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A New Iterative Algorithm to Solve Periodic Riccati Differential Equations With Sign Indefinite Quadratic Terms

Yantao Feng, Andras Varga, Brian D. O. Anderson, and Marco Lovera

***Abstract*—An iterative algorithm to solve periodic Riccati differential equations (PRDE) with an indefinite quadratic term is proposed. In our algorithm, we replace the problem of solving a PRDE with an indefinite quadratic term by the problem of solving a sequence of PRDEs with a negative semidefinite quadratic term which can be solved by existing methods. The global convergence and the local quadratic rate of convergence are both established. A numerical example is given to illustrate our algorithm.**

***Index Terms*—Periodic Riccati differential equations (PRDE).**

I. Introduction

In periodic H_∞ control [10], in order to obtain a feedback controller, typically we need to solve one or two PRDEs of the following form [10], [29]:

$$\begin{aligned} -\dot{\Pi}(t) &= A(t)^T\Pi(t) + \Pi(t)A(t) + C(t)^TC(t) \\ &- \Pi(t)\left(B_2(t)B_2(t)^T - B_1(t)B_1(t)^T\right)\Pi(t) \end{aligned} \tag{1}$$

where $A:\mathbb{R}^+\rightarrow\mathbb{R}^{n\times n}$, $B_1:\mathbb{R}^+\rightarrow\mathbb{R}^{n\times p}$, $B_2:\mathbb{R}^+\rightarrow\mathbb{R}^{n\times q}$, $C:\mathbb{R}^+\rightarrow\mathbb{R}^{r\times n}$ are piecewise continuous, locally integrable, T-periodic functions and $\Pi:\mathbb{R}^+\rightarrow\mathbb{R}^{n\times n}$ is the bounded symmetric positive semidefinite T-periodic stabilizing solution we seek. Here $\mathbb{R}^+$ denotes the set of nonnegative real numbers. Our interest is in providing a new type of solution algorithm to solve (1), which is built on recent developments for solving algebraic Riccati equations (AREs) with an indefinite quadratic term [23].

In [23], the problem of solving an H_∞-type ARE is replaced by the problem of solving a sequence of H_2-type AREs and each of them can be solved by some existing algorithms [21]; then the solution of the original ARE can be approximated by the sum of the solutions of the H_2-type AREs. Since AREs can be regarded as a special class of PRDEs, we are interested in extending the algorithm in [23] to solve H_∞-type PRDEs.

A key motivation of this paper comes from an increasing interest in addressing periodic control problems for linear time-periodic systems

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[26]. Indeed, periodic control design methods have been widely used in different fields, for example in biology [9], aerospace [5], [25].

In many applications in which periodic models arise (see, e.g., [17]), the averaging method [20] has been used to deal with time-variability. In the averaging method, the actual time-periodic dynamic model is approximated with a time-invariant one by taking time averages over one period. It can be shown [20] that the averaged model provides an acceptable approximation of the true periodic system as long as its dynamics are much slower than the excitation. Clearly, the advantage of this method is that the control problem becomes time-invariant; however the designer is left with the burden of verifying *a posteriori* that the designed controller actually stabilizes the original time-varying dynamics and achieves a satisfactory performance level, if at all possible [22].

A typical example to illustrate the advantages of periodic control design methods is satellite attitude control based on magnetic actuators. In magnetic satellite attitude regulation, control torques are generated by exploiting the interaction between a set of electromagnetic coils and the geomagnetic field of the earth. As the variability of the geomagnetic field along the orbit is almost periodic, the resulting linearized models turn out to be time-periodic. Classical methods for the design of magnetic attitude control laws rely on averaged models [17], [31], [32], where the use of averaging was specifically developed to deal with the stabilization problem for the coupled roll/yaw dynamics of a momentum biased spacecraft using a magnetic torquer aligned with the pitch axis. In such a situation, viable alternatives are several recently developed design methods able to handle fully periodic models, with the significant advantage of guaranteeing closed loop stability *a priori*.

For LTI systems, to obtain an H_2 or an H_∞ controller, one needs to solve AREs for which satisfactory algorithms are available. For linear continuous time-varying periodic systems, to obtain an H_2 or an H_∞ controller, one needs to solve PRDEs, which are significantly more difficult to solve than AREs. Although algorithms to solve PRDEs do exist, their state of maturity is not as advanced as those for solving AREs. The situation is better for periodic discrete-time systems [14], where several structure-exploiting and structure-preserving methods exist both for standard and descriptor periodic systems [7], [8], [18], [19], [27]. For continuous-time periodic systems, the interest to develop reliable algorithms to solve PRDEs has recently increased. In what follows,we shortly describe two direct methods, both of which can serve as core solvers for our new iterative algorithm.

The periodic multiple-shooting method [34] is essentially an invariant subspace approach which is based on discretization techniques. The continuous-time problem is turned into an equivalent discrete-time problem for which the above mentioned techniques for discrete-time systems are employed. To solve the PRDE a linear Hamiltonian system must be integrated. The importance of using special (symplectic) integration methods for solving the associated Hamiltonian system has recently been emphasized in [15]. The main computational ingredient for solving the discretized problem is the computation of an ordered periodic real Schur form [8], [18] of a cyclic product of symplectic matrices expressing the monodromy matrix associated with the Hamiltonian system. The multiple-shooting method described in [34] has a *fast*, structure-exploiting version and a slower, but structure-preserving version. Although both algorithms are numerically reliable, a potential problem with these methods is the lack of control of the achieved accuracy of the computed solution. A MATLAB implementation of the fast structure-exploiting version is available in the PERIODIC SYSTEMS Toolbox[1] [33] and can be used to solve both H_2-type PRDEs and H_∞-type PRDEs.

[1]The toolbox is a proprietary software of DLR. Contact author Dr. Andras Varga for licensing conditions.

The second method is the semidefinite programming (SDP) method, which is basically a convex optimization based approach [13] applicable to both H_2-type and H_∞-type PRDEs [15]. The SDP method is based on a *harmonic approximation* of the positive definite stabilizing solution of the PRDE as a weighted sum of trigonometric base functions [12]. By doing this approximation and reformulating the PRDE as a maximization problem, the problem of solving the PRDE is turned into a SDP problem with linear matrix inequality (LMI) constraints [13]. As revealed in a recent study [16], if the SDP method can be used to solve the PRDE, the computed solution can be usually determined with high accuracy [30]. One major limitation of the SDP method is the high storage requirement. A MATLAB implementation of this method is described in [13].

In this paper we propose a new iterative algorithm to solve H_∞-type PRDEs, where the original problem is equivalently replaced by the problem of solving a sequence of H_2-type PRDEs. The solution of the original PRDE can be approximated by the sum of solutions of H_2-type PRDEs, which can be obtained by any of the above mentioned methods. Since each iteration in the proposed algorithm relies on the use of a solver to solve an H_2-type PRDE, it is clear that the main benefit of using the new method is neither its computational efficiency nor computational storage savings. However, it is realistic to expect that our algorithm has a higher potential than the existing direct methods to determine limiting accuracy solutions of H_∞-type PRDEs, especially when ill-conditioning is the limiting factor on the achievable accuracy. This feature has been already verified in the linear time-invariant case [23].

The paper is organized as follows: Section II gives some definitions and preliminary results. Section III presents our main results which underly the proposed algorithm to solve (1). Also, a result stating the quadratic rate of convergence of our algorithm is presented. Section IV gives a numerical example which illustrates the quadratic convergence of the proposed algorithm to the limiting accuracy solution. Section V establishes our conclusions.

Notation: $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices; $\mathbb{R}^+$ denotes the set of nonnegative real numbers; $\mathbb{Z}$ denotes the set of integers with $\mathbb{Z}_{\geq a}$ denoting the set of integers greater or equal to $a \in \mathbb{R}$; $X(t) \geq 0$ means the matrix $X(t)$ is positive semidefinite for all $t \in \mathbb{R}^+$; $X(t) - Y(t) \geq 0$ means the matrix $X(t) - Y(t)$ is positive semidefinite for all $t \in \mathbb{R}^+$; $\dot{X}(t)$ denotes the derivative of the matrix $X(t)$; $\|X(t)\|$ denotes the Euclidean norm of the matrix $X(t)$; $X(t)^T$ denotes the transpose of the matrix $X(t)$; $\sup(S)$ denotes the least upper bound of a set S of real numbers. Throughout this paper, we define $Q(t) := B_2(t)B_2(t)^T - B_1(t)B_1(t)^T$ to simply the expression of (1).

II. DEFINITIONS AND PRELIMINARY RESULTS

Consider the periodic system described by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0 \tag{2}$$

$$y(t) = C(t)x(t) \tag{3}$$

where $t \in \mathbb{R}^+$, $x_0 \in \mathbb{R}^n$ is the initial state, $u(t) \in \mathbb{R}^m$ is the input value, $x(t) \in \mathbb{R}^n$ is the state value, and $y(t) \in \mathbb{R}^p$ is the output value. $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{r\times n}$ are piecewise continuous, locally integrable, T-periodic real matrices.

Definition 1: [1] The system (2) is said to be stabilizable (respectively, detectable) if there exists T-periodic $K : \mathbb{R}^+ \to \mathbb{R}^{m\times n}$ (respectively, T-periodic $L : \mathbb{R}^+ \to \mathbb{R}^{n\times r}$) piecewise continuous and bounded on $\mathbb{R}$ such that the system $\dot{x}(t) = (A(t) - B(t)K(t))x(t)$ (respectively, $\dot{x}(t) = (A(t) - L(t)C(t))x(t)$) is stable.[2]

[2]Here and later, we say a linear time-varying system $\dot{x}(t) = A(t)x(t)$ is stable if the solution $x(t, x_0, t_0)$ of the differential equation satisfies that $x(t, x_0, t_0) \to 0$ uniformly and asymptotically when $t \to +\infty$, see [24].

Definition 2: Let A, B_1, B_2, C be the matrix functions appearing in (1). If there exists a bounded symmetric T-periodic solution $\Pi(t)$ to PRDE (1) such that the system $\dot{x}(t) = (A(t) + B_1(t)B_1(t)^T\Pi(t) - B_2(t)B_2(t)^T\Pi(t))x(t)$ is stable, then $\Pi(t)$ is called a stabilizing solution of (1).

Definition 3: Let A, B_1, B_2, C be the real matrix functions appearing in (1). Suppose there exists a bounded symmetric positive semidefinite T-periodic stabilizing solution $\Pi(t)$ to (1). Let $P : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$. Let $\hat{A}_P : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ be defined as $\hat{A}_P(t) = A(t) + B_1(t)B_1(t)^T P(t) - B_2(t)B_2(t)^T P(t)$ for all $t \in \mathbb{R}^+$, and let $\bar{A}_P : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ be defined as $\bar{A}_P(t) = A(t) + B_1(t)B_1(t)^T P(t) - B_2(t)B_2(t)^T\Pi(t)$ for all $t \in \mathbb{R}^+$.

Lemma 4: Let A, B_1, B_2, C be the real matrix functions appearing in (1), and let M be the set of smooth mappings from $\mathbb{R}^+$ to $\mathbb{R}^{n\times n}$ and $P, Z \in M$. Define $F : M \longrightarrow M$

$$P(t) \longmapsto \dot{P}(t) + P(t)A(t) + A(t)^T P(t) - P(t)Q(t)P(t) + C(t)^T C(t). \quad (4)$$

If $P(t) = P(t)^T$ and $Z(t) = Z(t)^T$ for all $t \in \mathbb{R}^+$, then

$$F(P(t) + Z(t)) = F(P(t)) + \dot{Z}(t) + Z(t)\hat{A}_P(t) + \hat{A}_P(t)^T Z(t) - Z(t)Q(t)Z(t) \quad (5)$$

for all $t \in \mathbb{R}^+$, where $\hat{A}_P(t)$ is defined in Definition 3. Furthermore, if $P(t) = P(t)^T$ and $Z(t) = Z(t)^T$ for all $t \in \mathbb{R}^+$ and they satisfy

$$0 = \dot{Z}(t) + Z(t)\hat{A}_P(t) + \hat{A}_P(t)^T Z(t) - Z(t)B_2(t)B_2(t)^T Z(t) + F(P(t)) \quad \forall t \in \mathbb{R}^+ \quad (6)$$

then

$$F(P(t) + Z(t)) = Z(t)B_1(t)B_1(t)^T Z(t) \quad \forall t \in \mathbb{R}^+. \quad (7)$$

Proof: The first result can be obtained by direct computations; the second claim is then trivial. ■

Lemma 5: [6], [28] Consider the system defined by (2) and (3), and assume that (A, B) is stabilizable and (A, C) is detectable. Then, there exists a symmetric positive semidefinite T-periodic stabilizing solution $Z(t)$ satisfying the following PRDE:

$$-\dot{Z}(t) = A(t)^T Z(t) + Z(t)A(t) - Z(t)B(t)B(t)^T Z(t) + C(t)^T C(t). \quad (8)$$

Furthermore, $Z(t)$ is the unique stabilizing solution of (8) (i.e., there is no other stabilizing solution to (8)).

Proof: See [6]. ■

Lemma 6: [1]–[3], [6], [28] Let A, C be the real matrix functions appearing in the system defined by (2) and (3), and suppose that the pair (A, C) is detectable. Then

(i) the system $\dot{x}(t) = A(t)x(t)$ is stable if there exists a bounded symmetric differentiable matrix function $S : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ and positive constants λ_1 and λ_2 such that for all $t \in \mathbb{R}^+$ $0 < \lambda_1 I \leq S(t) \leq \lambda_2 I < \infty$, and such that

$$\dot{S}(t) + A(t)^T S(t) + S(t)A(t) = -C(t)^T C(t) \quad \forall t \in \mathbb{R}^+ \quad (9)$$

(ii) there exists a unique positive semidefinite symmetric differentiable solution to (9) if the system $\dot{x}(t) = A(t)x(t)$ is stable.

Proof: See [2], [3], [28] for (*i*); see [2], [3] for (*ii*). ■

Lemma 7: Suppose there exists a bounded symmetric stabilizing solution $\Pi(t)$ to (1); then $\Pi(t)$ is the unique stabilizing solution to (1) (i.e., there is no other stabilizing solution to (1)) and it is T-periodic. Furthermore, if $\Pi(t) \geq 0$ for all $t \in \mathbb{R}^+$, then the system $\dot{x}(t) = (A(t) - B_2(t)B_2(t)^T\Pi(t))x(t)$ is stable.

Proof: See [10]. ■

Lemma 8: Let A, B_1, B_2, C be the matrix functions appearing in (1), $P(t) = P(t)^T \in \mathbb{R}^{n\times n}$ for all $t \in \mathbb{R}^+$ and $Z(t) = Z(t)^T \in \mathbb{R}^{n\times n}$ for all $t \in \mathbb{R}^+$ satisfying (6), and a bounded stabilizing $\Pi(t) = \Pi(t)^T \in \mathbb{R}^{n\times n}$ for all $t \in \mathbb{R}^+$ satisfying (1), and let $\bar{A}_P$ be the function defined in Definition 3. Then

i) $\Pi(t) \geq (P(t) + Z(t))$ for all $t \in \mathbb{R}^+$ if $\bar{A}_P(t)$ is stable;

ii) $\bar{A}_{P+Z}(t)$ is stable if $\Pi(t) \geq (P(t) + Z(t))$ for all $t \in \mathbb{R}^+$.

Proof: The proof is in parallel with the proof of Lemma 2 in [23]. ■

III. Main Results and Algorithm

In this section, we set up the main theorem and give our algorithm.

Theorem 9: Let A, B_1, B_2, C be the real matrix functions appearing in (1). Suppose that (C, A) is detectable and (A, B_2) is stabilizable, and define $F : M \longrightarrow M$ as in (4). Suppose there exists a bounded symmetric positive semidefinite T-periodic stabilizing solution $\Pi(t)$ of PRDE (1). Then

(I) two square matrix function sequences $Z_k(t)$ and $P_k(t)$ can be defined for all $k \in \mathbb{Z}_{\geq 0}$ recursively as follows:

$$P_0(t) = 0 \quad \forall t \in \mathbb{R}^+ \quad (10)$$

$$A_k(t) = A(t) + B_1(t)B_1(t)^T P_k(t) - B_2(t)B_2(t)^T P_k(t) \forall t \in \mathbb{R}^+ \quad (11)$$

$Z_k(t) \geq 0$ is the unique T-periodic stabilizing solution of

$$-\dot{Z}_k(t) = Z_k(t)A_k(t) + A_k(t)^T Z_k(t) - Z_k(t)B_2(t)B_2(t)^T Z_k(t) + F(P_k(t)) \quad (12)$$

and then

$$P_{k+1}(t) = P_k(t) + Z_k(t) \quad \forall t \in \mathbb{R}^+ \quad (13)$$

(II) the two sequences $P_k(t)$ and $Z_k(t)$ in part (I) have the following properties:

1) $(A(t) + B_1(t)B_1(t)^T P_k(t), B_2(t))$ is stabilizable $\forall k \in \mathbb{Z}_{\geq 0}$;

2) $F(P_{k+1}(t)) = Z_k(t)B_1(t)B_1(t)^T Z_k(t) \forall k \in \mathbb{Z}_{\geq 0} \forall t \in \mathbb{R}^+$;

3) $A(t) + B_1(t)B_1(t)^T P_k(t) - B_2(t)B_2(t)^T P_{k+1}(t)$ is stable $\forall k \in \mathbb{Z}_{\geq 0}$;

4) $\Pi(t) \geq P_{k+1}(t) \geq P_k(t) \geq 0 \forall k \in \mathbb{Z}_{\geq 0} \forall t \in \mathbb{R}^+$;

(III) the limit $P_\infty(t) := \lim_{k\to\infty} P_k(t)$ exists for all $t \in \mathbb{R}^+$ with $P_\infty(t) \geq 0$ for all $t \in \mathbb{R}^+$. Furthermore, $P_\infty(t) = \Pi(t)$ is the unique T-periodic stabilizing solution of PRDE (1), which is also positive semidefinite.

Proof: We construct the sequence for $Z_k(t)$ and $P_k(t)$ to show results (*I*) and (*II*) together by an inductive argument.

Case $k = 0$: Since $P_0(t) = 0$ via (10), (*II*1) is trivially satisfied by assumption. Since (12) reduces to

$$-\dot{Z}_0(t) = Z_0(t)A(t) + A(t)^T Z_0(t) - Z_0(t)B_2(t)B_2(t)^T Z_0(t) + C(t)^T C(t) \quad (14)$$

by Lemma 5 there exists a unique T-periodic positive semidefinite and stabilizing solution $Z_0(t)$ for (14). Since $P_1(t) = P_0(t) + Z_0(t)$ for all $t \in \mathbb{R}^+$ via (13), we have $F(P_1(t)) = Z_0(t)B_1(t)B_1(t)^0 T Z_0(t)$ for all $t \in \mathbb{R}^+$ by Lemma 4 and (*II*2) is satisfied. Then by Lemma 5

$(A(t)-B_2(t)B_2(t)^TZ_0(t))$ is stable, hence (*II*3) is satisfied on noting that $P_0(t)=0$ and $P_1(t)=Z_0(t)$ for all $t\in\mathbb{R}^+$. (*II*4) can be shown by using Lemma 7 and Lemma 8 (see [23] for this argument).

Inductive Step for $k\in\mathbb{Z}_{\geq 0}$: We now consider an arbitrary $q\in\mathbb{Z}_{\geq 0}$, suppose that (*I*) and (*II*) are satisfied for $k=q\in\mathbb{Z}_{\geq 0}$, and show that (*I*) and (*II*) are also satisfied for $k=q+1$. Since $F(P_{q+1}(t))=Z_q(t)B_1(t)B_1(t)^TZ_q(t)\geq 0$ for all $t\in\mathbb{R}^+$ by inductive assumption (*II*2), sufficient conditions for the existence of a unique positive semidefinite T-periodic stabilizing solution $Z_{q+1}(t)$ to (12) are (use Lemma 5):

$(\alpha 1)$ $(A(t)+B_1(t)B_1(t)^TP_{q+1}(t), B_2(t))$ is stabilizable;

$(\beta 1)$ $(B_1(t)^TZ_q(t), A(t)+B_1(t)B_1(t)^TP_q(t)-B_2(t)B_2(t)^TP_{q+1}(t))$ is detectable.

We will now show the existence of $Z_{q+1}(t)$ is guaranteed via the following two points:

1) Since result (*II*4) holds by inductive assumptions, we have $\Pi(t)\geq P_{q+1}(t)$ for all $t\in\mathbb{R}^+$, and thus $(A(t)+B_1(t)B_1(t)^TP_{q+1}(t)-B_2(t)B_2(t)^T\Pi(t))$ is stable by Lemma 8 Part (*ii*). Hence $(A(t)+B_1(t)B_1(t)^TP_{q+1}(t), B_2(t))$ is stabilizable and thus condition $(\alpha 1)$ and result (*II*1) for $k=q+1$ are satisfied;
2) Since $(A(t)+B_1(t)B_1(t)^TP_q(t)-B_2(t)B_2(t)^TP_{q+1}(t))$ is stable by inductive assumption (*II*3), condition $(\beta 1)$ is also satisfied.

Since conditions $(\alpha 1)$ and $(\beta 1)$ hold, there exists a unique positive semidefinite T-periodic stabilizing solution $Z_{q+1}(t)$ for (12) with $k=q+1$. Then by using Lemma 4 and Lemma 8, we can prove that (*II*2),(*II*3),(*II*4) are satisfied for $k=q+1$ (see [23] for this argument).

Inductive Conclusion: (*III*) Since the sequence $P_k(t)$ is monotone for all $t\in\mathbb{R}^+$ and bounded above by $\Pi(t)$, the sequence converges to a limit $P_\infty(t)$ (see pp. 33–34 in [4] for the details), and convergence of the sequence $P_k(t)$ to $P_\infty(t)$ implies convergence of $Z_k(t)$ to 0 since

$$Z_\infty(t):=\lim_{k\to\infty}Z_k(t)=\lim_{k\to\infty}(P_{k+1}(t)-P_k(t))=0.$$

Then it is clear from (*II*2),(*II*3),(*II*4), $P_\infty(t)\geq 0$ is a stabilizing solution to $F(P_\infty(t))=0$. Since $\Pi(t)\geq 0$ is a stabilizing solution to $F(\Pi(t))=0$ and the stabilizing solution of the PRDE (1) is always unique (see Lemma 7), it is clear that $P_\infty(t)=\Pi(t)$ for all $t\in\mathbb{R}^+$. ■

The following corollary gives a condition under which there does *not* exist a stabilizing solution $\Pi(t)\geq 0$ to $F(\Pi(t))=0$.

Corollary 10: Let $A(t), B_1(t), B_2(t), C(t)$ be the functions appearing in (1). Suppose $(C(t), A(t))$ is detectable and $(A(t), B_2(t))$ is stabilizable, and let $\{P_k(t)\}$ and $F: M\longrightarrow M$ be defined as in Theorem 9. If $\exists k\in\mathbb{Z}_{\geq 0}$ such that $(A(t)+B_1(t)B_1(t)^TP_k(t), B_2(t))$ is not stabilizable, then there does not exist a stabilizing solution $\Pi(t)\geq 0$ to $F(\Pi(t))=0$.

Proof: Restatement of Theorem 9, implication (*II*1). ■

Based on Theorem 9 and Corollary 10 we can give an implementable algorithm to solve (1):

1) Let $P_0(t)=0$ and $k=0$.
2) $A_k(t)=A(t)+B_1(t)B_1(t)^TP_k(t)-B_2(t)B_2(t)^TP_k(t)$.
3) Construct the unique real symmetric T-periodic stabilizing solution $Z_k(t)\geq 0$ which satisfies

$$-\dot{Z}_k(t)=Z_k(t)A_k(t)+A_k(t)^TZ_k(t) \\ -Z_k(t)B_2(t)B_2(t)^TZ_k(t)+F(P_k(t)) \quad (15)$$

where $F(P_k(t))$ is given in (4), and by Theorem 9 Part *II*2) we have $F(P_k(t))=Z_{k-1}(t)B_1(t)B_1^TZ_{k-1}(t)$. Exit if no positive stabilizing solution exists.

4) Set $P_{k+1}(t)=P_k(t)+Z_k(t)$.
5) If $\sup_{t\in\mathbb{R}^+}\|Z_k(t)\|<\epsilon$, where ϵ is a prescribed tolerance, then set $\Pi(t)=P_{k+1}(t)$ and exit. Otherwise, increment k by one and go to step 2.

Remark: The algorithm exits at Step 3 if the pair $(A(t)+B_1(t)B_1(t)^TP_k(t), B_2(t))$ at the current iteration is not stabilizable (see Corollary 10).

The following theorem states the local quadratic rate of convergence of the proposed algorithm.

Theorem 11: Given the suppositions of Theorem 9, and two real function sequences $P_k(t), Z_k(t)$ as defined in Theorem 9 Part (*I*), then there exists an $\eta>0$ such that the rate of convergence of the sequence $P_k(t)$ is quadratic in the region $\sup_{t\in\mathbb{R}}\|P_k(t)-\Pi(t)\|<\eta$.

Proof: The proof is in parallel with the counterpart in [11], [23], and it is omitted here. ■

IV. Numerical Example

In this section, we provide a numerical example in satellite attitude control to verify our algorithm to solve H_∞ PRDEs. The main purpose of our study is to illustrate the quadratic convergence of our algorithm and its ability to compute solutions whose accuracy is comparable or potentially even better than of exiting algorithms [34].

A. The Model

The satellite model of this example is a slight modification of the model considered in [27].

The linearized state space model of the roll/yaw angular dynamics of the earth-pointing LEO satellite described in [27] is given by

$$\dot{x}(t)=\bar{A}x(t)+\bar{B}T_m(t)+\bar{B}T_d(t) \quad (16)$$

where the state vector $x(t)$ is formed with the roll and yaw angles and the corresponding angular rates, $T_m(t)$ and $T_d(t)$ are, respectively, the control and disturbance torques acting on the roll and yaw axes and $\bar{A}\in\mathbb{R}^{4\times 4}$ and $\bar{B}\in\mathbb{R}^{4\times 2}$ are constant matrices. We assume (see [27] for details) that the control torques are of the form $T_m(t)=\Gamma(t)u(t)$, where $\Gamma(t)\in\mathbb{R}^{2\times 2}$ is a known time-periodic, positive semi-definite matrix and $u(t)$ is the control signal. The external disturbance torques are of the form $T_d(t)=\Lambda(t)w(t)$, where $\Lambda(t)\in\mathbb{R}^{2\times 2}$ is a given periodic matrix, and $w(t)$ is the external disturbance signal.

Letting I_1 and I_3 be, respectively, the moments of inertia of the roll and yaw axes, the expressions of the constant matrices of the model (16) taken from [27] are

$$\bar{A}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\\frac{\omega_0h_s}{I_1}&0&0&\frac{h_s}{I_1}\\0&\frac{\omega_0h_s}{I_3}&-\frac{h_s}{I_3}&0\end{bmatrix},\quad \bar{B}=\begin{bmatrix}0&0\\0&0\\\frac{1}{I_1}&0\\0&\frac{1}{I_3}\end{bmatrix} \quad (17)$$

and the time-varying matrices entering in the definition of torques are

$$\Gamma(t)=\frac{\sin^2(i)}{1-z}\begin{bmatrix}2&0\\0&.5\end{bmatrix}+\frac{\sin^2(i)}{1-z+z\cos(2\omega_0t)} \\ \times\begin{bmatrix}-\frac{2\cos(2\omega_0t)}{(1-z)}&-\sin(2\omega_0t)\\-\sin(2\omega_0t)&\frac{(1-2z)\cos(2\omega_0t)}{2(1-z)}\end{bmatrix}, \\ \Lambda(t)=\begin{bmatrix}1&0\\0&1\end{bmatrix} \quad (18)$$

where i is the orbit inclination and we adopted the notation $z=-(3/2)\sin^2(i)$.

The numerical values of parameters according to [27] are: $I_1=81.7789\ \mathrm{kg\ m^2}$, $I_3=60.2566\ \mathrm{kg\ m^2}$, $h_s=-81.3491\ \mathrm{Nms}$, $\omega_0=$

TABLE I
ACCURACY OF SOLUTION AS FUNCTION OF N

N	16	128	256	512
Res	$3.7 \cdot 10^{-7}$	$1.7 \cdot 10^{-7}$	$1.4 \cdot 10^{-7}$	$1.5 \cdot 10^{-7}$
Times (sec)				
– single processor	15.7	15.9	16.8	18.5
– 8 local processors	2.4	2.5	2.6	3.1

0.00068860 rad/s, $T = 2\pi/\omega_0 = 9124.6$ s, $i = 108^\circ(\pi/180) = 1.8850$ rad.

B. The H_∞ Control Problem

A *full information* (also state feedback) periodic H_∞ control problem can be formulated as follows [10]: for the performance output defined as $z(t) = \begin{bmatrix} Cx(t) \\ u(t) \end{bmatrix}$ with $C = diag\{0.1, 0.1, 1, 1\}$, find a memoryless T-periodic controller of the form $u(t) = K(t)x(t)$ such that the H_∞ norm of the transfer operator T_{zw} from w to z is less than or equal to a prescribed positive attenuation level γ, i.e., $\|T_{zw}\|_\infty \le \gamma$. To obtain such a controller, one needs to solve

$$-\dot{\Pi}(t) = \bar{A}^T\Pi(t) + \Pi(t)\bar{A} + C^TC - \Pi(t)\left(\bar{B}\Gamma(t)\left(\bar{B}\Gamma(t)\right)^T - \frac{1}{\gamma^2}\bar{B}\Lambda(t)\left(\bar{B}\Lambda(t)\right)^T\right)\Pi(t) \tag{19}$$

for $\Pi(t)$. And we have $K(t) = -\Gamma(t)^T\bar{B}^T\Pi(t)$.

C. Solution of $\mathcal{H}_2$-Problems

The solution of $\mathcal{H}_2$-problems is necessary at each iteration of the proposed algorithm, including the initialization phase. For the solution of this problem we used the multiple-shooting algorithm of [34] which determines the periodic solution $\Pi(t)$ on a grid of N time values $t_i = (i-1)\Delta$, for $i = 1, \ldots, N$, where $\Delta = T/N$. The algorithm of [34] has been implemented as a MATLAB function `prcric`, which is now part of the Periodic Toolbox [33].

We evaluate

$$Res = \max_{i=1,\ldots,N} \frac{\left\|\overline{\Pi}(t_i) - \Pi(t_i)\right\|}{\left\|\overline{\Pi}(t_i)\right\|} \tag{20}$$

where $\Pi(t_i)$ is the computed solution with the algorithm of [34], while $\overline{\Pi}(t_i)$ is the solution obtained by integrating (19) backward in time from $t = t_{i+1}$ to $t = t_i$, with $\Pi(t)$ initialized with $\Pi(t_{i+1})$. For accuracy checks, we use tighter tolerances for the integration of ODEs.

For all numerical integrations of ODEs to solve (19), we employed the non-stiff solver `ode113` available in the standard MATLAB with a relative tolerance $RelTol = 10^{-5}$ and an absolute tolerance $AbsTol = 10^{-6}$, while for accuracy checks we used a tighter relative tolerance $RelTol = 10^{-10}$. In Table I the accuracy of solving (19) for $\gamma = \infty$ is presented for different values of N. In Table I we also present timing measurements on a DELL Precision T5500 desktop with two Intel Xenon X5550 quad-core processors running at 2.66 GHz.

In spite of the very large period, the multiple-shooting approach produced very accurate results for a large range of numbers of discretization points. The resulting values in Table I merely indicate that the accuracy of different solutions computed with a tolerance of $RelTol = 10^{-5}$ has actually about 7 exact decimal digits. Larger values of N involve internally more computations, and thus could induce more roundoff errors. This could explain some fluctuations in the accuracy at different values of N.

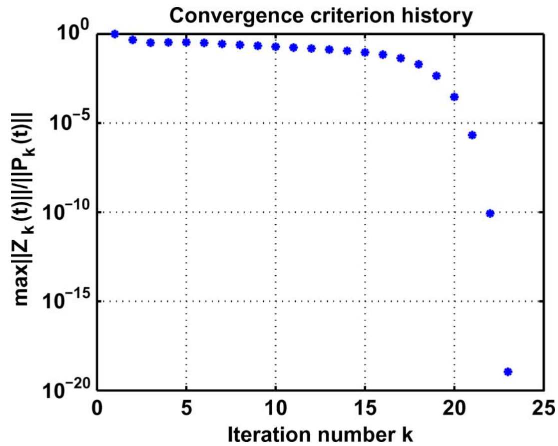

Fig. 1. History of the convergence criterion $\max\|Z_k(t)\|/\|P_k(t)\|$.

D. The Solution of the $\mathcal{H}_\infty$-Problem

The multiple-shooting algorithm of [34] can be equally used to solve the $\mathcal{H}_\infty$-Riccati (19) directly.

We employed a bisection search based γ-iteration starting with the bounds [1, 10] for γ. Finally, we arrived to locate the optimum in the range $\gamma_{opt} \in [1.7464, 1.7546]$, where the lower bound is still infeasible. Thus, to end the γ-iteration we can take $\gamma_{opt} = 1.7546$, which is a good approximation of the optimum value. For all values of γ above γ_{opt} the accuracy of the solution is very good with $Res < 10^{-6}$.

For $\gamma = 1.7546$ we employed the proposed iterative algorithm to compute the almost optimal solution of (19). Since high accuracy is necessary at all iterations, we used the same mesh with $N = 512$ grid points to solve the underlying H_2-type Riccati equation at Step 3). The updating of the solution at Step 4) happens only in the grid points and interpolation is used to obtain values of $A_k(t)$, $P_k(t)$ and $Z_k(t)$ between grid points.

As convergence criterion we used the condition involving the norms of correction terms

$$\max_{i=1,\ldots,N} \left(\|Z_k(t_i)\| / \|P_k(t_i)\|\right) < \epsilon_M$$

where $\epsilon_M = 2.22 \cdot 10^{-16}$ is the double precision unit roundoff (machine precision). With $RelTol = 10^{-10}$, the algorithm performed 24 iterations until convergence and the iteration history shows that neither $\|Z_k(t)\|$ nor $\|Z_k(t)\|/\|P_k(t)\|$ have a monotonically decreasing behavior. However, the final quadratic convergence towards the limiting accuracy solution can be easily seen from the plot of the iteration history of the convergence criterion in Fig. 1.

In Fig. 2 we plotted only the element $P_{11}(t)$ computed using the linear interpolation and cubic spline interpolation based methods. As it can be observed, the use of cubic-splines-based interpolation allows to significantly improve the accuracy of linear interpolation to approximate higher peak values. In spite of some penalty for speed, the solution obtained using spline based interpolation has a satisfactory accuracy with $Res = 3.46 \cdot 10^{-7}$.

The total time to solve the problem using parallelization on 8 local processors takes 144 seconds when employing linear interpolation and 227 seconds when employing spline based interpolation to obtain values of $P_k(t)$ and $Z_k(t)$ between grid points.

V. CONCLUSION

In this paper, an iterative algorithm to compute the stabilizing solutions of H_∞-type PRDEs is given. By using the proposed algorithm,

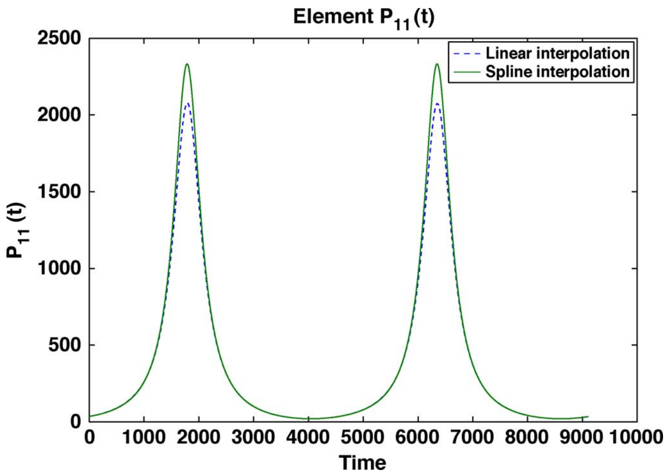

Fig. 2. Element $P_{11}(t)$ computed using linear and spline interpolation.

we can compute the solution of the original H_∞-type PRDE by using the solutions of a sequence of H_2-type PRDEs. The main appeals of our iterative algorithm are the simple choice of the initial approximation, and the local quadratic rate of convergence towards the solution. The provided simulation results illustrate the potential of the proposed algorithm to produce limiting accuracy solutions.

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