Integrated Computational Algorithm for Solving $\mathcal{H}_\infty$-Optimal FDI Problems

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Abstract: The synthesis problem of fault detection and isolation (FDI) filters can be formulated as a model matching problem and solved using an $\mathcal{H}_\infty$-norm optimization approach. A systematic procedure is proposed to choose appropriate filter specifications which guarantee the existence of proper and stable solutions of the model matching problem. This selection is integral part of a numerically reliable computational method to design $\mathcal{H}_\infty$-optimal FDI filters. The proposed design approach is completely general, being applicable to both continuous- and discrete-time systems, and can easily handle even unstable and/or improper systems.

Keywords: Fault detection, fault isolation, numerical algorithms, linear systems.

1. THE APPROXIMATE FDI PROBLEM

Consider additive fault models described by input-output representations of the form

$$\begin{aligned}\mathbf{y}(\lambda) = \; & G_u(\lambda)\mathbf{u}(\lambda) + G_d(\lambda)\mathbf{d}(\lambda) + G_w(\lambda)\mathbf{w}(\lambda) \\ & + G_f(\lambda)\mathbf{f}(\lambda)\end{aligned} \tag{1}$$

where $\mathbf{y}(\lambda)$, $\mathbf{u}(\lambda)$, $\mathbf{d}(\lambda)$, $\mathbf{w}(\lambda)$, and $\mathbf{f}(\lambda)$ are Laplace- or Z-transformed vectors of the p-dimensional system output vector $y(t)$, m_u-dimensional control input vector $u(t)$, m_d-dimensional disturbance vector $d(t)$, m_w-dimensional noise vector $w(t)$ and m_f-dimensional fault vector $f(t)$, respectively, and where $G_u(\lambda)$, $G_d(\lambda)$, $G_w(\lambda)$ and $G_f(\lambda)$ are the corresponding *transfer-function matrices* (TFMs). According to the system type, λ is either s, the complex variable in the Laplace-transform in the case of a continuous-time system or z, the complex variable in the Z-transform in the case of a discrete-time system. For complete generality of our problem setting, we will allow that these TFMs are general non-proper rational matrices for which we will not *a priori* assume any further properties.

A linear residual generator (or fault detection filter) processes the measurable system outputs $y(t)$ and control inputs $u(t)$ and generates the residual signals $r(t)$ which serve for decision making on the presence or absence of faults. The input-output form of this filter is

$$\mathbf{r}(\lambda) = Q(\lambda)\begin{bmatrix}\mathbf{y}(\lambda)\\ \mathbf{u}(\lambda)\end{bmatrix} \tag{2}$$

where $Q(\lambda)$ is the TFM of the filter. For a physically realizable filter, $Q(\lambda)$ must be *proper* (i.e., only with finite poles) and *stable* (i.e., only with poles having negative real parts for a continuous-time system or magnitudes less than one for a discrete-time system). The (dynamic) *order* of $Q(\lambda)$ (also known as *McMillan degree*) is the dimension of the state vector of a minimal state-space realization of $Q(\lambda)$. The dimension q of the residual vector $r(t)$ depends on the fault detection problem to be solved.

The residual signal $r(t)$ in (2) generally depends via the system outputs $y(t)$ of all system inputs $u(t)$, $d(t)$, $w(t)$ and $f(t)$. The residual generation system, obtained by replacing in (2) $\mathbf{y}(\lambda)$ by its expression in (1), is given by

$$\begin{aligned}r(\lambda) = \; & R_u(\lambda)\mathbf{u}(\lambda) + R_d(\lambda)\mathbf{d}(\lambda) + R_w(\lambda)\mathbf{w}(\lambda) \\ & + R_f(\lambda)\mathbf{f}(\lambda)\end{aligned} \tag{3}$$

where

$$\begin{aligned}&[\, R_u(\lambda) | R_d(\lambda) | R_w(\lambda) | R_f(\lambda)\,] := \\ &\qquad Q(\lambda)\left[\begin{array}{c|c|c|c} G_u(\lambda) & G_d(\lambda) & G_w(\lambda) & G_f(\lambda)\\ I_{m_u} & 0 & 0 & 0\end{array}\right]\end{aligned}$$

For a successfully designed filter $Q(\lambda)$, the corresponding residual generation system (3) is proper and stable and achieves specific fault detection requirements.

For the solution of fault detection problems it is always possible to completely decouple the control input $u(t)$ from the residuals $r(t)$ by requiring $R_u(\lambda) = 0$. Regarding the disturbance input $d(t)$ and noise input $w(t)$ we aim to impose a similar condition on the disturbances input $d(t)$ by requiring $R_d(\lambda) = 0$, while minimizing simultaneously the effect of noise input $w(t)$ on the residual (e.g., by minimizing the norm of $R_w(\lambda)$). Thus, from a practical synthesis point of view, the distinction between $d(t)$ and $w(t)$ lies solely in the way these signals are treated when solving the residual generator synthesis problem.

Let $M_r(\lambda)$ be a suitably chosen reference model (i.e., stable, proper, diagonal and invertible) representing the desired TFM from the faults to residuals. We want to achieve that $\mathbf{r}(\lambda) \approx M_r(\lambda)\mathbf{f}(\lambda)$, that is, each residual $r_i(t)$ is influenced mainly by fault $f_i(t)$. Our formulation of the *approximate fault detection and isolation problem* (AFDIP) extends the formulation of the model-matching approach of Chen and Patton (1999); Blanke et al. (2003) by requiring to determine a stable and proper filter $Q(\lambda)$ such that the following conditions are fulfilled:

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$$
\begin{array}{ll}
(i) & R_u(\lambda) = 0, \\
(ii) & R_d(\lambda) = 0, \\
(iii) & R_f(\lambda) \approx M_r(\lambda), \;\text{ with }\; R_f(\lambda) \;\text{ stable;} \\
(iv) & R_w(\lambda) \approx 0, \;\text{ with }\; R_w(\lambda) \;\text{ stable.}
\end{array} \quad (4)
$$

The *exact fault detection and isolation problem* (EFDIP) requiring $R_f(\lambda) = M_r(\lambda)$ can be easily included in this formulation and corresponds to $m_w = 0$, while the formulation of the AFDIP in (Chen and Patton, 1999; Blanke et al., 2003) corresponds to $m_d = 0$.

It is straightforward to show that for the solution of the AFDIP, the solvability conditions are those for the solvability of the EFDIP stated in (Frank and Ding, 1994). For a proof see (Varga, 2010).

Theorem 1. For the system (1) there exists a stable, diagonal, proper, and invertible $M_r(\lambda)$ such that the AFIDP is solvable if and only if

$$
rank\,[\,G_f(\lambda)\; G_d(\lambda)\,] = m_f + rank\, G_d(\lambda) \quad (5)
$$

The solution of the AFDIP can be addressed by solving the approximate model matching problem $\mathbf{r}(\lambda) \approx M_r(\lambda)\mathbf{f}(\lambda)$ by minimizing the $\mathcal{H}_\infty$-norm of the residual error

$$
\mathcal{R}(\lambda) := F(\lambda) - Q(\lambda)G(\lambda), \quad (6)
$$

where $F(\lambda) = [\,M_r(\lambda)\; 0\; 0\; 0\,]$ and

$$
G(\lambda) = \begin{bmatrix} G_f(\lambda) & G_w(\lambda) & G_d(\lambda) & G_u(\lambda) \\ 0 & 0 & 0 & I_{m_u} \end{bmatrix} \quad (7)
$$

The computation of the $\mathcal{H}_\infty$-optimal solution involves determining a *stable* and *proper* $Q(\lambda)$ such that $\mathcal{R}(\lambda)$ is stable and proper and $\|\mathcal{R}(\lambda)\|_\infty$ is minimized.

This $\mathcal{H}_\infty$ model matching problem can be easily reformulated as a standard $\mathcal{H}_\infty$-norm minimization based "controller" synthesis problem (Zhou et al., 1996; Blanke et al., 2003) by defining a suitable 2×2 blocks partitioned generalized plant $P(\lambda)$. To solve the optimal $\mathcal{H}_\infty$-synthesis problem, standard software tools exist, as for example, the function `hinfsyn` available in the MATLAB Robust Control Toolbox. The main problem when employing standard tools (like `hinfsyn`), is that, although a stable and proper solution $Q(\lambda)$ of the AFDIP may exist, this solution can not be computed because of the presence of technical assumptions. For example, this function is not applicable in the case of unstable systems (including improper systems as well), because the stabilizability condition is not fulfilled by any state-space realization of $P(\lambda)$ due to the particular form of the $(1,2)$ block ($P_{12}(\lambda) = I$). Also, the presence of zeros of the $(2,1)$ block $P_{21}(\lambda)$ on the boundary of stability domain (i.e., the extended imaginary axis for continuous-time systems or the unit circle in the origin for discrete-time systems) prevents the applicability of standard tools.

To face the above limitations, it is necessary to develop synthesis procedures for which no such limitations exist. The key parameter to guarantee the stability and properness of the detector is $M_r(\lambda)$, the desired TFM relating the faults to the residuals. The choice of $M_r(\lambda)$ is not obvious and often $M_r(\lambda)$ results from an exact nominal synthesis. However, in (Varga, 2005) a procedure has been proposed, where the choice of suitable $M_r(\lambda)$ is part of the solution. In this paper, we refine this procedure, by proposing an integrated computational approach to the detector synthesis. An important feature of the proposed approach is that it relies on successive updating of an initial fault detector. The underlying state space computations employ explicit least order realizations of the detector, thus a least final order of the detector is guaranteed. Moreover, all structural features of the intermediary results can be exploited in the next computational steps, which overall leads to highly efficient structure exploiting computations.

2. ENHANCED MODEL-MATCHING APPROACH

In this section we propose an enhanced version of the algorithm of Varga (2005), where we exploit the additional structure in the model (1) owing to the separation of the unknown inputs in two components $d(t)$ and $w(t)$. Moreover, by using a new parametrization of the detector, we derive an integrated computational approach based on detector updating techniques. We describe in what follows the main stages of the overall computational procedure.

First stage: Consider $Q(\lambda)$ in a factored form

$$
Q(\lambda) = Q_1(\lambda)N_l(\lambda), \quad (8)
$$

where $N_l(\lambda)$ is a left proper rational nullspace basis satisfying

$$
N_l(\lambda) \begin{bmatrix} G_d(\lambda) & G_u(\lambda) \\ 0 & I_{m_u} \end{bmatrix} = 0 \quad (9)
$$

and $Q_1(\lambda)$ is a factor to be further determined. With this choice, it follows that $Q(\lambda)$ automatically fulfills the first two conditions in (4). The existence of $N_l(\lambda)$ is guaranteed provided condition (5) is fulfilled. The resulting $N_l(\lambda)$ has maximal row rank $p - r_d$, where $r_d = rank\, G_d(\lambda)$. Moreover, we can choose $N_l(\lambda)$ stable and such that both $N_f(\lambda)$ and $N_w(\lambda)$ defined as

$$
[\,N_f(\lambda)\; N_w(\lambda)\,] := N_l(\lambda) \begin{bmatrix} G_f(\lambda) & G_w(\lambda) \\ 0 & 0 \end{bmatrix} \quad (10)
$$

are proper and stable TFMs (Varga, 2008). However, as it will be apparent, enforcing the stability condition is not necessary at this stage. We can easily check now the solvability of the AFDIP by verifying that

$$
rank\, N_f(\lambda) = m_f \quad (11)
$$

To fulfill the last two conditions in (4) we can solve a $\mathcal{H}_\infty$-norm minimization problem for $\|\widetilde{\mathcal{R}}(\lambda)\|_\infty$ to determine $Q_1(\lambda)$, where

$$
\widetilde{\mathcal{R}}(\lambda) = M(\lambda)\widetilde{F}(\lambda) - Q_1(\lambda)\widetilde{G}(\lambda), \quad (12)
$$

with $\widetilde{G}(\lambda) = [\,N_f(\lambda)\;\; N_w(\lambda)\,]$ and $\widetilde{F}(\lambda) = [\,M_r(\lambda)\;\; O\,]$. Here, $M_r(\lambda)$ is the TFM of a given reference model (i.e., stable, proper, diagonal, invertible), while $M(\lambda)$ is a free updating factor with the same properties (to be determined). Thus, the solution of the AFDIP using the $\mathcal{H}_\infty$ model matching approach involves choosing an appropriate $M(\lambda)$ such that the resulting $Q(\lambda)$ in (8) is stable and proper, the error residual $\widetilde{\mathcal{R}}(\lambda)$ is finite, and $\|\widetilde{\mathcal{R}}(\lambda)\|_\infty$ is minimized by the choice of $Q_1(\lambda)$.

Let ℓ be the rank of the $(p - r_d) \times (m_f + m_w)$ TFM $\widetilde{G}(\lambda)$. If $\ell < p - r_d$ (i.e., $\widetilde{G}(\lambda)$ has no full row rank), we can take instead $N_l(\lambda)$, ℓ linear combinations of basis vectors of the form $W(\lambda)N_l(\lambda)$, which ensures that $W(\lambda)\widetilde{G}(\lambda)$ has full row rank ℓ. A suitable choice of the $\ell\times(p-r_d)$ TFM $W(\lambda)$ which also minimizes the McMillan degree of $W(\lambda)N_l(\lambda)$ is described in (Varga, 2008).

Second stage: We compute a quasi-co-outer–inner factorization

$$\widetilde{G}(\lambda) = [\, G_{o,1}(\lambda)\ 0\,] \begin{bmatrix} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{bmatrix} := G_o(\lambda)G_i(\lambda), \qquad (13)$$

where $G_i(\lambda)$ is a $(m_f+m_w)\times(m_f+m_w)$ inner TFM and $G_{o,1}(\lambda)$ is an $\ell\times\ell$ invertible TFM. Recall that a square TFM $G_i(\lambda)$ is *inner* (and simultaneously *co-inner*) if it has only stable poles and satisfies $G_i(\lambda)G_i^*(\lambda)=I$, where $G_i^*(s):=G_i^T(-s)$ in a continuous-time setting and $G_i^*(z):=G_i^T(1/z)$ in a discrete-time setting. The *quasi-co-outer* factor $G_o(\lambda)$ may contain besides stable zeros, also zeros which lie on the boundary of the stability domain.

We can update the parametrization (8) of the detector by choosing $Q_1(\lambda)$ of the form

$$Q_1(\lambda) = Q_2(\lambda)G_{o,1}^{-1}(\lambda) \qquad (14)$$

where $Q_2(\lambda)$ is to be determined. Using (13) and (14), we can express $\widetilde{\mathcal{R}}(\lambda)$ in (12) as $\widetilde{\mathcal{R}}(\lambda)=\overline{\mathcal{R}}(\lambda)G_i(\lambda)$, with

$$\overline{\mathcal{R}}(\lambda) = \left[\, M(\lambda)\overline{F}_1(\lambda) - Q_2(\lambda) \,\middle|\, M(\lambda)\overline{F}_2(\lambda)\,\right], \qquad (15)$$

where $\overline{F}_1(\lambda):=\widetilde{F}(\lambda)G_{i,1}^*(\lambda)$ and $\overline{F}_2(\lambda):=\widetilde{F}(\lambda)G_{i,2}^*(\lambda)$. Since $G_i(\lambda)$ is inner, we have $\|\widetilde{\mathcal{R}}(\lambda)\|_\infty=\|\overline{\mathcal{R}}(\lambda)\|_\infty$.

Third stage: For $M(\lambda)=I_{m_f}$, we determine an appropriate $Q_2(\lambda)$ as the solution of the suboptimal two-blocks minimum distance problem

$$\|[\,\overline{F}_1(\lambda) - Q_2(\lambda)\ \ \overline{F}_2(\lambda)\,]\|_\infty < \gamma, \qquad (16)$$

where $\gamma_{opt}<\gamma\leq\gamma_{opt}+\varepsilon$, with ε an arbitrary user specified (accuracy) tolerance for the least achievable value γ_{opt} of γ. With the following lower and upper bounds for γ_{opt}

$$\gamma_l = \|\overline{F}_2(\lambda)\|_\infty, \quad \gamma_u = \left\|\left[\,\overline{F}_1(\lambda) \,\middle|\, \overline{F}_2(\lambda)\,\right]\right\|_\infty \qquad (17)$$

such a γ-suboptimal solution $Q_2(\lambda)$ can be computed using the bisection-based γ-iteration approach (Francis, 1987).

For a given $\gamma>\gamma_l$, we compute first the spectral factorization (Zhou et al., 1996)

$$\gamma^2 I - \overline{F}_2(\lambda)\overline{F}_2^*(\lambda) = V(\lambda)V^*(\lambda), \qquad (18)$$

where $V(\lambda)$ is biproper, stable and minimum-phase. Further, we compute the additive decomposition

$$L_s(\lambda) + L_u(\lambda) = V^{-1}(\lambda)\overline{F}_1(\lambda), \qquad (19)$$

where $L_s(\lambda)$ is the stable part and $L_u(\lambda)$ is the unstable part. If $\gamma>\gamma_{opt}$, the two-blocks problem (16) is equivalent to the one-block problem (Francis, 1987)

$$\left\|V^{-1}(\lambda)\left(\overline{F}_1(\lambda) - Q_2(\lambda)\right)\right\|_\infty \leq 1 \qquad (20)$$

and $\gamma_H:=\|L_u^*(\lambda)\|_H<1$ ($\|\cdot\|_H$ denotes the Hankel norm of a stable TFM). In this case we readjust $\gamma_u=\gamma$. If $\gamma_H\geq 1$, we readjust $\gamma_l=\gamma$. Then, for $\gamma=(\gamma_l+\gamma_u)/2$ we redo the factorization (18) and decomposition (19). This process is repeated until $\gamma_u-\gamma_l\leq\varepsilon$.

If $\gamma_u\geq\gamma>\gamma_{opt}$, the stable solution of (20) is

$$Q_2(\lambda) = V(\lambda)(L_s(\lambda) + Q_{2,s}(\lambda)), \qquad (21)$$

where, for any γ_1 satisfying $1\geq\gamma_1>\gamma_H$, $Q_{2,s}(\lambda)$ is the stable solution of the γ_1-suboptimal Nehari problem

$$\|L_u(\lambda) - Q_{2,s}(\lambda)\|_\infty < \gamma_1. \qquad (22)$$

Fourth stage: For $M(\lambda)=I_{m_f}$ and given a solution $Q_2(\lambda)$ of the minimum distance problem (16), the resulting $\mathcal{H}_\infty$ optimal detector is

$$\widehat{Q}(\lambda) = Q_2(\lambda)G_{o,1}^{-1}(\lambda)N_l(\lambda) \qquad (23)$$

Since $G_{o,1}(\lambda)$ is only a quasi co-outer factor, it can still have unstable zeros on the boundary of the stability domain. Thus, these zeros may appears as poles of $\widehat{Q}(\lambda)$. To ensure that the final detector is proper and stable, the resulting $Q(\lambda)$ can be chosen as $Q(\lambda)=M(\lambda)\widehat{Q}(\lambda)$, where $M(\lambda)$ is chosen such that $Q(\lambda)$ is proper and stable, and the norm condition (16) is still fulfilled when replacing $Q_2(\lambda)$ by $M(\lambda)Q_2(\lambda)$, $\overline{F}_1(\lambda)$ by $M(\lambda)\overline{F}_1(\lambda)$, and $\overline{F}_2(\lambda)$ by $M(\lambda)\overline{F}_2(\lambda)$. For example, to ensure properness, $M(\lambda)$ is chosen diagonal with the diagonal terms $m_j(\lambda)$, $j=1,\ldots,m_f$ having the form

$$m_j(\lambda) = \frac{1}{(\tau s+1)^{k_j}} \quad \text{or} \quad m_j(z) = \frac{1}{z^{k_j}}$$

for continuous- or discrete-time settings, respectively. Both above factors have unit $\mathcal{H}_\infty$-norm.

The high-level computations in terms of TFMs in the enhanced $\mathcal{H}_\infty$ synthesis procedure can be performed via state-space models based reliable numerical computations, which are described in the next section.

3. COMPUTATIONAL ISSUES

For computations we employ an equivalent *descriptor* state space realization of the input-output model (1),

$$\begin{aligned} E\lambda x(t) &= Ax(t)+B_u u(t)+B_d d(t)+B_w w(t)+B_f f(t) \\ y(t) &= Cx(t)+D_u u(t)+D_d d(t)+D_w w(t)+D_f f(t) \end{aligned} \qquad (24)$$

with the n-dimensional state vector $x(t)$, where $\lambda x(t)=\dot{x}(t)$ or $\lambda x(t)=x(t+1)$ depending on the type of the system, continuous or discrete, respectively. In general, the square matrix E can be singular, but we will assume that the linear pencil $A-\lambda E$ is regular. For systems with proper TFMs in (1), we can always choose a *standard* state space realization with $E=I$. In general, we can assume that the representation (24) is minimal, that is, the descriptor pair $(A-\lambda E,[\,B_u\ B_d\ B_w\ B_f\,])$ is *controllable* and $(A-\lambda E, C)$ is *observable*.

The corresponding TFMs of the model in (1) are

$$\begin{aligned} G_u(\lambda) &= C(\lambda E-A)^{-1}B_u + D_u \\ G_d(\lambda) &= C(\lambda E-A)^{-1}B_d + D_d \\ G_w(\lambda) &= C(\lambda E-A)^{-1}B_w + D_w \\ G_f(\lambda) &= C(\lambda E-A)^{-1}B_f + D_f \end{aligned} \qquad (25)$$

or in an equivalent notation

$$[\,G_u(\lambda)\ G_d(\lambda)\ G_w(\lambda)\ G_f(\lambda)\,] := \left[\begin{array}{c|cccc} A-\lambda E & B_u & B_d & B_w & B_f \\ \hline C & D_u & D_d & D_w & D_f \end{array}\right]$$

In what follows we discuss the solution of computational problems at each stage.

Stage 1 and Stage 2: These stages are described in details in Varga (2010), and are based on reliable methods to compute minimal proper rational nullspace bases (Varga, 2008) and the quasi-co-outer–inner factorization (Oară and Varga, 2000; Oară, 2005). At the end of *Stage 2* we obtain the state-space realization of the $\ell \times \ell$ proper quasi-co-outer factor $G_{o,1}(\lambda)$ in the form

$$G_{o,1}(\lambda) = \left[\begin{array}{c|c} \widetilde{A} - \lambda\widetilde{E} & \overline{B}_o \\ \hline \widetilde{C} & \overline{D}_o \end{array}\right] \tag{26}$$

with $\widetilde{E}$ invertible. The system with the TFM $G_{o,1}(\lambda)$ may have besides the stable zeros, also zeros on the imaginary axis (including infinity) in the continuous-time case or on the unit circle in the discrete-time case. The $(m_f + m_w) \times (m_f + m_w)$ TFM of the inner factor $G_i(\lambda)$ is proper and stable and assume that its inverse (i.e., its conjugated TFM) has a state space realization of the form

$$G_i^*(\lambda) = [\, G_{i,1}^* \; G_{i,2}^* \,] = \left[\begin{array}{c|cc} A_i - \lambda E_i & B_{i,1} & B_{i,2} \\ \hline C_i & D_{i,1} & D_{i,2} \end{array}\right]$$

where all generalized eigenvalues of the pair (A_i, E_i) are unstable. For a continuous-time system $E_i = I$, but this can not be assumed, in general, for a discrete-time system unless $G_i(\lambda)$ does not have poles in the origin. If $G_i(\lambda)$ has poles in the origin, the resulting E_i is singular and thus the pair (A_i, E_i) has also infinite (unstable) generalized eigenvalues.

Let the quadruple (A_r, B_r, C_r, D_r) describe the state space realization of $[\, M_r(\lambda)\; 0 \,]$. Then, the state space realization of $[\, \overline{F}_1(\lambda)\; \overline{F}_2(\lambda) \,]$ has the form

$$[\, \overline{F}_1(\lambda)\; \overline{F}_2(\lambda) \,] = \left[\begin{array}{cc|cc} A_r - \lambda I & B_r C_i & B_r D_{i,1} & B_r D_{i,2} \\ 0 & A_i - \lambda E_i & B_{i,1} & B_{i,2} \\ \hline C_r & D_r C_i & D_r D_{i,1} & D_r D_{i,2} \end{array}\right],$$

where A_r has only stable eigenvalues, while the pair (A_i, E_i) has only unstable generalized eigenvalues. From (8) and (14), the detection filter $Q(\lambda)$ has the product form

$$Q(\lambda) = Q_2(\lambda) G_{o,1}^{-1}(\lambda) N_l(\lambda) := Q_2(\lambda)\overline{Q}(\lambda),$$

where $Q_2(\lambda)$ has to be determined to minimize $\|\overline{\mathcal{R}}(\lambda)\|_\infty$ in (15) and $\overline{Q}(\lambda)$ is a partial detector with an explicit descriptor realization of the form

$$\overline{Q}(\lambda) = \left[\begin{array}{cc|c} \widetilde{A} - \lambda\widetilde{E} & \overline{B}_o & \widetilde{B}_{yu} \\ \widetilde{C} & \overline{D}_o & \widetilde{D}_{yu} \\ \hline 0 & -I_\ell & 0 \end{array}\right]$$

Stage 3: At this stage we need to perform the γ-iteration to solve the suboptimal two-blocks minimum distance problem (16). To start, we have to compute the $\mathcal{L}_\infty$-norms in (17) to obtain γ_l and γ_u. For this purpose, efficient algorithms can be employed based on extensions of the method of Bruinsma and Steinbuch (1990) (for which standard numerical tools are available in MATLAB). Note that for computing γ_u we can exploit that $\gamma_u = \|M_r(\lambda)\|_\infty$ as a consequence of the all-pass property of $G_i^*(\lambda)$.

The main computation at this stage is the computation of the spectral factorization (18), which involves two steps. Firstly, we compute a right coprime factorization of $\overline{F}_2(\lambda)$ with inner denominator such that $\overline{F}_2(\lambda) = \overline{N}_2(\lambda)\overline{M}_2^{-1}(\lambda)$, where $\overline{M}_2(\lambda)$ is inner. It follows that $\overline{F}_2(\lambda)\overline{F}_2^*(\lambda) = \overline{N}_2(\lambda)\overline{N}_2^*(\lambda)$. This computation needs to be performed only once and suitable algorithms for this purpose have been proposed in (Varga, 1998) for standard systems or in (Oară and Varga, 1998) for discrete-time descriptor systems. In both cases, the resulting $\overline{N}_2(\lambda)$ has a standard system realization of the form

$$\overline{N}_2(\lambda) = \left[\begin{array}{cc|c} A_r - \lambda I & B_r \overline{C}_i & B_r \overline{D}_{i,2} \\ 0 & \overline{A}_i - \lambda I & \overline{B}_{i,2} \\ \hline C_r & D_r \overline{C}_i & D_r \overline{D}_{i,2} \end{array}\right]$$

Secondly, we solve the spectral factorization problem

$$\gamma^2 I - \overline{N}_2(\lambda)\overline{N}_2^*(\lambda) = V(\lambda)V^*(\lambda)$$

using methods described in (Zhou et al., 1996) to obtain a realization of $V^{-1}(\lambda)\overline{F}_1(\lambda)$ of the form

$$V^{-1}(\lambda)\overline{F}_1(\lambda) = \left[\begin{array}{cc|c} \overline{A}_{11} - \lambda I & \overline{A}_{12} & \overline{B}_1 \\ 0 & A_i - \lambda E_i & \overline{B}_2 \\ \hline \overline{C}_1 & \overline{C}_2 & \overline{D} \end{array}\right],$$

where $\overline{A}_{11}$ has only stable eigenvalues, while the pair (A_i, E_i) has only unstable eigenvalues. This computation involves the solution of an algebraic Riccati equation at each iteration. For details see (Zhou et al., 1996).

To compute the spectral separation (19) we perform a similarity transformation to obtain the transformed pole pencil

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{A}_{11} - \lambda I & \overline{A}_{12} \\ 0 & A_i - \lambda E_i \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$$

in a block diagonal form by annihilating its (1,2) block. This comes down to solve the Sylvester system of matrix equations

$$\begin{aligned} 0 &= XA_i + \overline{A}_{11}Y + \overline{A}_{12} \\ 0 &= XE_i + Y \end{aligned}$$

After applying the transformations to the input and output matrices we obtain

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} = \begin{bmatrix} \overline{B}_1 + X\overline{B}_2 \\ \overline{B}_2 \end{bmatrix},$$

$$[\, \overline{C}_1\; \overline{C}_2 \,] \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = [\, \overline{C}_1\; \overline{C}_1 Y + \overline{C}_2 \,]$$

The stable and unstable terms are given by

$$L_s(\lambda) = \left[\begin{array}{c|c} \overline{A}_{11} - \lambda I & \overline{B}_1 + X\overline{B}_2 \\ \hline \overline{C}_1 & \overline{D} \end{array}\right], \; L_u(\lambda) = \left[\begin{array}{c|c} A_i - \lambda E_i & \overline{B}_2 \\ \hline \overline{C}_1 Y + \overline{C}_2 & 0 \end{array}\right]$$

The computation of the Hankel-norm $\gamma_H = \|L_u^*(\lambda)\|_H$ can be performed using standard algorithms for proper and stable systems.

The computation of the Nehari approximation can be done using the algorithm of Glover (1984) for continuous-time systems. For discrete-time systems, the same algorithm is applicable after performing a bilinear transformation to map the exterior of the unit circle to the open right half plane. A suitable transformation and its inverse transformation are $z = \frac{1+s}{1-s}$ and $s = \frac{z-1}{z+1}$, respectively. Note that in the case of an improper $L_u(z)$, all infinite poles go to $s = 1$

and therefore the "equivalent" continuous-time system will be proper.

In the light of the cancelation theory for continuous-time two-blocks problems of Limebeer and Halikias (1988), pole-zero cancelations occur when forming $Q_2(s)$ in (21). In accordance with this theory, the expected order of $Q_2(s)$ is $n_r + n_i - 1$, where n_r and n_i are the McMillan degrees of $M_r(s)$ and $G_i(s)$, respectively. It is conjectured that similar cancelations will occur also for discrete-time systems, where a cancelation theory for two-blocks problems is still missing. Although we were not able to derive an explicit minimal state-space realization of $Q_2(\lambda)$, we can safely employ minimal realization procedures which exploits that the resulting $Q_2(\lambda)$ is stable. Balancing related methods are especially well suited for this computation, as for example, the square-root balancing-free method (Varga, 1992, Algorithm MR6).

Stage 4: The resulting detector $\widehat{Q}(\lambda)$ in (23) may be improper and/or unstable, and therefore we need to determine diagonal and stable $M(\lambda)$ having the least McMillan degree such that $M(\lambda)Q_2(\lambda)\overline{Q}(\lambda)$ is proper and stable. For this purpose we can solve proper and stable factorizations problems for each row of $Q_2(\lambda)\overline{Q}(\lambda)$, for which we can build descriptor state space realizations using the results from the previous step. Suitable state-space algorithms for this purpose are described in (Varga, 1998).

4. ILLUSTRATIVE EXAMPLE

We consider the robust actuator fault detection and isolation example of (Varga, 2010). The fault system (1) has a standard state space realization (24) with $E = I$ and

$$A(\delta_1,\delta_2) = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.5(1+\delta_1) & 0.6(1+\delta_2) \\ 0 & -0.6(1+\delta_2) & -0.5(1+\delta_1) \end{bmatrix}$$

$$B_u = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_d = 0,\ \ B_f = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D_u = 0,\ D_d = 0,\ D_f = 0.$$

In the expression of $A(\delta_1,\delta_2)$, δ_1 and δ_2 are uncertainties in the real and imaginary parts of the two complex conjugated eigenvalues $\lambda_{1,2} = -0.5 \pm j0.6$ of the nominal value $A(0,0)$. The FDI filter is aimed to provide robust fault detection and isolation of actuator faults in the presence of these parametric uncertainties.

We reformulate the problem by assimilating δ_1 and δ_2 with fictitious noise inputs. We take A in (24) simply as the nominal value $A(0,0)$ and additionally define

$$B_w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D_w = 0.$$

At *Stage 1* we choose as left nullspace basis

$$N_l(s) = [\, I \ \ -G_u(s)\,] = \left[\begin{array}{c|cc} A - sI & 0 & -B_u \\ \hline C & I & -D_u \end{array}\right]$$

and the corresponding $N_f(s)$ and $N_w(s)$ are simply $N_f(s) = G_f(s)$ and $N_w(s) = G_w(s)$.

Note that $[\, N_f(s)\ \ N_w(s)\,]$ has two zeros at infinity. At *Stage 2* we compute the quasi-co-outer-inner factorization of $[\, N_f(s)\ N_w(s)\,]$. The resulting realization of $G_{o,1}(s)$ in (26) has the matrices $\widetilde{A} = A$, $\widetilde{E} = I$, $\widetilde{C} = C$ and

$$\overline{B}_0 = \begin{bmatrix} -1.2405 & -0.2781 \\ -1.2052 & 0.1402 \\ -0.3603 & -1.3850 \end{bmatrix}, \quad \overline{D}_0 = 0$$

As expected, $G_{o,1}(s)$ has also two zeros at infinity and a stable zero at -1.1336. This stable zero is also the only pole of the 4×4 inner factor $G_i(s)$.

The descriptor realization of the resulting $\overline{Q}(s)$ is

$$\overline{Q}(s) = \left[\begin{array}{cc|cc} A - sI & \overline{B}_o & 0 & -B_u \\ C & \overline{D}_o & I & -D_u \\ \hline 0 & -I & 0 & 0 \end{array}\right]$$

and has an improper TFM with two poles at ∞.

With $M_r(s) = I_2$, we compute $\overline{F}_1(s)$ and $\overline{F}_2(s)$ as

$$[\, \overline{F}_1(s)\ \ \overline{F}_2(s)\,] = \left[\begin{array}{c|cc} A_i - sI & B_{i,1} & B_{i,2} \\ \hline C_i & D_{i,1} & D_{i,2} \end{array}\right]$$

where

$$A_i = 1.134, \quad C_i = \begin{bmatrix} 0.0623 \\ 0.7413 \end{bmatrix},$$

$$B_{i,1} = [\ 0.04246 \quad 0.5032\ ], \quad B_{i,2} = [\ -0.7575 \quad -1.523\ ],$$

$$D_{i,1} = \begin{bmatrix} -0.8314 & 0.2423 \\ -0.3914 & -0.3112 \end{bmatrix}, \quad D_{i,2} = \begin{bmatrix} 0.4477 & -0.2226 \\ -0.7625 & -0.4105 \end{bmatrix}$$

Both $\overline{F}_1(s)$ and $\overline{F}_2(s)$ are thus represented by first order systems with an unstable pole at 1.1336.

At *Stage 3*, the γ-iteration starts with $\gamma_l = 0.9239$ and $\gamma_u = 1$ and ends with $\gamma = 0.9239 \quad (= \gamma_l)$ for which the corresponding $\gamma_H = 0.5233$. The optimal Nehari approximation of the unstable part $L_u(s)$ has order zero, and the corresponding norm $\left\| V^{-1}(\lambda)\left(\overline{F}_1(\lambda) - Y(\lambda)\right)\right\|_\infty = \gamma_H$. Full cancelation takes place when forming $Q_2(s)$ in (21), which thus results a constant gain

$$Q_2(s) = \begin{bmatrix} -0.8362 & 0.1854 \\ -0.3935 & -0.3371 \end{bmatrix}$$

This order fully agrees with the degree theory of Limebeer and Halikias (1988).

Finally, at *Stage 4* we choose $M(s) = \frac{10}{s+10} I_2$ to make $Q(s)$ proper and stable. The expression of the detector $Q(s)$ can be written down explicitly as

$$Q(s) = M(s)Q_2(s)G_{o,1}^{-1}(s)[\, I \ \ -G_u(s)\,]$$

which has a standard system realization of order 3. Note that the orders of the realizations of the individual factors are respectively 2, 0, 5, and 3, which sum together to 10. The resulting low order (in fact least possible order) clearly illustrates the advantage of the integrated algorithm, which allows, via explicitly computable realizations, to obtain at each step least order representations of the detector.

For completeness, we give the resulting state-space representation of the detector

$$Q(s) = \left[\begin{array}{c|c} A_Q - sI & B_Q \\ \hline C_Q & D_Q \end{array}\right]$$

with

$$A_Q = \begin{bmatrix} 10.0284 & 0.4410 & 3.2395 \\ -0.0070 & -10.1089 & -0.7999 \\ -0.0790 & -1.2256 & 0.9963 \end{bmatrix}$$

$$B_Q = \begin{bmatrix} 2.6035 & -0.5571 & 0.1846 & 0.3369 \\ -1.3316 & 3.4294 & 0.5806 & 0.1852 \\ 0.1780 & 0.3022 & 0.0473 & -0.3354 \end{bmatrix}$$

$$C_Q = \begin{bmatrix} -11.5159 & -21.6025 & -5.3846 \\ 36.0717 & 13.1418 & 12.5640 \end{bmatrix}$$

$$D_Q = \begin{bmatrix} -6.3099 & 8.2359 & 0 & 0 \\ 11.3898 & -5.8839 & 0 & 0 \end{bmatrix}$$

We evaluated the step responses of the parameter dependent residual generation system (of the form (3)) from the faults and control inputs on a uniform grid for both δ_1 and δ_2 in the range $[-0.25, 0.25]$, with $N = 5 \times 5$ values. The resulting parametric step responses can be seen in Fig. 1. As it can be observed, with an appropriate choice of the detection threshold, the detection and isolation of constant faults can be reliably performed in the presence of parametric uncertainties.

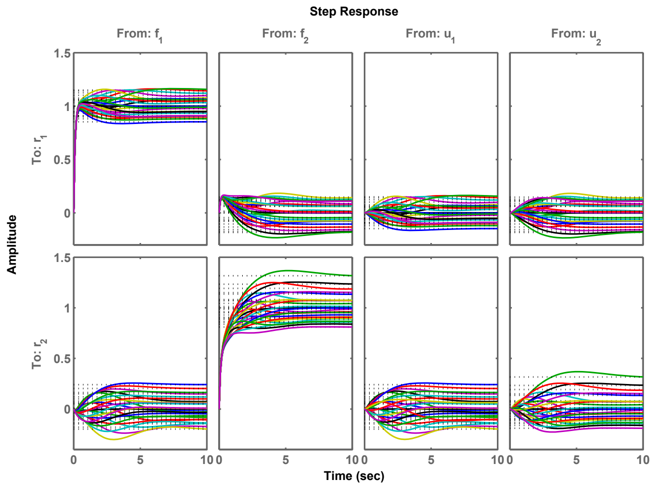

Fig. 1. Parametric step response analysis

5. CONCLUSIONS

We proposed a general computational approach to solve the $\mathcal{H}_\infty$-norm optimal FDI filter design problem. The new approach reformulates the filter design problem as an equivalent model matching problem for which an integrated computational algorithm is proposed which is able to solve this problem in the most general setting. In this way, the technical difficulties often encountered by the existing methods when trying to reduce the approximation problems to standard $\mathcal{H}_\infty$-norm synthesis problems are completely avoided. For example, the presence of zeros on the boundary of stability domains or problems with non-full rank and even improper transfer-function matrices can be easily handled. The underlying main computational algorithms are based on descriptor system representations and rely on orthogonal matrix pencil reductions. For all basic computations, reliable numerical software tools are available for MATLAB in the Descriptor Systems Toolbox Varga (2000) and in the current version of the FAULT DETECTION Toolbox Varga (2006, 2009).

REFERENCES

Blanke, M., Kinnaert, M., Lunze, J., and Staroswiecki, M. (2003). *Diagnosis and Fault-Tolerant Control*. Springer-Verlag, Berlin.

Bruinsma, N. and Steinbuch, M. (1990). A fast algorithm to compute the H_∞-norm of a transfer function. *Systems & Control Letters*, 14, 287–293.

Chen, J. and Patton, R.J. (1999). *Robust Model-Based Fault Diagnosis for Dynamic Systems.* Kluwer Academic Publishers, London.

Francis, B.A. (1987). *A Course in H^∞ Theory.* Springer-Verlag, New York.

Frank, P.M. and Ding, X. (1994). Frequency domain approach to optimally robust residual generation and evaluation for model-based fault diagnosis. *Automatica*, 30, 789–804.

Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds. *Int. J. Control*, 39, 1115–1193.

Limebeer, D. and Halikias, G. (1988). A controller degree bound for $\mathcal{H}^\infty$-optimal control problems of the second kind. *SIAM J. Control Optim.*, 26, 646–677.

Oară, C. (2005). Constructive solutions to spectral and inner-outer factorizations with respect to the disk. *Automatica*, 41, 1855 – 1866.

Oară, C. and Varga, A. (1998). Least order coprime factorizations of rational matrices: the canonical case. *Proc. CDC'98, Tampa, FL*, 2745–2750.

Oară, C. and Varga, A. (2000). Computation of general inner-outer and spectral factorizations. *IEEE Trans. Automat. Control*, 45, 2307–2325.

Varga, A. (1992). Minimal realization procedures based on balancing and related techniques. In F. Pichler and R.M. Diaz (eds.), *Computer Aided Systems Theory - EUROCAST'91*, volume 585 of *Lect. Notes Comp. Scie.*, 733–761. Springer Verlag, Berlin.

Varga, A. (1998). Computation of coprime factorizations of rational matrices. *Lin. Alg. & Appl.*, 271, 83–115.

Varga, A. (2000). A DESCRIPTOR SYSTEMS toolbox for MATLAB. *Proc. CACSD'2000 Symposium, Anchorage, Alaska.*

Varga, A. (2005). Numerically reliable methods for optimal design of fault detection filters. *Proc. of CDC'05, Seville, Spain.*

Varga, A. (2006). A FAULT DETECTION toolbox for MATLAB. *Proc. of CACSD'06, Munich, Germany.*

Varga, A. (2008). On computing nullspace bases – a fault detection perspective. *Proc. IFAC 2008 World Congress, Seoul, Korea.*, 6295–6300.

Varga, A. (2009). Linear FDI-Techniques and Software Tools. FAULT DETECTION Toolbox V0.9 – Technical Documentation, German Aerospace Center (DLR), Institute of Robotics and Mechatronics.

Varga, A. (2010). Integrated algorithm for solving $\mathcal{H}_2$-optimal fault detection and isolation problems. Proc. of SYSTOL'2010, Nice, France.

Zhou, K., Doyle, J.C., and Glover, K. (1996). *Robust and Optimal Control.* Prentice Hall.