141.

[20] ——, *The generalized eigenstructure problem in linear systems theory*, IEEE Trans. Automat. Control, 26 (1981), pp. 111–129.

[21] ——, *A generalized eigenvalue approach for solving Riccati equations*, SIAM J. Sci. Stat. Comput., 2 (1981), pp. 121–135.

[22] A. Varga, *Computation of transfer function matrices of generalized state-space models*, Int. J. Control, 50 (1989), pp. 2543–2561.

[23] ——, *Computation of irreducible generalized state-space realizations*, Kybernetika, 26 (1990), pp. 89–106.

[24] ——, *Minimal realization procedures based on balancing and related techniques*, in Computer Aided Systems Theory - EUROCAST'91, F. Pichler and R. Moreno Diaz, eds., vol. 585 of Lect. Notes Comp. Scie., Springer Verlag, Berlin, 1992, pp. 733–761.

[25] ——, *Computation of coprime factorizations of rational matrices*, Lin. Alg. & Appl., 271 (1998), pp. 83–115.

[26] ——, *A* Descriptor Systems *toolbox for* Matlab, in Proc. CACSD'2000 Symposium, Anchorage, Alaska, 2000.

[27] ——, *Computing generalized inverse systems using matrix pencil methods*, Int. J. of Applied Mathematics and Computer Science, 11 (2001), pp. 1055–1068.

[28] ——, *Computation of least order solutions of linear rational equations*, in Proc. of MTNS'04, Leuven, Belgium, 2004.

[29] ——, *New computational approach for the design of fault detection and isolation filters*, in Advances in Automatic Control, M. Voicu, ed., vol. 754 of The Kluwer International Series in Engineering and Computer Science, Kluwer Academic Publishers, 2004, pp. 367–381.

[30] ——, *Numerically reliable methods for optimal design of fault detection filters*, in Proc. of CDC'05, Seville, Spain, 2005.

[31] ——, *A* fault detection *toolbox for* Matlab, in Proc. of CACSD'06, Munich, Germany, 2006.

[32] ——, *On designing least order residual generators for fault detection and isolation*, in Proc. 16th Internat. Conf. on Control Systems and Computer Science, Bucharest, Romania, 2007, pp. 323–330.

[33] ——, *On computing nullspace bases – a fault detection perspective*, in Proc. IFAC 2008 World Congress, Seoul, Korea., 2008, pp. 6295–6300.

[34] ——, *General computational approach for optimal fault detection*, in Proc. SAFEPROCESS'2009, Barcelona, Spain, 2009.

[35] ——, *Linear FDI-Techniques and Software Tools*, Fault Detection Toolbox V0.9 – Technical Documentation, German Aerospace Center (DLR), Institute of Robotics and Mechatronics, 2009.

[36] ——, *Integrated algorithm for solving $\mathcal{H}_2$-optimal fault detection and isolation problems*, in Proc. of SYSTOL'2010, Nice, France, 2010.

[37] ——, *Integrated computational algorithm for solving $\mathcal{H}_\infty$-optimal FDI problems*, in Proc. IFAC 2011 World Congress, Milano, Italy, 2011.

[38] ——, *Linear FDI-Techniques and Software Tools*, Fault Detection Toolbox V1.0 – Technical Documentation, German Aerospace Center (DLR), Institute of Robotics and Mechatronics, 2011.

[39] ——, *On computing minimal proper nullspace bases with applications in fault detection*, in Numerical Linear Algebra in Signals, Systems and Control, P. Van Dooren, S. P. Bhattacharyya, R. H. Chan, V. Olshevsky, and A. Routray, eds., vol. 80 of Lecture Notes in Electrical Engineering, Springer Verlag, Berlin, 2011, pp. 433–465.

[40] A. Varga and B. D. O. Anderson, *Accuracy-enhancing methods for balancing-related frequency-weighted model and controller reduction*, Automatica, 39 (2003), pp. 919–927.

[41] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1996.