Integrated algorithm for solving $\mathcal{H}_2$-optimal fault detection and isolation problems

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***Abstract*— The design problem of fault detection and isolation filters can be formulated as a model matching problem and solved using an $\mathcal{H}_2$-norm minimization approach. A systematic procedure is proposed to choose appropriate filter specifications which guarantee the existence of proper and stable solutions of the model matching problem. This selection is integral part of a numerically reliable computational method to design of $\mathcal{H}_2$-optimal fault detection filters. The proposed design approach is completely general, being applicable to both continuous- and discrete-time systems, and can easily handle even unstable and/or improper systems.**

I. THE FAULT DETECTION AND ISOLATION PROBLEM

Consider additive fault models described by input-output representations of the form

$$\mathbf{y}(\lambda) = G_u(\lambda)\mathbf{u}(\lambda)+G_d(\lambda)\mathbf{d}(\lambda)+G_w(\lambda)\mathbf{w}(\lambda)+G_f(\lambda)\mathbf{f}(\lambda), \quad (1)$$

where $\mathbf{y}(\lambda)$, $\mathbf{u}(\lambda)$, $\mathbf{d}(\lambda)$, $\mathbf{w}(\lambda)$, and $\mathbf{f}(\lambda)$ are Laplace- or Z-transformed vectors of the p-dimensional system output vector $y(t)$, m_u-dimensional control input vector $u(t)$, m_d-dimensional disturbance vector $d(t)$, m_w-dimensional noise vector $w(t)$ and m_f-dimensional fault vector $f(t)$, respectively, and where $G_u(\lambda)$, $G_d(\lambda)$, $G_w(\lambda)$ and $G_f(\lambda)$ are the *transfer-function matrices* (TFMs) from the control inputs to outputs, disturbance inputs to outputs, noise inputs to outputs, and fault inputs to outputs, respectively. According to the system type, the frequency variable λ is either s, the complex variable in the Laplace-transform in the case of a continuous-time system or z, the complex variable in the Z-transform in the case of a discrete-time system. For most of practical applications, the TFMs $G_u(\lambda)$, $G_d(\lambda)$, $G_w(\lambda)$ and $G_f(\lambda)$ are proper rational matrices. However, for complete generality of our problem setting, we will allow that these TFMs are general non-proper rational matrices for which we will not *a priori* assume any further properties (e.g., stability, full rank).

A linear residual generator (or fault detection filter) processes the measurable system outputs $y(t)$ and control inputs $u(t)$ and generates the residual signals $r(t)$ which serve for decision making on the presence or absence of faults. The input-output form of this filter is

$$\mathbf{r}(\lambda) = Q(\lambda)\begin{bmatrix} \mathbf{y}(\lambda) \\ \mathbf{u}(\lambda) \end{bmatrix} \quad (2)$$

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where $Q(\lambda)$ is the TFM of the filter. For a physically realizable filter, $Q(\lambda)$ must be *proper* (i.e., only with finite poles) and *stable* (i.e., only with poles having negative real parts for a continuous-time system or magnitudes less than one for a discrete-time system). The (dynamic) *order* of $Q(\lambda)$ (also known as *McMillan degree*) is the dimension of the state vector of a minimal state-space realization of $Q(\lambda)$. The dimension q of the residual vector $r(t)$ depends on the fault detection problem to be solved.

The residual signal $r(t)$ in (2) generally depends via the system outputs $y(t)$ of all system inputs $u(t)$, $d(t)$, $w(t)$ and $f(t)$. The residual generation system, obtained by replacing in (2) $\mathbf{y}(\lambda)$ by its expression in (1), is given by

$$r(\lambda) = R_u(\lambda)\mathbf{u}(\lambda)+R_d(\lambda)\mathbf{d}(\lambda)+R_w(\lambda)\mathbf{w}(\lambda)+R_f(\lambda)\mathbf{f}(\lambda) \quad (3)$$

where

$$[\, R_u(\lambda) | R_d(\lambda) | R_w(\lambda) | R_f(\lambda)\,] := Q(\lambda)\left[\begin{array}{c|c|c|c} G_u(\lambda) & G_d(\lambda) & G_w(\lambda) & G_f(\lambda) \\ I_{m_u} & 0 & 0 & 0 \end{array}\right]$$

For a successfully designed filter $Q(\lambda)$, the corresponding residual generation system is proper and stable and achieves specific fault detection requirements.

For the solution of fault detection problems it is always possible to completely decouple the control input $u(t)$ from the residuals $r(t)$ by requiring $R_u(\lambda) = 0$. Regarding the disturbance input $d(t)$ and noise input $w(t)$ we aim to impose a similar condition on the disturbances input $d(t)$ by requiring $R_d(\lambda) = 0$, while minimizing simultaneously the effect of noise input $w(t)$ on the residual (e.g., by minimizing the norm of $R_w(\lambda)$). Thus, from a practical synthesis point of view, the distinction between $d(t)$ and $w(t)$ lies solely in the way these signals are treated when solving the residual generator synthesis problem.

More precisely, the disturbance inputs in $d(t)$ are additive effects from which exact decoupling of the residuals is presumably possible and is targeted in the detector synthesis. On the other hand, the noise input vector $w(t)$ contains everything else, including proper random noise or "ordinary" disturbances in excess of those which may be exactly decoupled. It may even contain fictive inputs which model the effect of parametric uncertainties in the process model. This distinction between $d(t)$ and $w(t)$ allows to address the solution of both exact and approximate fault detection problems using a unique computational framework.

Let $M_r(\lambda)$ be a suitably chosen reference model (i.e., stable, proper, diagonal and invertible) representing the desired

TFM from the faults to residuals. We want to achieve that $\mathbf{r}(\lambda) \approx M_r(\lambda)\mathbf{f}(\lambda)$, that is, each residual $r_i(t)$ is influenced mainly by fault $f_i(t)$. Our formulation of the *approximate fault detection and isolation problem* (AFDIP) extends the formulation of the model-matching approach of [1], [2] by requiring to determine a stable and proper filter $Q(\lambda)$ such that the following conditions are fulfilled:

$$\begin{array}{ll} (i) & R_u(\lambda) = 0, \\ (ii) & R_d(\lambda) = 0, \\ (iii) & R_f(\lambda) \approx M_r(\lambda), \text{ with } R_f(\lambda) \text{ stable;} \\ (iv) & R_w(\lambda) \approx 0, \text{ with } R_w(\lambda) \text{ stable.} \end{array} \tag{4}$$

The *exact fault detection and isolation problem* (EFDIP) requiring $R_f(\lambda) = M_r(\lambda)$ is included in this formulation and corresponds to $m_w = 0$, while the formulation of the AFDIP in [1], [2] corresponds to $m_d = 0$.

It is straightforward to show that for the solution of the AFDIP, the solvability conditions are those for the solvability of the EFDIP stated in [3].

Theorem 1: For the system (1) there exists a stable, diagonal, proper, and invertible $M_r(\lambda)$ such that the AFIDP is solvable if and only if

$$\textit{rank}\,[\,G_f(\lambda)\ G_d(\lambda)\,] = m_f + \textit{rank}\,G_d(\lambda) \tag{5}$$

Proof: The condition (5) is necessary according to [3] to guarantee that an exact solution exists in the case of no noise (i.e., when $w(t) = 0$). To prove sufficiency, let $Q(\lambda)$ be a solution to the EFDIP. $Q(\lambda)$ is also a solution of the AFDIP, provided the corresponding $R_w(\lambda)$ is stable and proper. If this is not the case, then we can choose a suitable diagonal $M(\lambda)$ (i.e., stable, proper and invertible) such that $M(\lambda)R_w(\lambda)$ is stable and proper. Then $M(\lambda)Q(\lambda)$ is a solution for $M_r(\lambda)$ replaced by $M(\lambda)M_r(\lambda)$. ■

Generically, the condition (5) is fulfilled if $p \geq m_f + m_d$, which implies that the system must have a sufficiently large number of measurements. For the case $m_d = 0$ considered in [1], [2], this condition reduces to the simple left invertibility condition:

$$\textit{rank}\,G_f(\lambda) = m_f \tag{6}$$

In the next section we describe the solution of the AFDIP by solving an approximate model-matching problem using $\mathcal{H}_2$-norm minimization techniques.

II. The $\mathcal{H}_2$-optimal model-matching approach

Consider $Q(\lambda)$ in a factored form $Q(\lambda) = \overline{Q}(\lambda)N_l(\lambda)$, where $N_l(\lambda)$ is a proper left rational nullspace basis satisfying

$$N_l(\lambda)\left[\begin{array}{cc} G_d(\lambda) & G_u(\lambda) \\ 0 & I_{m_u} \end{array}\right] = 0 \tag{7}$$

and $\overline{Q}(\lambda)$ is a factor to be further determined. With this choice it follows that $Q(\lambda)$ automatically fulfills the first two conditions in (4). The existence of $N_l(\lambda)$ is guaranteed provided condition (5) is fulfilled. The resulting $N_l(\lambda)$ has maximal row rank $p - r_d$, where $r_d = \textit{rank}\,G_d(\lambda)$. Moreover, we can choose $N_l(\lambda)$ stable and such that both $N_f(\lambda)$ and $N_w(\lambda)$ defined as

$$[\,N_f(\lambda)\ N_w(\lambda)\,] := N_l(\lambda)\left[\begin{array}{cc} G_f(\lambda) & G_w(\lambda) \\ 0 & 0 \end{array}\right] \tag{8}$$

are proper and stable TFMs [4].

To fulfill the last two conditions in (4) we can solve a $\mathcal{H}_2$-norm minimization problem to determine $\overline{Q}(\lambda)$ such that

$$\big\|[\,\overline{Q}(\lambda)N_f(\lambda) - M_r(\lambda)\ \ \overline{Q}(\lambda)N_w(\lambda)\,]\big\|_2 = \min$$

This $\mathcal{H}_2$ model matching problem can be easily reformulated as a standard $\mathcal{H}_2$-norm minimization based "controller" synthesis problem [5] as shown in Fig. 1. Here, the underlying equations are

$$\begin{array}{rl} \mathbf{e}(\lambda) = & \mathbf{r}(\lambda) - M_r(\lambda)\mathbf{f}(\lambda) \\ \overline{\mathbf{y}}(\lambda) = & N_f(\lambda)\mathbf{f}(\lambda) + N_w(\lambda)\mathbf{w}(\lambda) \\ \mathbf{r}(\lambda) = & \overline{Q}(\lambda)\overline{\mathbf{y}}(\lambda) \end{array}$$

and lead to the following definition of the generalized plant

$$P(\lambda) = \left[\begin{array}{c|c} P_{11}(\lambda) & P_{12}(\lambda) \\ \hline P_{21}(\lambda) & P_{22}(\lambda) \end{array}\right] := \left[\begin{array}{cc|c} -M_r(\lambda) & 0 & I \\ \hline N_f(\lambda) & N_w(\lambda) & 0 \end{array}\right]$$

The minimization of the $\mathcal{H}_2$-norm of the TFM from $[\,f^T(t)\ \ w^T(t)\,]^T$ to $e(t)$) via an optimal $\overline{Q}(\lambda)$ is thus formally a standard $\mathcal{H}_2$-synthesis problem for which software tools exist, as for example, the function `h2syn` available in the MATLAB Robust Control Toolbox. The main problem when employing standard tools like `h2syn`, is that, although a stable and proper solution of the AFDIP may exist, this solution can not be computed because of the presence of technical assumptions which must be fulfilled.

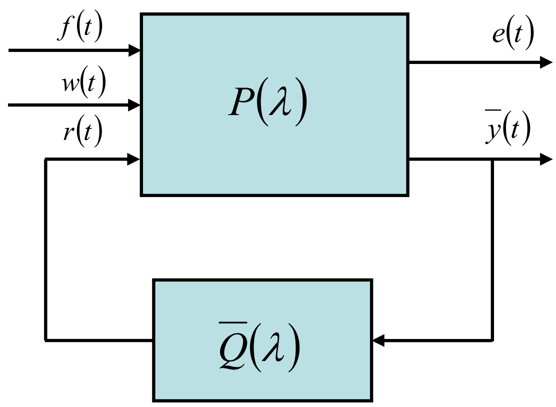

Fig. 1. Standard $\mathcal{H}_2$ synthesis setting.

To face the above limitations, it is necessary to develop general synthesis procedures for which no such limitations exist. The key parameter to guarantee the stability and properness of the detector is $M_r(\lambda)$, the desired TFM relating the faults to the residuals. The choice of $M_r(\lambda)$ is not obvious and can be even the object of an optimization based choice [6]. Often good candidates for $M_r(\lambda)$ result from exact (nominal) synthesis. However, in [7] a procedure has been proposed, where the choice of suitable $M_r(\lambda)$ is part of the solution. In this paper, we refine this procedure, by proposing an integrated approach to the detector synthesis. An important feature of the proposed computational approach is that it relies on repeated updating of an initial fault detector. The underlying state space computations employ explicit least order realizations of the successive detectors, thus a least order of the final detector is guaranteed. Since the successive steps are strongly connected, all structural features of the computed intermediary results can be exploited in the next steps. This leads to an integrated computational procedure based on highly efficient structure exploiting computations.

III. Enhanced model-matching procedure

In this section we propose an enhanced version of the algorithm of [7], where we exploit the additional structure in the model (1) owing to the separation of the unknown inputs in two components $d(t)$ and $w(t)$. Moreover, by using a new parametrization of the detector, we derive an integrated computational approach based on detector updating techniques. We describe in what follows the three main stages of the overall computational procedure.

The first stage has been already partly described in the previous section. Assuming the EFDIP is solvable when $w(t) = 0$, we compute $N_l(\lambda)$, a proper left nullspace basis satisfying (7). We choose for the detector the parametric form

$$Q(\lambda) = \overline{Q}(\lambda)N_l(\lambda), \tag{9}$$

where $\widetilde{Q}(\lambda)$ has to be determined. The basis $N_l(\lambda)$ can be determined such that both $N_f(\lambda)$ and $N_w(\lambda)$ defined in (8) are proper and stable TFMs. However, as it will be apparent, enforcing the stability condition is not necessary at this stage. We can easily check at this stage the solvability of the AFDIP by verifying that

$$\text{rank}\, N_f(\lambda) = m_f \tag{10}$$

The choice (9) of $Q(\lambda)$ enforces that $R_u(\lambda) = 0$ and $R_d(\lambda) = 0$ in (3). With $N_f(\lambda)$ and $N_w(\lambda)$ defined as in (8), we can formulate a slightly modified $\mathcal{H}_2$ model matching problem to minimize $\|\mathcal{R}(\lambda)\|_2$, where

$$\mathcal{R}(\lambda) = M(\lambda)F(\lambda) - \overline{Q}(\lambda)G(\lambda), \tag{11}$$

with $G(\lambda) = [\, N_f(\lambda)\ \ N_w(\lambda)\,]$ and $F(\lambda) = [\, M_r(\lambda)\ \ O\,]$. Here, $M_r(\lambda)$ is the TFM of a given reference model (i.e., stable, proper, diagonal, invertible), while $M(\lambda)$ is a free updating factor with the same properties (to be determined). Thus, the solution of the AFDIP using the $\mathcal{H}_2$ model matching approach involves choosing an appropriate $M(\lambda)$ such that $\|\mathcal{R}(\lambda)\|_2$ is finite and the resulting $Q(\lambda)$ is stable and proper.

Let ℓ be the rank of the $(p - r_d) \times (m_f + m_w)$ TFM $G(\lambda) = [\, N_f(\lambda)\ \ N_w(\lambda)\,]$. If $\ell < p - r_d$ (i.e., $G(\lambda)$ has no full row rank), we can take instead $N_l(\lambda)$, ℓ linear combinations of basis vectors of the form $W(\lambda)N_l(\lambda)$, which ensures that $W(\lambda)G(\lambda)$ has full row rank ℓ. A suitable choice of the $\ell \times (p - r_d)$ TFM $W(\lambda)$ which also minimizes the McMillan degree of $W(\lambda)N_l(\lambda)$ is described in [4].

The second stage involves the determination of a *column compressed* quasi-co-outer–inner factorization

$$G(\lambda) = [\, G_{o,1}(\lambda)\ 0\,] \left[\begin{array}{c} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{array} \right] := G_o(\lambda)G_i(\lambda), \tag{12}$$

where $G_i(\lambda)$ is a $(m_f + m_w) \times (m_f + m_w)$ inner TFM and $G_{o,1}(\lambda)$ is an $\ell \times \ell$ invertible TFM. Recall that a square $G_i(\lambda)$ is *inner* (and simultaneously *co-inner*) if it has only stable poles and satisfies $G_i(\lambda)G_i^*(\lambda) = I$, where $G_i^*(s) := G_i^T(-s)$ in a continuous-time setting and $G_i^*(z) := G_i^T(1/z)$ in a discrete-time setting. The *quasi-co-outer* factor $G_o(\lambda)$ may contain besides stable zeros, also zeros which lie on the extended imaginary axis for a continuous-time system or on the unit circle for a discrete-time system.

We can update the parametrization (9) of the detector by choosing $\overline{Q}(\lambda)$ of the form

$$\overline{Q}(\lambda) = \widehat{Q}(\lambda)G_{o,1}^{-1}(\lambda) \tag{13}$$

where $\widehat{Q}(\lambda)$ is to be determined. Using (12) and (13), we can express $\mathcal{R}(\lambda)$ in (11) as $\mathcal{R}(\lambda) = \widehat{\mathcal{R}}(\lambda)G_i(\lambda)$, with

$$\widehat{\mathcal{R}}(\lambda) = \left[\ M(\lambda)\widehat{F}_1(\lambda) - \widehat{Q}(\lambda)\ \middle|\ M(\lambda)\widehat{F}_2(\lambda)\ \right]$$

where $\widehat{F}_1(\lambda) := F(\lambda)G_{i,1}^*(\lambda)$ and $\widehat{F}_2(\lambda) := F(\lambda)G_{i,2}^*(\lambda)$.

In the third stage we choose

$$\widehat{Q}(\lambda) = M(\lambda)[\widehat{F}_1(\lambda)]_+,$$

where $M(\lambda)$ is a stable, proper, diagonal and invertible TFM chosen to ensure that

$$Q(\lambda) := M(\lambda)[\widehat{F}_1(\lambda)]_+ G_{o,1}^{-1}(\lambda)N_l(\lambda)$$

is proper and stable and $M(\lambda)\widehat{F}_2(\lambda)$ is stable and strictly proper in the continuous-time case and proper in the discrete-time case. Here, $[\cdot]_+$ denotes the stable part of the underlying TFM. With this choice, it follows that

$$\|\mathcal{R}(\lambda)\|_2 = \|\widehat{\mathcal{R}}(\lambda)\|_2$$

is finite. The computation of appropriate $M(\lambda)$ can be done using the stable and proper coprime factorization algorithm of [9]. These ideas are summarized in the following conceptual procedure:

Procedure FDIH2: $\mathcal{H}_2$-synthesis of FDI filters
Inputs: $G_u(\lambda)$, $G_d(\lambda)$, $G_w(\lambda)$, $G_f(\lambda)$, $M_r(\lambda)$;
Outputs: $Q(\lambda)$, $M(\lambda)$, $R_f(\lambda)$, $R_w(\lambda)$.

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1) Compute an $(p - r_d) \times (p + m)$ rational left nullspace basis $Q(\lambda)$ satisfying

$$Q(\lambda) \left[\begin{array}{cc} G_d(\lambda) & G_u(\lambda) \\ 0 & I_{m_u} \end{array} \right] = 0 \tag{14}$$

and compute

$$R_f(\lambda) = Q(\lambda) \begin{bmatrix} G_f(\lambda) \\ 0 \end{bmatrix}, \quad R_w(\lambda) = Q(\lambda) \begin{bmatrix} G_w(\lambda) \\ 0 \end{bmatrix}$$

If $\mathit{rank}\, R_f(\lambda) \neq m_f$ **Exit** (no solution exists).

2) If $\ell = \mathit{rank}\, [\, R_f(\lambda)\; R_w(\lambda)\,] < p - r_d$, compute a $\ell \times (p - r_d)$ $W(\lambda)$ such that $W(\lambda)N_l(\lambda)$ is proper, it has least possible McMillan degree and $W(\lambda)[\, R_f(\lambda)\;\; R_w(\lambda)\,]$ has full row rank. Compute $Q(\lambda) \leftarrow W(\lambda)Q(\lambda)$, $R_f(\lambda) \leftarrow W(\lambda)R_f(\lambda)$ and $R_w(\lambda) \leftarrow W(\lambda)R_w(\lambda)$.
3) Compute the *quasi* co-outer–inner factorization

$$[\, R_f(\lambda)\; R_w(\lambda)\,] = [\, G_{o,1}(\lambda)\; 0\,] \left[\begin{array}{c} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{array} \right],$$

where $[\, G_{i,1}^T(\lambda)\;\; G_{i,2}^T(\lambda)\,]^T$ is square and inner, and $G_{o,1}(\lambda)$ is a $\ell \times \ell$ invertible TFM which has only stable zeros, excepting possible zeros on the imaginary axis (including infinity) for a continuous-time system or on the unit circle for a discrete-time system. Compute $Q(\lambda) \leftarrow G_{o,1}^{-1}(\lambda)Q(\lambda)$, $R_f(\lambda) \leftarrow G_{o,1}^{-1}(\lambda)R_f(\lambda)$, $R_w(\lambda) \leftarrow G_{o,1}^{-1}(\lambda)R_w(\lambda)$.
4) Compute $\widehat{F}_1(\lambda) = [\, M_r(\lambda)\;\; O\,]G_{i,1}^*(\lambda)$ and $\widehat{F}_2(\lambda) = [\, M_r(\lambda)\; O\,]G_{i,2}^*(\lambda)$.
5) Compute a diagonal $M(\lambda)$ having the least McMillan degree such that $M(\lambda)[\widehat{F}_1(\lambda)]_+ Q(\lambda)$ is proper and stable, and $M(\lambda)\widehat{F}_2(\lambda)$ is stable and strictly proper for a continuous-time system or proper for a discrete-time system. Compute $Q(\lambda) \leftarrow M(\lambda)[\widehat{F}_1(\lambda)]_+ Q(\lambda)$, $R_f(\lambda) \leftarrow M(\lambda)[\widehat{F}_1(\lambda)]_+ R_f(\lambda)$, $R_w(\lambda) \leftarrow M(\lambda)[\widehat{F}_1(\lambda)]_+ R_w(\lambda)$.

The high-level computations in terms of TFMs in the **Procedure FDIH2** can be performed via state-space models based reliable numerical computations, which are described in the next section.

IV. Computational issues

For computations we employ an equivalent *descriptor* state space realization of the input-output model (1),

$$\begin{array}{rl} E\lambda x(t) \!\!&= Ax(t) + B_u u(t) + B_d d(t) + B_w w(t) + B_f f(t) \\ y(t) \!\!&= Cx(t) + D_u u(t) + D_d d(t) + D_w w(t) + D_f f(t) \end{array} \quad (15)$$

with the n-dimensional state vector $x(t)$, where $\lambda x(t) = \dot{x}(t)$ or $\lambda x(t) = x(t+1)$ depending on the type of the system, continuous or discrete, respectively. In general, the square matrix E can be singular, but we will assume that the linear pencil $A - \lambda E$ is regular. For systems with proper TFMs in (1), we can always choose a *standard* state space realization where $E = I$. In general, we can assume that the representation (15) is minimal, that is, the descriptor pair $(A - \lambda E, C)$ is *observable* and the pair $(A - \lambda E, [\, B_u\; B_d\; B_w\; B_f\,])$ is *controllable*. The corresponding TFMs of the model in (1) are

$$\begin{array}{rcl} G_u(\lambda) &=& C(\lambda E - A)^{-1} B_u + D_u \\ G_d(\lambda) &=& C(\lambda E - A)^{-1} B_d + D_d \\ G_w(\lambda) &=& C(\lambda E - A)^{-1} B_w + D_w \\ G_f(\lambda) &=& C(\lambda E - A)^{-1} B_f + D_f \end{array} \quad (16)$$

or in an equivalent notation

$$\big[\, G_u(\lambda)\; G_d(\lambda)\; G_w(\lambda)\; G_f(\lambda)\, \big] := \left[\begin{array}{c|c} A - \lambda E & B_u\; B_d\; B_w\; B_f \\ \hline C & D_u\; D_d\; D_w\; D_f \end{array} \right]$$

At Step 1 we employ recently developed synthesis algorithms based on rational nullspace methods [4], to obtain the preliminary $(p - r_d) \times (m + p)$ filter $Q(\lambda)$ as a proper left nullspace basis satisfying (14) and the corresponding $R_f(\lambda)$ and $R_w(\lambda)$ with realizations of the form

$$\big[\, Q(\lambda)\; R_f(\lambda)\; R_w(\lambda)\, \big] = \left[\begin{array}{c|c} \widetilde{A} - \lambda \widetilde{E} & \widetilde{B}_{yu}\; \widetilde{B}_f\; \widetilde{B}_w \\ \hline \widetilde{C} & \widetilde{D}_{yu}\; \widetilde{D}_f\; \widetilde{D}_w \end{array} \right], \quad (17)$$

where $\widetilde{E}$ is invertible (thus all TFMs are proper) and the pair $(\widetilde{A}, \widetilde{E})$ has only finite generalized eigenvalues which can be arbitrarily placed.

At Step 2, if $\ell < p - r_d$ we can determine using minimum dynamic covers techniques a suitable $\ell \times (p - r_d)$ prefilter $W(\lambda)$ such that $W(\lambda)Q(\lambda)$ has the least possible order and $W(\lambda)[\, R_f(\lambda)\;\; R_w(\lambda)\,]$ has full row rank. The state-space realization of $W(\lambda)\big[\, Q(\lambda)\; R_f(\lambda)\; R_w(\lambda)\,\big]$ has still the form (17) and can be obtained by using updating techniques described in [4].

For the computation of the quasi-co-outer–inner factorization of $[\, R_f(\lambda)\; R_w(\lambda)\,]$ at Step 3 we employ the dual of the algorithm of [10] for the continuous-time case and the dual of the algorithm of [11] for the discrete-time case. In both cases, the quasi-co-outer factor $G_{o,1}(\lambda)$ is obtained in the form

$$G_{o,1}(\lambda) = \left[\begin{array}{c|c} \widetilde{A} - \lambda \widetilde{E} & \overline{B}_o \\ \hline \widetilde{C} & \overline{D}_o \end{array} \right] \quad (18)$$

where $\overline{B}_o$ and $\overline{D}_o$ are matrices with ℓ columns. The system with the TFM $G_{o,1}(\lambda)$ may have besides the stable zeros (partly resulted from the column compression), also zeros on the imaginary axis (including infinity) in the continuous-time case or on the unit circle in the discrete-time case. The $(m_f + m_w) \times (m_f + m_w)$ TFM of the inner factor is proper and stable and assume that its inverse (i.e., its conjugated TFM) has a state space realization of the form

$$G_i^*(\lambda) = \left[\begin{array}{c|c} A_i - \lambda E_i & B_i \\ \hline C_i & D_i \end{array} \right]$$

To compute the updated filter $\overline{Q}(\lambda) := G_{o,1}^{-1}(\lambda)Q(\lambda)$ as well as $\overline{R}_f(\lambda) := G_{o,1}^{-1}(\lambda)R_f(\lambda)$ and $\overline{R}_w(\lambda) := G_{o,1}^{-1}(\lambda)R_w(\lambda)$, we can solve the linear rational system of equations

$$G_{o,1}(\lambda)\big[\overline{Q}(\lambda)\,\overline{R}_f(\lambda)\,\overline{R}_w(\lambda)\big] = \big[Q(\lambda)\,R_f(\lambda)\,R_w(\lambda)\big] \quad (19)$$

Observe that $G_{o,1}(\lambda)$, $Q(\lambda)$, $R_f(\lambda)$ and $R_w(\lambda)$ have descriptor realizations which share the same state, descriptor and output matrices. Using these state space realizations, the linear rational equation (19) can be equivalently solved (see [12]) by computing first the solution $X(\lambda)$ of

$$\left[\begin{array}{cc} \widetilde{A} - \lambda \widetilde{E} & \overline{B}_o \\ \widetilde{C} & \overline{D}_o \end{array} \right] X(\lambda) = \left[\begin{array}{ccc} \widetilde{B}_{yu} & \widetilde{B}_f & \widetilde{B}_w \\ \widetilde{D}_{yu} & \widetilde{D}_f & \widetilde{D}_w \end{array} \right]$$

and then

$$\left[\; \overline{Q}(\lambda) \quad \overline{R}_f(\lambda) \quad \overline{R}_w(\lambda) \;\right] = \left[\; 0 \quad I_\ell \;\right] X(\lambda)$$

With the invertible system matrix

$$S_o(\lambda) = \left[\begin{array}{cc} \widetilde{A} - \lambda \widetilde{E} & \overline{B}_o \\ \widetilde{C} & \overline{D}_o \end{array} \right] \qquad (20)$$

we obtain

$$\left[\, \overline{Q}(\lambda)\; \overline{R}_f(\lambda)\; \overline{R}_w(\lambda) \,\right] = \left[\, 0 \;\; I_\ell \,\right] S_o^{-1}(\lambda) \left[\begin{array}{ccc} \widetilde{B}_{yu} & \widetilde{B}_f & \widetilde{B}_w \\ \widetilde{D}_{yu} & \widetilde{D}_f & \widetilde{D}_w \end{array} \right]$$

from which descriptor state space realizations of updated $Q(\lambda) \leftarrow \overline{Q}(\lambda)$, $R_f(\lambda) \leftarrow \overline{R}_f(\lambda)$ and $R_w(\lambda) \leftarrow \overline{R}_w(\lambda)$ can be easily read out.

To compute at Step 5 a suitable $M(\lambda)$ which guarantees that the norm of $\mathcal{R}(\lambda)$ in (11) is finite and the resulting final detector is proper and stable we can solve proper and stable factorizations problems for each row of the compound TFM $[\,[\widehat{F}_1(\lambda)]_+ Q(s) \;\; \widehat{F}_2(\lambda)\,]$ for which we can build immediately descriptor state space realizations. Suitable state-space algorithms for this purpose are described in [9].

V. Illustrative example

We consider the robust actuator fault detection and isolation example of [13]. The fault system (1) has a standard state space realization (15) with $E = I$ and

$$A(\delta_1,\delta_2) = \left[\begin{array}{ccc} -0.8 & 0 & 0 \\ 0 & -0.5(1+\delta_1) & 0.6(1+\delta_2) \\ 0 & -0.6(1+\delta_2) & -0.5(1+\delta_1) \end{array} \right]$$

$$B_u = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{array} \right],\; B_d = 0,\;\; B_f = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{array} \right],\; C = \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right]$$

$$D_u = 0,\; D_d = 0,\; D_f = 0.$$

In the expression of $A(\delta_1,\delta_2)$, δ_1 and δ_2 are uncertainties in the real and imaginary parts of the two complex conjugated eigenvalues $\lambda_{1,2} = -0.5 \pm j0.6$ of the nominal value $A(0,0)$. The fault detector filter is aimed to provide robust fault detection and isolation of actuator faults in the presence of these parametric uncertainties.

We reformulate the problem by assimilating δ_1 and δ_2 with fictitious noise inputs. We take A in (15) simply as the nominal value $A(0,0)$ and additionally define

$$B_w = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{array} \right], \quad D_w = 0.$$

At *Step 1* we choose as initial detector

$$Q(s) = [\, I \;\; -G_u(s) \,] = \left[\begin{array}{c|cc} A - sI & 0 & -B_u \\ \hline C & I & -D_u \end{array} \right]$$

and the corresponding $R_f(s)$ and $R_w(s)$ are simply $R_f(s) = G_f(s)$ and $R_w(s) = G_w(s)$. Since $R_f(s)$ is invertible, Step 2 is skipped.

Note that $[\, R_f(s) \;\; R_w(s) \,]$ has two zeros at infinity. At Step 3 we compute the quasi-co-outer-inner factorization of $[\, R_f(s)\; R_w(s) \,]$. The resulting realization of $G_{o,1}$ in (18) has the matrices

$$\overline{B}_0 = \left[\begin{array}{cc} -1.2405 & -0.2781 \\ -1.2052 & 0.1402 \\ -0.3603 & -1.3850 \end{array} \right], \quad \overline{D}_0 = 0$$

As expected, $G_{o,1}(s)$ has also two zeros at infinity and a stable zero at -1.1336. This stable zero is also the only pole of the 4×4 inner factor $G_i(s)$.

The descriptor realization of the updated $Q(s)$ is

$$Q(s) = \left[\begin{array}{cc|cc} A - sI & \overline{B}_o & 0 & -B_u \\ C & \overline{D}_o & I & -D_u \\ \hline 0 & -I & 0 & 0 \end{array} \right]$$

While the updated detector $Q(s)$ is improper (having two infinite poles), the updated $R_f(s)$ and $R_w(s)$ can alternatively be expressed as

$$[\, R_f(s)\; R_w(s) \,] = [\, I \;\; 0 \,] \left[\begin{array}{c} G_{i,1}(s) \\ G_{i,2}(s) \end{array} \right] = G_{i,1}(s)$$

and therefore have minimal realizations which are stable standard systems (as parts of the inner factor).

With $M_r(s) = I_2$, we compute $\widehat{F}_1(s)$ and $\widehat{F}_2(s)$ as

$$\begin{array}{rcl} [\, \widehat{F}_1(s) \;\; \widehat{F}_2(s) \,] & = & [\, I \;\; 0 \,][\, G_{i,1}^*(s) \;\; G_{i,2}^*(s) \,] \\ & = & \left[\begin{array}{c|cc} \widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\ \hline \widehat{C} & \widehat{D}_1 & \widehat{D}_2 \end{array} \right] \end{array}$$

where

$$\widehat{A} = 1.134, \quad \widehat{C} = \left[\begin{array}{c} 0.0623 \\ 0.7413 \end{array} \right],$$

$$\widehat{B}_1 = \left[\begin{array}{cc} 0.04246 & 0.5032 \end{array} \right], \quad \widehat{B}_2 = \left[\begin{array}{cc} -0.7575 & -1.523 \end{array} \right],$$

$$\widehat{D}_1 = \left[\begin{array}{cc} -0.8314 & 0.2423 \\ -0.3914 & -0.3112 \end{array} \right], \quad \widehat{D}_2 = \left[\begin{array}{cc} 0.4477 & -0.2226 \\ -0.7625 & -0.4105 \end{array} \right]$$

Both $\widehat{F}_1(s)$ and $\widehat{F}_2(s)$ are represented by first order systems with an unstable eigenvalue at 1.1336.

We choose $M(s)$ at Step 5 of the form

$$M(s) = K \left[\begin{array}{cc} \dfrac{10}{s+10} & 0 \\ 0 & \dfrac{10}{s+10} \end{array} \right]$$

where K is a scaling matrix. For this example, we determined K to ensure that the resulting DC-gain of the TFM from faults to residuals is the identity matrix. The expression of the detector $Q(s)$ can be written down explicitly as

$$Q(s) = M(s) \widehat{D}_1 G_{o,1}^{-1}(s) [\, I \;\; -G_u(s) \,]$$

which has a standard system realization of order 3. Note that the orders of the realizations of the individual factors are respectively 2, 0, 5, and 3, which sum together to 10. The resulting low order (in fact the least possible order) clearly illustrates the advantage of the integrated algorithm, which allows, via explicitly computable realizations, to obtain at each step least order representations of the detector. In this way, performing repeated minimal realizations can be completely avoided.

For completeness, we give the resulting state-space representation of the detector

$$Q(s) = \left[\begin{array}{c|c} A_Q - sI & B_Q \\ \hline C_Q & D_Q \end{array}\right]$$

with

$$A_Q = \begin{bmatrix} -10.0147 & -0.4346 & -3.0643 \\ -0.0057 & -10.1691 & -1.1925 \\ 0.0433 & 1.2836 & -0.9498 \end{bmatrix}$$

$$B_Q = \begin{bmatrix} -2.4464 & 0.4409 & -0.1912 & -0.3260 \\ -1.6712 & 3.6086 & 0.5794 & 0.1533 \\ -0.1336 & -0.3443 & -0.0512 & 0.3378 \end{bmatrix}$$

$$C_Q = \begin{bmatrix} -9.6340 & -21.1930 & -5.3805 \\ 36.3688 & 13.7607 & 12.5546 \end{bmatrix}$$

$$D_Q = \begin{bmatrix} -6.3099 & 8.2359 & 0 & 0 \\ 11.3898 & -5.8839 & 0 & 0 \end{bmatrix}$$

The corresponding residual norm is $\|\mathcal{R}(\lambda)\|_2 = 7.9203$.

In Figure 2 we present the results of a Monte Carlo analysis of step responses of the parameter dependent residual generation system (of the form (3)) from the fault and control inputs for 100 random samples of δ_1 and δ_2 in the range $[-0.25, 0.25]$. The simulations have been performed using the original parameter uncertain state-space model. As it can be observed, with an appropriate choice of the detection threshold, the detection and isolation of constant faults can be reliably performed in the presence of parametric uncertainties.

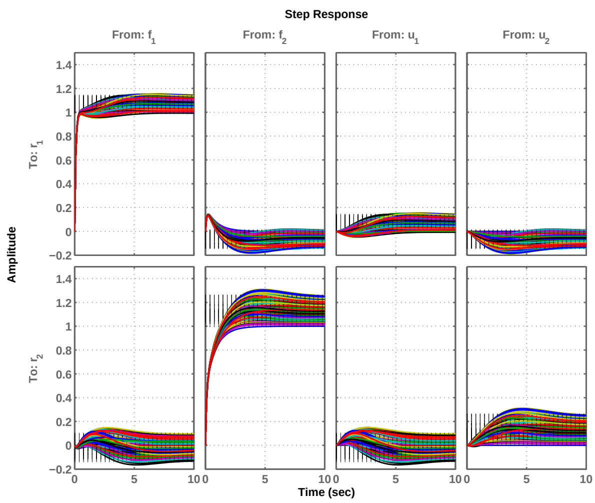

Fig. 2. Monte Carlo step response analysis of detector robustness for uncertainties in the real and imaginary parts of the complex conjugated eigenvalues

VI. Conclusions

We proposed a general computational approach to solve the $\mathcal{H}_2$-norm optimal FDI filter design problem. The new approach reformulates the filter design problem as an equivalent model matching problem for which an integrated algorithm is proposed which is able to solve this problem in the most general setting. In this way, the technical difficulties often encountered by the existing methods when trying to reduce the approximation problems to standard $\mathcal{H}_2$-norm synthesis problems are completely avoided. For example, the presence of zeros or poles on the boundary of stability domains or problems with non-full rank and even improper transfer-function matrices can be easily handled. The underlying main computational algorithms are based on descriptor system representations and rely on orthogonal matrix pencil reductions. For all basic computations, reliable numerical software tools are available for MATLAB in the Descriptor Systems Toolbox [14] and in the current version of the FAULT DETECTION Toolbox [15], [16]. The proposed algorithm represents an integrated alternative approach to the exact synthesis method proposed in [8].

References

[1] J. Chen and R. J. Patton. *Robust Model-Based Fault Diagnosis for Dynamic Systems*. Kluwer Academic Publishers, London, 1999.

[2] M. Blanke, M. Kinnaert, J. Lunze, and M. Staroswiecki. *Diagnosis and Fault-Tolerant Control*. Springer-Verlag, Berlin, 2003.

[3] P. M. Frank and X. Ding. Frequency domain approach to optimally robust residual generation and evaluation for model-based fault diagnosis. *Automatica*, 30:789–804, 1994.

[4] A. Varga. On computing nullspace bases – a fault detection perspective. *Proc. IFAC 2008 World Congress, Seoul, Korea.*, pp. 6295–6300, 2008.

[5] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1996.

[6] H. Niemann and J. Stoustrup. Design of fault detectors using H_∞ optimization. *Proc. of CDC'00, Sydney, Australia*, pp. 4237–4238, 2000.

[7] A. Varga. Numerically reliable methods for optimal design of fault detection filters. *Proc. of CDC'05, Seville, Spain*, 2005.

[8] A. Varga. New computational approach for the design of fault detection and isolation filters. In M. Voicu, editor, *Advances in Automatic Control*, vol. 754 of *The Kluwer International Series in Engineering and Computer Science*, pp. 367–381, Kluwer Academic Publishers, 2004.

[9] A. Varga. Computation of coprime factorizations of rational matrices. *Lin. Alg. & Appl.*, 271:83–115, 1998.

[10] C. Oară and A. Varga. Computation of general inner-outer and spectral factorizations. *IEEE Trans. Automat. Control*, 45:2307–2325, 2000.

[11] C. Oară. Constructive solutions to spectral and inner-outer factorizations with respect to the disk. *Automatica*, 41:1855 – 1866, 2005.

[12] A. Varga. Computation of least order solutions of linear rational equations. *Proc. of MTNS'04, Leuven, Belgium*, 2004.

[13] A. Edelmayer and J. Bokor. Optimal $\mathcal{H}_\infty$ scaling for sensitivity optimization of detection filters. *Int. J. Robust Nonlinear Control*, 12:749–760, 2002.

[14] A. Varga. A DESCRIPTOR SYSTEMS toolbox for MATLAB. *Proc. CACSD'2000 Symposium, Anchorage, Alaska*, 2000.

[15] A. Varga. A FAULT DETECTION toolbox for MATLAB. *Proc. of CACSD'06, Munich, Germany*, 2006.

[16] A. Varga. Linear FDI-Techniques and Software Tools. FAULT DETECTION Toolbox V0.9 – Technical Documentation, German Aerospace Center (DLR), Institute of Robotics and Mechatronics, 2009.