On solving non-standard $\mathcal{H}_-/\mathcal{H}_{2/\infty}$ fault detection problems

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***Abstract*— We discuss solvability issues of $\mathcal{H}_-/\mathcal{H}_{2/\infty}$ optimal fault detection problems in the most general setting. A solution approach is presented which successively reduces the initial problem to simpler ones. The last computational step generally may involve the solution of a non-standard $\mathcal{H}_-/\mathcal{H}_{2/\infty}$ optimization problem for which we discuss possible solution approaches. Using an appropriate definition of the $\mathcal{H}_-$-index, we provide a complete solution of this problem in the case of $\mathcal{H}_2$-norm. Furthermore, we discuss the solvability issues in the case of $\mathcal{H}_\infty$-norm.**

I. INTRODUCTION

The fault detection problem consists in detecting via so-called residual signals (or residuals) the occurrence of any fault in a system in the presence of arbitrary control and disturbance inputs acting on that system. The residuals are generated by a residual generator filter having as inputs the measured outputs and the controlled inputs. For the exact or approximate solution of fault detection problems we need to synthesize residual generator filters which are highly sensitive to all faults in the presence of controls and disturbances acting on the system. This problem has been widely studied using different problem settings and different solution approaches. For a comprehensive account of existing methods see for example the monographs of [1], [2].

The conditions for the exact solvability of the fault detection problem are frequently not fulfilled in practical applications. This is usually the case, for example, when robustness aspects are addressed by recasting uncertain parameters as additional (artificial) disturbance inputs. Thus, in most of applications only approximate solutions of the fault detection problem can be aimed to be computed, where the goal is to design residual generators which minimize the effects of disturbances on the residuals, while simultaneously maximizing the effect of faults. The approximate solution of the fault detection problem has been addressed by many authors [3], [4], [5], [6], [7], [8] by solving various multi-objective optimization problems, as for example, the $\mathcal{H}_-/\mathcal{H}_\infty$ and $\mathcal{H}_-/\mathcal{H}_2$ optimal fault detection problems. A common feature of some of the proposed solution approaches is that they usually rely on various technical assumptions which, although allows the derivation of explicit analytical solutions, are not really necessary for the solution of the problem. Moreover, often the connections between the exact and approximate solutions are obscured either by employing inappropriate optimization criteria or completely ignoring available structural information.

We discuss computational issues in solving the $\mathcal{H}_-/\mathcal{H}_{2/\infty}$ optimal fault detection problems in the most general setting. A computational approach is presented which successively reduces the initial problem to simpler ones. The last computational step generally may involve the solution of a non-standard $\mathcal{H}_-/\mathcal{H}_{2,\infty}$ optimization problem for which we discuss possible computational approaches. Using an appropriate definition of the $\mathcal{H}_-$-index, we provide a complete solution of this problem in the case of $\mathcal{H}_2$-norm. Furthermore, we discuss the solvability issues in the case of $\mathcal{H}_\infty$-norm.

II. THE FAULT DETECTION PROBLEM

Consider additive fault models described by input-output representations of the form

$$\mathbf{y}(\lambda)=G_u(\lambda)\mathbf{u}(\lambda)+G_d(\lambda)\mathbf{d}(\lambda)+G_w(\lambda)\mathbf{w}(\lambda)+G_f(\lambda)\mathbf{f}(\lambda), \tag{1}$$

where $\mathbf{y}(\lambda)$, $\mathbf{u}(\lambda)$, $\mathbf{d}(\lambda)$, $\mathbf{w}(\lambda)$, and $\mathbf{f}(\lambda)$ are Laplace- or Z-transformed vectors of the p-dimensional system output vector $y(t)$, m_u-dimensional control input vector $u(t)$, m_d-dimensional disturbance vector $d(t)$, m_w-dimensional noise vector $w(t)$ and m_f-dimensional fault vector $f(t)$, respectively, and where $G_u(\lambda)$, $G_d(\lambda)$, $G_w(\lambda)$ and $G_f(\lambda)$ are the *transfer-function matrices* (TFMs) from the control inputs to outputs, disturbance inputs to outputs, noise inputs to outputs, and fault inputs to outputs, respectively. According to the system type, the frequency variable λ is either s, the complex variable in the Laplace-transform in the case of a continuous-time system or z, the complex variable in the Z-transform in the case of a discrete-time system. For most of practical applications, the TFMs $G_u(\lambda)$, $G_d(\lambda)$, $G_w(\lambda)$ and $G_f(\lambda)$ are proper rational matrices. However, for complete generality of our problem setting, we will allow that these TFMs are general non-proper rational matrices for which we will not *a priori* assume any further properties (e.g. full rank).

A linear residual generator (or fault detection filter) processes the measurable system outputs $y(t)$ and control inputs $u(t)$ and generates the residual signals $r(t)$ which serve for decision making on the presence or absence of faults. The input-output form of this filter is

$$\mathbf{r}(\lambda) = R(\lambda)\begin{bmatrix} \mathbf{y}(\lambda) \\ \mathbf{u}(\lambda) \end{bmatrix} \tag{2}$$

where $R(\lambda)$ is the TFM of the filter. For a physically realizable filter, $R(\lambda)$ must be *proper* (i.e., only with finite

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poles) and *stable* (i.e., only with poles having negative real parts for a continuous-time system or magnitudes less than one for a discrete-time system). The (dynamic) *order* of $R(\lambda)$ (also known as *McMillan degree*) is the dimension of the state vector of a minimal state-space realization of $R(\lambda)$. The dimension q of the residual vector $r(t)$ depends on the fault detection problem to be solved, and can be either given or determined during the solution process.

The residual signal $r(t)$ in (2) generally depends via the system outputs $y(t)$ of all system inputs $u(t)$, $d(t)$, $w(t)$ and $f(t)$. The residual generation system is obtained by replacing in (2) $\mathbf{y}(\lambda)$ by its expression in (1)

$$\mathbf{r}(\lambda) = R_u(\lambda)\mathbf{u}(\lambda)+R_d(\lambda)\mathbf{d}(\lambda)+R_w(\lambda)\mathbf{w}(\lambda)+R_f(\lambda)\mathbf{f}(\lambda) \tag{3}$$

where

$$\begin{aligned}&[\, R_u(\lambda)|R_d(\lambda)|R_w(\lambda)|R_f(\lambda)\,] := \\ &\qquad R(\lambda)\left[\begin{array}{c|c|c|c} G_u(\lambda) & G_d(\lambda) & G_w(\lambda) & G_f(\lambda) \\ I_{m_u} & 0 & 0 & 0 \end{array}\right]\end{aligned}$$

For a successfully designed filter $R(\lambda)$, the corresponding residual generation system is proper and stable and achieves specific fault detection requirements (e.g., decoupling of control and disturbance inputs from the residuals).

In this paper we consider the solution of the following *approximate fault detection problem* (AFDP): For given $\gamma > 0$, determine $\beta > 0$ and a physically realizable linear residual generator filter of the form (2) such that we have:

(i) $R_u(\lambda) = 0$ and $R_d(\lambda) = 0$;
(ii) $\|R_w(\lambda)\|_{2/\infty} \leq \gamma$;
(iii) $\|R_f(\lambda)\|_{2/\infty-} \geq \beta$.

The condition (i) means that the control input $u(t)$ and disturbance input $d(t)$ are fully decoupled from the residual signal $r(t)$. In consequence, condition (ii) requires that in the absence of faults the effects of noise on the residuals is bounded. The third condition (iii) requires that in the absence of noise, the residual signal $r(t)$ is sensitive to all faults. Precise statements will depend on the definition used for $\|\cdots\|_{2/\infty-}$. It follows that maximizing β/γ is a meaningful goal to achieve maximum sensitivity to faults versus noise.

Several definitions of the $\|\cdot\|_{2/\infty-}$ index are used in the literature. The definitions used in [3], [4], [5], [6], [7] are in terms of the least singular values the frequency-response of $R_f(\lambda)$ and therefore are meaningful only when $m_f \leq p$. In this paper we use the following definition for this index

$$\|R_f(\lambda)\|_{2/\infty-} = \min_j \|R_{f_j}(\lambda)\|_{2/\infty}, \tag{4}$$

where $R_{f_j}(\lambda)$ is the j-th column of $R_f(\lambda)$. The requirement $\|R_f(\lambda)\|_{2/\infty-} > 0$ merely asks for nonzero columns $R_{f_j}(\lambda)$ and thus is equivalent to a (*weak*) *fault detectability* condition.

The above formulation of AFDP includes the *exact fault detection problem* (EFDP) when $m_w = 0$, as well as the alternative formulations in [3], [5], [6] when $m_d = 0$. Moreover, this formulation of the AFDP also covers structured residuals, where part of the disturbance signals in $d(t)$ are faults which must be decoupled from the residuals [1].

III. The Solution Approach

In this section we present an approach to solve the formulated AFDP by solving the following optimization problem.
Problem AFDPO: Determine a proper and stable $R(\lambda)$ which achieves the conditions in (i) ($R_u(\lambda) = 0$ and $R_d(\lambda) = 0$), and additionally solves the optimization problem

$$\beta := \max\{\|R_f(\lambda)\|_{2/\infty-} : \|R_w(\lambda)\|_{2/\infty} \leq \gamma\} > 0 \tag{5}$$

The described approach is similar to those proposed in [3], [5], [6], [8] and consists of three successive steps which reduce the original optimization problem to equivalent but simpler ones. We describe these steps and point out the distinctions between our approach and the existing ones.

Step 1: We choose a detector $R(\lambda)$ of the form

$$R(\lambda) = Q(\lambda)R_1(\lambda),$$

where $R_1(\lambda)$ is a $q_1 \times (p + m_u)$ stable and proper TFM which solves the EFDP assuming $w(t) \equiv 0$, and $Q(\lambda)$ is a $q \times q_1$ TFM to be determined together with q_1, such that $q_1 \leq q$ (the number of outputs of the detector). We compute

$$N_f(\lambda) = R_1(\lambda)\left[\begin{array}{c} G_f(\lambda) \\ 0 \end{array}\right], \;\; \widetilde{M}_w(\lambda) = R_1(\lambda)\left[\begin{array}{c} G_w(\lambda) \\ 0 \end{array}\right]$$

Furthermore, it is possible to choose (see below) $q_1 = r_w := \operatorname{rank} \widetilde{M}_w(\lambda)$. The optimization problem (5) becomes in this case to determine a $q \times q_1$ proper and stable TFM $Q(\lambda)$ which solves

$$\beta = \max\{\|Q(\lambda)N_f(\lambda)\|_{2/\infty-} : \|Q(\lambda)\widetilde{M}_w(\lambda)\|_{2/\infty} \leq \gamma\} > 0 \tag{6}$$

To determine $R_1(\lambda)$, we can use the nullspace based techniques of [9], [10], and we set $R_1(\lambda) = WN_l(\lambda)$, where $N_l(\lambda)$ is a $(p-r_d)\times(p+m_u)$ TFM representing a rational proper left nullspace basis of

$$G(\lambda) := \left[\begin{array}{cc} G_u(\lambda) & G_d(\lambda) \\ I_{m_u} & 0 \end{array}\right], \tag{7}$$

with $r_d = \operatorname{rank} G_d(\lambda)$, and W is a $r_w \times (p - r_d)$ constant matrix chosen such that $\widetilde{M}_w(\lambda)$ has full row rank $r_w > 0$ and the fault detectability condition $\|N_f(\lambda)\|_{2/\infty-} > 0$ is fulfilled. Note that $p - r_d$ is also the dimension of a left nullspace of $G_d(\lambda)$ and is an upper bound on the achievable maximum number of independent residual outputs.

At the end of this step, the resulting $R_1(\lambda)$, $N_f(\lambda)$ and $\widetilde{M}_w(\lambda)$ are guaranteed to be proper and rational TFMs, although the original model (1) may be non-proper. Moreover, the associated dynamics (i.e., poles) can be arbitrarily assigned. Note that the case of non-proper systems has been only addressed in [8] in the context of solving the AFDPO.

Remark. This computational step has been employed in [8] as a preliminary processing step, to reduce the initial problem with control and disturbance inputs to a simpler one with only noise and fault inputs. Therefore, if there are no noise inputs, the exact solution can be targeted by skipping the next steps. A necessary and sufficient condition for the existence of the nullspace basis is $p > r_d$, which in the absence of disturbance input is automatically fulfilled. If this condition is not fulfilled, then some of disturbance inputs in d can be redefined as noise inputs. In [3], [5], [6], only the case without disturbance input is considered, thus the exact solution can not be computed even if one exists. In contrast to [3], [5], [6], our approach is also seamlessly applicable to improper systems.

Step 2: We compute the *quasi*-outer-inner factorization of $\widetilde{M}_w(\lambda)$ in the form

$$\widetilde{M}_w(\lambda) = M_{wo}(\lambda) M_{wi}(\lambda),$$

where the quasi-outer factor $M_{wo}(\lambda)$ is a $q_1 \times q_1$ invertible TFM which has only stable zeros, excepting possible zeros on the boundary of the stability domain (i.e., the imaginary axis, including infinity, for a continuous-time system or the unit circle for a discrete-time system), and $M_{wi}(\lambda)$ is inner (i.e., $M_{wi}(\lambda)M_{wi}^*(\lambda) = I_q$ with $M_{wi}^*(s) = M_{wi}^T(-s)$ in the continuous-time case, and $M_{wi}^*(z) = M_{wi}^T(1/z)$ in the discrete-time case).

The optimization problem (6) can be equivalently reformulated as

$$\beta = \max\{\|Q(\lambda)N_f(\lambda)\|_{2/\infty-} : \|Q(\lambda)M_{wo}(\lambda)\|_{2,\infty} \leq \gamma\} > 0 \tag{8}$$

For the computation of the quasi-outer-inner factorization of $\widetilde{M}_w(\lambda)$ we employ the dual of the algorithm of [11] for the continuous-time case and the dual of the algorithm of [12] for the discrete-time case. Both algorithms rely on state-space computations and are completely general, being able to cope with systems having zeros on the boundary of the stability domain.

At the end of this step, the resulting $M_{wo}(\lambda)$ is square, stable, and proper. Its poles are the same as those of $\widetilde{M}_w(\lambda)$. The zeros of $M_{wo}(\lambda)$ are stable, excepting those zeros inherited from $\widetilde{M}_w(\lambda)$ which lie on the boundary of the stability domain.

Remark. This computational step is common to several existing approaches [3], [5], [6], where only the *standard case* is addressed, when the resulting outer factor $M_{wo}(\lambda)$ is minimum-phase and invertible. In this case, the solution can be easily computed analytically, by solving standard spectral factorization techniques. In this way, the *non-standard case*, when $\widetilde{M}_w(\lambda)$ has zeros on the boundary of the stability domain, is explicitly excluded in [3], [5], [6] via technical assumptions. This computational step has been employed in [8] to address the non-standard case, by employing suitable factorization algorithms [11], [12] able to determine quasi-outer factors.

Step 3: This step addresses the solution of the optimization problem (6) and is described in detail in the next section. Both the standard and non-standard cases are considered. The main improvements over the methods in [3], [5], [6], [8] consist in providing computable solutions for several non-standard cases considered in what follows.

IV. The Optimization Step

In this section we will consider the four problems that result from the two choices of the two norms. Given $M_{wo}(\lambda) \in \mathcal{H}_\infty$ and $N_f(\lambda) \in \mathcal{H}_\infty$, find $Q(\lambda) \in \mathcal{H}_\infty$ to maximize β such that:

Problem $\mathrm{P}(\infty/\infty)$:
$$\|Q(\lambda)M_{wo}(\lambda)\|_\infty \leq 1 \ \text{ and } \left\|Q(\lambda)N_{f_j}(\lambda)\right\|_\infty \geq \beta \ \forall\ j$$
Problem $\mathrm{P}(\infty/2)$:
$$\|Q(\lambda)M_{wo}(\lambda)\|_\infty \leq 1 \ \text{ and } \left\|Q(\lambda)N_{f_j}(\lambda)\right\|_2 \geq \beta \ \forall\ j$$
Problem $\mathrm{P}(2/\infty)$:
$$\|Q(\lambda)M_{wo}(\lambda)\|_2 \leq 1 \ \text{ and } \left\|Q(\lambda)N_{f_j}(\lambda)\right\|_\infty \geq \beta \ \forall\ j$$
Problem $\mathrm{P}(\infty/\infty)$:
$$\|Q(\lambda)M_{wo}(\lambda)\|_2 \leq 1 \ \text{ and } \left\|Q(\lambda)N_{f_j}(\lambda)\right\|_2 \geq \beta \ \forall\ j$$

A. Problems P(∞/∞) and P($\infty/2$)

These problems are well studied when $M_{wo}^{-1} \in \mathcal{H}_\infty$ with the optimal solution being $Q = M_{wo}^{-1}$ since for $\lambda = j\omega$ or $e^{j\omega T}$, $Q^*Q \leq {M_{wo}^*}^{-1}M_{wo}^{-1}$ implies $N_{f_j}^* Q^* Q N_{f_j} \leq N_{f_j}^* {M_{wo}^*}^{-1} M_{wo}^{-1} N_{f_j}$ and hence $\left\|QN_{f_j}\right\| \leq \left\|M_{wo}^{-1}N_{f_j}\right\|$ for both choices of norm and hence this optimal solution is in fact independent of the objective defined by N_f (see [5], [6], [3]).

In the case when $M_{wo}^{-1} \notin \mathcal{H}_\infty$ the objective might be unbounded and the optimal solution may be improper or have poles approaching the stability boundary. An approach to this case is to consider the sequence $M_\epsilon(\lambda) \in \mathcal{H}_\infty$ for $\epsilon > 0$, $\epsilon \to 0$ the stable minimum phase spectral factor satisfying,

$$M_\epsilon(\lambda)M_\epsilon^*(\lambda) \ = \ \epsilon I + M_{wo}(\lambda)M_{wo}^*(\lambda) \tag{9}$$
$$\text{and let } Q_\epsilon(\lambda) \ = \ M_\epsilon(\lambda)^{-1} \ \in \mathcal{H}_\infty \text{ for } \epsilon > 0 \tag{10}$$

which can be calculated from solving the relevant Riccati equation (for example Theorem 13.19 in [13]). It is easily shown that $\|Q_\epsilon(\lambda)M_{wo}(\lambda)\|_\infty < 1$ for all $\epsilon > 0$. For all frequencies except any zeros of M_{wo} on the stability boundary, this sequence will approach the upper bound and hence will approach the infimum as $\epsilon \to 0$.

B. Problem P($2/\infty$)

This case can be discounted because a sequence, $Q_{2\epsilon}$ exists making $\|Q_{2\epsilon}M_{wo}\|_2 < 1$ but with $\left\|Q_{2\epsilon}N_{f_j}\right\|_\infty \to \infty$ as $\epsilon \to 0$ [5]. In continuous time if $N_{f_j}(0) \neq 0$ then such a sequence could be obtained from $Q_{2\epsilon} = \frac{\sqrt{\epsilon}}{(s+\epsilon)}$ with $\|Q_{2\epsilon}\|_2 = 1/\sqrt{2}$ and $Q_{2\epsilon}(0) = 1/\sqrt{\epsilon}$. Hence $\left\|Q_{2\epsilon}N_{f_j}\right\|_\infty$ increases as $1/\sqrt{\epsilon}$ but with $\|Q_{2\epsilon}\|_2$ remaining bounded. Analogous $Q_{2\epsilon}$ can be derived for non-zero frequencies and the discrete-time case. This calculation also indicates that the condition $\left\|QN_{f_j}\right\|_\infty \geq \beta$ is not necessarily a good indicator of fault detection since the gain may only occur over a very narrow frequency range for the fault signal.

C. Problem P($2/2$)

This problem is considered in for example [14] where $\|QN_f\|_2$ was considered and it is shown that the optimal solution finds a single frequency where the ratio of gains from w and from f can be minimized and the optimal Q then is a narrow band filter centered on this frequency. In our case with a constraint for each fault signal $\|QN_{f_j}\|_2 \geq \beta$ the solution can be more involved but it is still the case that the gain of the optimal Q is concentrated on at most m_f frequencies. The analysis is similar to that given in [15]

Theorem 4.1: Given $M_{wo},\ M_{wo}^{-1},\ N_f\ \in \mathcal{H}_\infty$, then there exists a sequence, Q_k approaching the optimal solution to:

$$\inf_{Q\in\mathcal{H}_\infty} \|QM_{wo}\|_2 \text{ s.t. } \|QN_{f_j}\|_2 \geq 1 \text{ for } j = 1\cdots m_f$$

such that

1) $Q_k(\lambda) \to 0$ for all $\lambda(= j\omega/e^{j\theta})$ except at most m_f frequencies, $\hat{\lambda}_1, ..., \hat{\lambda}_{m_f}$.
2) $\sum_{i=1}^{m_f} \text{rank } Q_k(\hat{\lambda}_i) \leq m_f$.
3) $\text{rank } Q_k(\hat{\lambda}_i) \leq \lfloor \sqrt{m_f} \rfloor$.

Proof: An outline of the proof is now given. Firstly approximating the $\mathcal{H}_2$-norm by a summation gives the problem, (for $\lambda_i = j\omega_i$ or $e^{j\omega_i T}$)

$$\min_{Q\in\mathcal{H}_\infty} \sum_{i=1}^{N-1} \text{trace } M_{wo}^*(\lambda_i)Q^*(\lambda_i)Q(\lambda_i)M_{wo}(\lambda_i)(\omega_{i+1}-\omega_i)$$
$$\text{s.t. } \sum_{i=1}^{N-1} N_{f_j}^*(\lambda_i)Q^*(\lambda_i)Q(\lambda_i)N_{f_j}(\lambda_i)(\omega_{i+1}-\omega_i) \geq \pi\ \forall j$$

Now suppose that an optimal solution is given by $Q^*(\lambda_i)Q(\lambda_i) =: Z_i = Z_i^* \geq 0$, and without loss of generality that $Z_i = \text{diag}(z_{i,1}, \cdots, z_{i,r_i}, 0\cdots 0)$ with $z_{i,k} > 0$. In these coordinates optimising over $z_{i,k}$ is a standard LP and hence has an optimal basic feasible solution, i.e. with at most m_f non-zero $z_{i,k}$ values, and this gives items 1) and 2).

For item 3) consider the problem: $\min_{Z\geq 0}$ trace M^*ZM s.t. $N_j^*ZN_j = b_j\ j = 1\cdots m_f$. Suppose Z^o is an optimal solution and $\text{rank}(Z^o) = r$. Without loss of generality assume $Z^o = \begin{bmatrix} Z_1^o & 0 \\ 0 & 0 \end{bmatrix}$ with $Z_1^o > 0$ and consider $Z^\epsilon = \begin{bmatrix} Z_1^\epsilon & 0 \\ 0 & 0 \end{bmatrix}$ where $Z_1^\epsilon = Z_1^o + \epsilon\Delta_1$. If $r^2 > m_f$ then there exists $\Delta_1 = \Delta_1^* \neq 0$ such that $N_j^* \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} N_j = 0,\ \forall j$ (since this corresponds to m_f real linear equations in r^2 real unknowns). Hence $N_j^*Z^\epsilon N_j = b_j\ \forall j$ and for Z^o to be optimal we must have that trace $M^*Z^\epsilon M =$ trace M^*Z^oM for all ϵ. In addition it can be shown that there exists an ϵ_o such that $Z_1^{\epsilon_o} \geq 0$ but is singular. This has demonstrated that the rank of Z^o can be reduced by at least one whilst maintaining the value of the objective and constraints and this can be continued until $r^2 \leq m_f$.

■

D. Optimisation over a Fixed Structure Affine in Parameters

In this section we consider the problems when $Q(\lambda)$ or indeed $R(\lambda)$ is given by $D + C(\lambda I - A)^{-1}B$ with $\begin{bmatrix} A & B \end{bmatrix}$ given and $\begin{bmatrix} C & D \end{bmatrix}$ to be optimized. We will demonstrate that Problems P($\infty/2$) and P($2/2$) both reduce to Linear Matrix Inequalities, whereas Problem P(∞/∞) does not obviously reduce to a convex problem and a non-convex simple example is presented.

Firstly, let

$$X = \begin{bmatrix} C & D \end{bmatrix}^* \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

and consider the continuous-time case, then $\|X^{1/2}L_1(s)\|_\infty \leq 1$, where $L_1(s) = D_1 + C_1(sI - A_1)^{-1}B_1$ is equivalent to the LMI in $P \geq 0$ (e.g. see [16]):

$$\begin{bmatrix} A_1^*P + PA_1 & PB_1 \\ B_1^*P & -I \end{bmatrix} + \begin{bmatrix} C_1^* \\ D_1^* \end{bmatrix} X \begin{bmatrix} C_1 & D_1 \end{bmatrix} \leq 0 \tag{11}$$

The condition $\|X^{1/2}L_2(s)\|_2 \leq (\geq)\beta$ where $L_2(s) = C_2(sI - A_2)^{-1}B_2$ is equivalent to:

$$\text{trace } C_2YC_2^*X \leq (\geq)\beta^2 \tag{12}$$

where $Y = Y^* \geq 0$ satisfies the Lyapunov equation: $A_2Y + YA_2^* + B_2B_2^* = 0$.

Analogous LMI's are available for the discrete-time case.

Also note that although the results are stated for optimizing over Q where $R = QR_1$ and R_1 is chosen such that $R_1 \begin{bmatrix} G_u & G_d \\ I & 0 \end{bmatrix} = 0$, optimization could be performed on R directly given a suitable choice of $\begin{bmatrix} A & B \end{bmatrix}$ for R.

1) Problem P($\infty/2$): In the continuous time case the constraint that $\|QM_{wo}\|_\infty < 1$ and $\|QN_{f_j}\|_2 \geq \beta$ is equivalent to the LMI's given in (11) and (12) with $L_1(s) = \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} M_{wo}(s)$ and $L_{2,j}(s) = \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} N_{f_j}(s)$. The convex optimization problem is then to maximize β^2 subject to these LMI constraints. If $\text{rank}(X^{1/2}) > m_w$ then a spectral factorization to give a Q with m_w outputs can be performed.

2) Problem P($2/2$): In the continuous-time case (12) gives a set of m_f linear inequalities in $X \geq 0$. Appropriate values for (A, B) can be obtained from the result of Theorem 4.1. The finite summation approximation gives a set of LMI's in N matrices $Z_i > 0$ which are of modest size, $m_w \times m_w$. The eigenvalues of A can be taken as lightly damped variations to $\pm j\hat{\lambda}_i$ and the total state dimension need be no greater than $2m_f$.

3) Problem P(∞/∞): Whereas the constraint $\|QM_{wo}\|_\infty \leq 1$ is convex in Q, the constraint that $\|QN_{f_j}\|_\infty > \beta$ is not convex in Q and it is not apparent how this could be re-parameterized into a convex problem.

Example 4.2: Consider the simple discrete-time example with

$$M_{wo}(z) = \begin{bmatrix} \frac{1}{1+az^{-1}} & 0 \\ 0 & \frac{1}{1-az^{-1}} \end{bmatrix}$$
$$N_f(z) = \begin{bmatrix} z^{-1} & 1 \\ 1 & -z^{-1} \end{bmatrix}$$

and for some $0 < a < 1$ and Q constrained to be a constant. Let $Q^*Q = X = X^* = \begin{bmatrix} x_1 & x_2 \\ x_2^* & x_3 \end{bmatrix} \geq 0$ then

$$\begin{aligned} X &\leq M_{wo}^{-1}(e^{j\theta})^* M_{wo}^{-1}(e^{j\theta}) =: L(\theta) \\ L(\theta) &= \begin{bmatrix} (1 + a^2 + 2a\cos(\theta)) & 0 \\ 0 & (1 + a^2 - 2a\cos(\theta)) \end{bmatrix} \\ &= \frac{1 + \cos(\theta)}{2} L(0) + \frac{1 - \cos(\theta)}{2} L(\pi), \end{aligned}$$

Hence $X \leq L(\theta)\ \forall\ \theta$ iff $X \leq L(0)$ and $X \leq L(\pi)$. The constraints from QN_f are then

$$\begin{bmatrix} e^{j\theta_1} & 1 \end{bmatrix} X \begin{bmatrix} e^{j\theta_1} & 1 \end{bmatrix}^* \geq \beta^2 \text{ for some } \theta_1$$

$$\text{and } \begin{bmatrix} 1 & -e^{j\theta_2} \end{bmatrix} X \begin{bmatrix} 1 & -e^{j\theta_2} \end{bmatrix}^* \geq \beta^2 \text{ for some } \theta_2$$

both of which reduce to

$$x_1 + x_3 + 2|x_2| \geq \beta^2$$

The optimal solution can then be shown to be $x_1 = x_3 = \frac{(1-a^2)^2}{2(1+a^2)} = \pm x_2$ and there are two distinct global minima. Hence the problem is not convex in X.

Note that the non-convexity is due to the inequalities being "for some θ_1 and θ_2"whereas if it were "for all θ_1 and θ_2" then this more stringent condition would be convex in X and β^2 and is the formulation used in a number of papers, e.g. [4].

V. Illustrative example

We consider the robust fault detection example of [8]. The fault system (1) is proper and has a state space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t) + B_f f(t) \\ y(t) &= Cx(t) + D_u u(t) + D_w w(t) + D_f f(t) \end{aligned} \quad (13)$$

with

$$A = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.5(1+\delta_1) & 0.6(1+\delta_2) \\ 0 & -0.6(1+\delta_2) & -0.5(1+\delta_1) \end{bmatrix}$$

$$B_u = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_w = 0_{3\times 2},\ \ B_f = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},\ D_u = 0_{2\times 2},\ D_w = 0_{2\times 2},\ D_f = 0_{2\times 2}.$$

In the expression of A, δ_1 and δ_2 are uncertainties in the real and imaginary parts of the two complex conjugated eigenvalues $\lambda_{1,2} = -0.5 \pm j0.6$. The fault detector filter is aimed to provide robust fault detection of actuator faults in the presence of these parametric uncertainties.

We reformulate the problem by assimilating δ_1 and δ_2 with fictitious noise inputs. We replace A in (13) simply with its nominal value for $\delta_1 = 0$ and $\delta_2 = 0$, and additionally redefine

$$B_w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We wish to design a residual generator to detect actuator faults, which provides the largest possible gap β/γ. For this we employ a detector with two outputs (i.e., we choose $q = 2$).

At *Step 1* we can choose as initial detector

$$R_1(s) = [\, I_2\ \ -G_u(s) \,] = \left[\begin{array}{c|cc} A - sI & 0 & -B_u \\ \hline C & I & -D_u \end{array} \right]$$

It follows that $N_f(s) = G_f(s)$ and the resulting $\widetilde{M}_w(s) = G_w(s)$ is invertible and has three invariant zeros at $\{-0.8, \infty, \infty\}$. The finite zero -0.8 is in fact a non-controllable eigenvalue of A in the pair (A, B_w).

At *Step 2* no inner-outer factorization is necessary and we simply set $M_{wo}(s) = G_w(s)$, and $G_{wi}(s) = I_2$.

At *Step 3*, firstly we note that $M_{wo}^{-1} \notin \mathcal{H}_\infty$ and hence we use the spectral factorization given in (9) and obtain a $Q_{1,\epsilon_1}(s)$ from (10) with $\epsilon_1 = 0.01$. This will be approximately optimal for both problems P($\infty/2$) and P(∞/∞).

In Figure 1 we present the step responses from control inputs, $u(t)$, and from the fault inputs, $f(t)$, to the detector outputs, $r(t)$, for a grid of δ_1 and δ_2 with values: $[-0.25, -0.125, 0, 0.125, 0.25]$. Note that since $r(t)$ is two-dimensional for convenience just $\|r(t)\|$ is plotted. We note that this detector has fast initial response to the fault.

Secondly we illustrate the approximate solution to Problem(2/2). Theorem 4.1 applied to a grid of N frequency points $\omega_c(i) = (2/T_s)\tan((i-1)\pi/(2N))$ for $i = 1, \cdots, N$, with $N = 200$ and $T_s = 0.1$. The optimization over 200 2×2 matrices is straightforward and gives that there is only one active constraint (for $j = 2$) and hence only one critical frequency which is given by $\hat{\omega} = 0$ and the resulting $Z^o(\hat{\omega})$ has rank one, say $Z^o(\hat{\omega}) = vv^*$. This optimal solution can now be approximated by $Q_{2,\epsilon_2}(s) = \sqrt{\frac{\epsilon_2(\omega_c(2)-\omega_c(1))}{\pi}} \frac{1}{s+\epsilon_2} v$ and a value of $\epsilon_2 = 0.3$ gives a reasonable compromise between optimality and speed of response. The results are presented in Figure 2. It is seen that the response is somewhat slower and the steady state gains similar to the Problem($\infty/2$) case.

These two solutions for Q_{1,ϵ_1} and Q_{2,ϵ_2} can also be compared with the ratios of the various gains as follows (recalling that large values are better):

	$j = 1$	$j = 2$
$\|Q_{1,\epsilon_1} N_{f_j}\|_2 / \|Q_{1,\epsilon_1} M_{wo}\|_\infty$	4.7401	2.7312
$\|Q_{2,\epsilon_2} N_{f_j}\|_2 / \|Q_{2,\epsilon_2} M_{wo}\|_\infty$	0.6835	0.5829
$\|Q_{1,\epsilon_1} N_{f_j}\|_2 / \|Q_{1,\epsilon_1} M_{wo}\|_2$	1.4176	0.8168
$\|Q_{2,\epsilon_2} N_{f_j}\|_2 / \|Q_{2,\epsilon_2} M_{wo}\|_2$	1.7863	1.5235
$\|Q_{1,\epsilon_1} N_{f_j}\|_\infty / \|Q_{1,\epsilon_1} M_{wo}\|_\infty$	2.3621	1.7706
$\|Q_{2,\epsilon_2} N_{f_j}\|_\infty / \|Q_{2,\epsilon_2} M_{wo}\|_\infty$	1.9173	1.7766

It is noted that the gain from f_2 is always smaller than that from f_1 and hence gives the active constraint. As would be expected Q_{1,ϵ_1} is significantly better for problem P($\infty/2$) and the Q_{2,ϵ_2} is significantly better for problem P($2/2$). In this simple case the gain at $\omega = 0$ is critical and the solution to Problem P($2/2$) is also optimal for Problem P(∞/∞) since in this case it can be shown that a rank one solution can be chosen. The McMillan degree of the $Q_{1,\epsilon_1} R_1$ is 3 and that for $Q_{2,\epsilon_2} R_1$ is 4.

As can be observed, with an appropriate choice of the detection threshold, the detection of constant faults can be reliably performed in the presence of parametric uncertainties in either case. In addition similar results can be obtained using the results from section IV-D and indeed optimization over $R(s)$ directly. Modifications to be particularly sensitive to certain types of fault, or to include speed of response, which are not well-described in the unweighted $\mathcal{H}_{\infty/2}$-norms, can also be addressed.

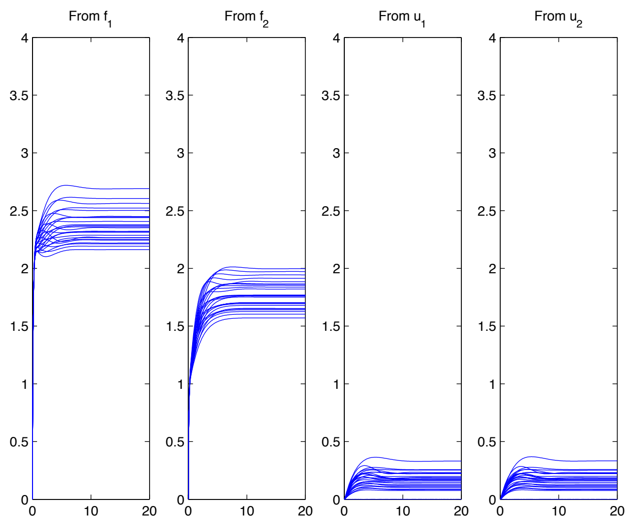

Fig. 1. Step response analysis of detector robustness for uncertainties in the real and imaginary parts of the complex conjugated eigenvalues, ($\|r(t)\|$ is plotted) - Problems P($\infty/2$) and P(∞/∞) approximate solution.

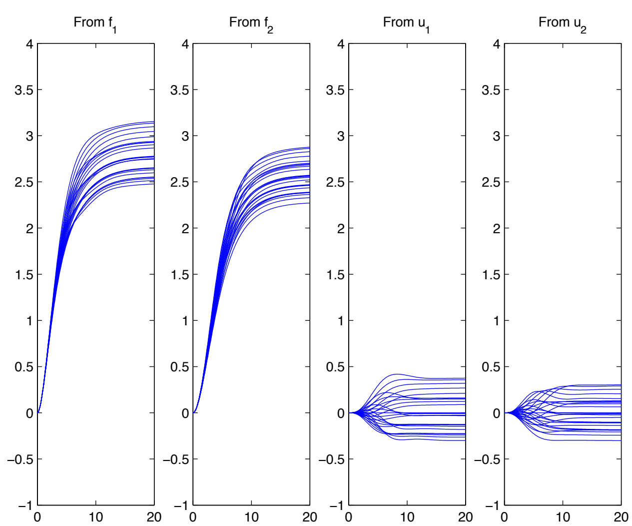

Fig. 2. Step response analysis of detector robustness for uncertainties in the real and imaginary parts of the complex conjugated eigenvalues - problem P(2/2) approximate solution.

VI. Conclusions

In this paper, the solution of approximate fault detection problems has been addressed via an optimization-based approach, where all technical assumptions, often made in the literature, have been eliminated. The aim of the optimization-based setup is to maximise the detection gap defined as the ratio of the least gain from each single fault input to the residual to the gain from the noise input to the residual. All gain combinations defined in terms of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms have been analysed in non-standard cases when zeros on the stability boundary are present. In addition, when there is a fixed form for the detector that is affine in some parameters, similar properties have been demonstrated and suitable computational techniques have been proposed. In our presentation we considered only norm definitions involving the full range of frequency values (both in continuous- and discrete-time settings). Since the maximum gap might occur at a frequency where the fault has little energy, the appropriateness of the choice of a particular norm combination must be assessed for each concrete fault detection problem. To eliminate non-sense results, occasionally norm definitions involving only bounded or even finite frequency domains seem to be more appropriate. For this case, adequate algorithmic solutions need still to be developed.

References

[1] J. Gertler. *Fault Detection and Diagnosis in Engineering Systems*. Marcel Dekker, New York, 1998.

[2] J. Chen and R. J. Patton. *Robust Model-Based Fault Diagnosis for Dynamic Systems*. Kluwer Academic Publishers, London, 1999.

[3] S. X. Ding, P. M. Frank, E. L. Ding, and T. Jeinsch. A unified approach to the optimization of fault detection systems. *Int. J. of Adaptive Contr. and Signal Processing*, 14:725–745, 2000.

[4] I. M. Jaimoukha, Z. Li, and V. Papakos. A matrix factorization solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem. *Automatica*, 42:1907–1912, 2006.

[5] N. Liu and K. Zhou. Optimal solutions to multiobjective robust fault detection problems. In *Proc. of CDC'07, New Orleans, USA*, 2007.

[6] N. Liu and K. Zhou. Optimal robust fault detection for linear discrete time systems. *Journal of Control Science and Engineering*, vol. 2008, Article ID 829459, 16 pages, doi:10.1155/2008/829459, 2008.

[7] P. Zhang and S. X. Ding. An integrated trade-off design of observer based fault detection systems. *Automatica*, 44:1886–1894, 2008.

[8] A. Varga. General computational approach for optimal fault detection. In Proc. SAFEPROCESS'09, Barcelona, Spain, 2009.

[9] A. Varga. On computing least order fault detectors using rational nullspace bases. In *Proc. of IFAC Symp. SAFEPROCESS'2003, Washington D.C.*, 2003.

[10] A. Varga. On computing nullspace bases – a fault detection perspective. In *Proc. IFAC 2008 World Congress, Seoul, Korea.*, pages 6295–6300, 2008.

[11] C. Oară and A. Varga. Computation of general inner-outer and spectral factorizations. *IEEE Trans. Automat. Control*, 45:2307–2325, 2000.

[12] C. Oară. Constructive solutions to spectral and inner-outer factorizations with respect to the disk. *Automatica*, 41:1855 – 1866, 2005.

[13] K. Zhou, J. Doyle and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1995.

[14] P. M. Frank. Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy - a survey and some new results. *Automatica*, 26: 459-474,1990.

[15] W. Hoeffding. The extrema of the expected value of a function of independent random variables. *The annals of Mathematical Statistics*, 26: 268-275,1955.

[16] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.