On computing normalized coprime factorizations of periodic systems

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***Abstract*— A numerically reliable state space algorithm is proposed for computing normalized coprime factorizations of periodic descriptor systems. A preprocessing step is used in the algorithm to convert the initial problem for possibly non-causal systems into a simpler problem for causal periodic systems. The main computational ingredient here is the computation of a coprime factorization with causal factors which is addressed by computing right annihilators of an appropriately extended system via periodic pencil manipulation algorithms. The solution of the normalized coprime factorization problem for a causal system involves the solution of a generalized periodic Riccati equation. The proposed two steps approach is completely general, being applicable to periodic systems with time-varying dimensions.**

I. Introduction

Normalized periodic coprime factorizations are important is several application domains. An important area is the robust stabilization of periodic systems under perturbations in the coprime factors. Here, the definition of suitable robustness measures using gap or ν-gap metric for periodic systems strongly relies on the use of normalized periodic coprime factorizations [1]. Another application area is the extension of model reduction methods for stable systems to unstable systems, as for example, by reducing the normalized coprime factors using balancing related methods [2].

A state-space algorithm to compute normalized periodic coprime factorizations for standard (causal) periodic systems with constant dimensions has been proposed in [3]. The main computation in this algorithm is the solution of an appropriate periodic Riccati equation to solve a spectral factorization problem. Computational algorithms for more general systems (e.g., non-causal and/or with time-varying dimensions) can be easily devised by using lifting-based approaches in conjunction with methods applicable to arbitrary descriptor systems [4], [5]. However, the excessive computational effort and the lack of structure exploiting/preserving make these approaches generally not recommendable.

In this paper we propose a numerically reliable state-space algorithm to compute normalized periodic coprime factorizations of discrete-time standard or descriptor periodic systems (both causal and non-causal). The proposed algorithm is lifting-free and is completely general being applicable to systems with time-varying state, input and output dimensions. The method can be seen both as an extension as well as an enhancement of the algorithm proposed in [4] for arbitrary descriptor systems. Similarly to the algorithm of [4], a preprocessing step is used to convert the initial problem for possibly non-causal systems into a simpler problem for causal periodic systems. The main computational ingredient here is the computation of a coprime factorization with causal factors which is addressed by computing right annihilators of an extended system. Periodic pencil manipulation algorithms using periodic orthogonal transformations underly this computational step. Next, the normalized coprime factorization problem for a causal system is solved, which involves the solution of a generalized periodic Riccati equation. The main computation here is the determination of a stable periodic deflating subspace of a certain periodic pencil, by using algorithms relying on orthogonal transformations. The numerical reliability of the overall computational procedure is guaranteed by the exclusive use of orthogonal transformations in both computational steps.

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II. Preliminaries

We consider linear periodic time-varying discrete-time descriptor systems of the form

$$\begin{array}{rcl} E_k x(k+1) & = & A_k x(k) + B_k u(k) \\ y(k) & = & C_k x(k) + D_k u(k) \end{array} \qquad (1)$$

where the matrices $E_k \in \mathbb{R}^{\nu_k \times n_{k+1}}$, $A_k \in \mathbb{R}^{\nu_k \times n_k}$, $B_k \in \mathbb{R}^{\nu_k \times m_k}$, $C_k \in \mathbb{R}^{p_k \times n_k}$, $D_k \in \mathbb{R}^{p_k \times m_k}$ are periodic with period $N \geq 1$. The periodic system (1) will be alternatively denoted by the periodic quintuple $\mathcal{S} := (E_k, A_k, B_k, C_k, D_k)$. The vector of state-dimensions $\mathbf{n} = [\, n_1,\ n_2,\ \cdots, n_N \,]$ characterizes the state-space order of the periodic system. We use similar notations $\mathbf{m} = [\, m_1,\ m_2,\ \cdots, m_N \,]$ for the input dimensions and $\mathbf{p} = [\, p_1,\ p_2,\ \cdots, p_N \,]$ for the output dimensions. The triple $(\mathbf{n}, \mathbf{m}, \mathbf{p})$ describes compactly the state-space, input and output dimensions of the periodic system. We assume that $\sum_{k=1}^{N} \nu_k = \sum_{k=1}^{N} n_k$ is fulfilled. Periodic systems with time-varying state dimensions have been considered in several works (see for example [6], [7], [8], [9]), while periodic systems with time-varying input and output dimensions are common for the description of multirate periodic systems (see [1] and the literature cited therein). In what follows we summarize some notations and definitions for periodic systems used throughout the paper.

A. Causal periodic systems

The case of *causal systems*, when $\nu_k = n_{k+1}$ and E_k are invertible matrices, plays an important role in most of applications. For a causal system (1), we denote the *monodromy matrix* at time k by

$$\Psi_{E_k^{-1}A_k} := E_{k+N-1}^{-1} A_{k+N-1} E_{k+N-2}^{-1} A_{k+N-2} \cdots E_k^{-1} A_k.$$

The eigenvalues of $\Psi_{E_k^{-1}A_k}$ are called *characteristic multipliers* of the periodic pair (E_k, A_k). Let $\mathbb{D}^-$ and $\mathbb{D}^+$ be two open subsets of the complex plane $\mathbb{C}$ representing the interior and the exterior of the unit disk centered in the origin, respectively. We say that $\mathcal{S}$ is *stable* (or equivalently the periodic pair (E_k, A_k) is *stable*) if all characteristic multipliers have moduli less than one (i.e., $\Lambda(\Psi_{E_k^{-1}A_k}) \subset \mathbb{D}^-$). Note that $\Psi_{E_k^{-1}A_k}$ has always $n_k - \underline{n}$ null eigenvalues, where $\underline{n} := \min\{n_k\}$. The rest of $\underline{n}$ eigenvalues of $\Psi_{E_k^{-1}A_k}$ form the *core* characteristic multipliers and are the same for all values of k. For *non-causal systems* (e.g., with E_k singular or even non-square), similar notions like finite and infinite characteristic multipliers can be defined using lifting-based representations (see for example [7]).

The definitions used for *reachability/stabilizability* and *observability/detectability* of causal periodic descriptor systems are those of [6], which also apply for the more general non-causal case. For causal systems, we will use the following more intuitive definitions of stabilizability and detectability. The causal periodic system $\mathcal{S}$ is *stabilizable* if there exists a periodic F_k of appropriate dimensions such that the periodic pair $(E_k, A_k + B_kF_k)$ is *stable*. Similarly, the causal periodic system $\mathcal{S}$ is *detectable* if there exists a periodic L_k of appropriate dimensions such that the periodic pair $(E_k, A_k + L_kC_k)$ is *stable*.

B. Similarity transformations

Two periodic systems $\mathcal{S} := (E_k, A_k, B_k, C_k, D_k)$ and $\widetilde{\mathcal{S}} := (\widetilde{E}_k, \widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, D_k)$ are called *similar* if the matrices of their state-space representations are related by

$$\widetilde{E}_k = U_kE_kV_{k+1}, \widetilde{A}_k = U_kA_kV_k, \widetilde{B}_k = U_kB, \widetilde{C}_k = C_kV_k,$$

with U_k and V_k N-periodic nonsingular matrices. Two similar systems have the same input-output map.

C. System couplings

The *series coupling* of two periodic systems $\mathcal{S}_1 = (E_k^{(1)}, A_k^{(1)}, B_k^{(1)}, C_k^{(1)}, D_k^{(1)})$ with dimensions $(\mathbf{n}_1, \mathbf{m}_1, \mathbf{p}_1)$ and $\mathcal{S}_2 = (E_k^{(2)}, A_k^{(2)}, B_k^{(2)}, C_k^{(2)}, D_k^{(2)})$ with dimensions $(\mathbf{n}_2, \mathbf{m}_2, \mathbf{m}_1)$ we denote with $\mathcal{S}_1 \star \mathcal{S}_2 := \left(\widehat{E}_k, \widehat{A}_k, \widehat{B}_k, \widehat{C}_k, \widehat{D}_k\right)$ and has a realization of order $\mathbf{n}_1 + \mathbf{n}_2$ with the state-space matrices

$$\widehat{E}_k = \begin{bmatrix} E_k^{(1)} & 0 \\ 0 & E_k^{(2)} \end{bmatrix}, \quad \widehat{A}_k = \begin{bmatrix} A_k^{(1)} & B_k^{(1)}C_k^{(2)} \\ 0 & A_k^{(2)} \end{bmatrix},$$

$$\widehat{B}_k = \begin{bmatrix} B_k^{(1)}D_k^{(2)} \\ B_k^{(2)} \end{bmatrix}, \quad \widehat{C}_k = \begin{bmatrix} C_k^{(1)} & D_k^{(1)}C_k^{(2)} \end{bmatrix},$$

$$\widehat{D}_k = D_k^{(1)}D_k^{(2)}$$

The *parallel coupling* of two periodic systems $\mathcal{S}_1$ and $\mathcal{S}_2$ we denote by $\mathcal{S}_1 \oplus \mathcal{S}_2 = \left(\widehat{E}_k, \widehat{A}_k, \widehat{B}_k, \widehat{C}_k, \widehat{D}_k\right)$ and has a realization of order $\mathbf{n}_1 + \mathbf{n}_2$ with the state-space matrices

$$\widehat{E}_k = \begin{bmatrix} E_k^{(1)} & 0 \\ 0 & E_k^{(2)} \end{bmatrix}, \quad \widehat{A}_k = \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & A_k^{(2)} \end{bmatrix},$$

$$\widehat{B}_k = \begin{bmatrix} B_k^{(1)} \\ B_k^{(2)} \end{bmatrix}, \quad \widehat{C}_k = [\, C_k^{(1)} \; C_k^{(2)} \,], \quad \widehat{D}_k = D_k^{(1)} + D_k^{(2)}$$

We denote by $[\, \mathcal{S}_1 \;\; \mathcal{S}_2 \,]$ and $\begin{bmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{bmatrix}$ the input and output concatenated couplings, respectively, which have obvious state-space representations. For example, $[\, \mathcal{S}_1 \;\; \mathcal{S}_2 \,] := \left(\widehat{E}_k, \widehat{A}_k, \widehat{B}_k, \widehat{C}_k, \widehat{D}_k\right)$ has the realization

$$\widehat{E}_k = \begin{bmatrix} E_k^{(1)} & 0 \\ 0 & E_k^{(2)} \end{bmatrix}, \quad \widehat{A}_k = \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & A_k^{(2)} \end{bmatrix},$$

$$\widehat{B}_k = \begin{bmatrix} B_k^{(1)} & 0 \\ 0 & B_k^{(2)} \end{bmatrix}, \quad \widehat{C}_k = \begin{bmatrix} C_k^{(1)} & C_k^{(2)} \end{bmatrix},$$

$$\widehat{D}_k = \begin{bmatrix} D_k^{(1)} & D_k^{(2)} \end{bmatrix}$$

D. Inversion and conjugation

The inverse of $\mathcal{S} = (E_k, A_k, B_k, C_k, D_k)$ for invertible D_k is

$$\mathcal{S}^{-1} = (E_k, A_k - B_kD_k^{-1}C_k, B_kD_k^{-1}, -D_k^{-1}C_k, D_k^{-1}) \quad (2)$$

and has order $\mathbf{n}$. More general inverses of periodic systems are discussed in [9].

The conjugate system $\mathcal{S}^\sim$ is defined as $\mathcal{S}^\sim = (\widetilde{E}_k, \widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k)$, where (see [10])

$$\widetilde{E}_k = \begin{bmatrix} A_k^T & 0 \\ -B_k^T & 0 \end{bmatrix}, \quad \widetilde{A}_k = \begin{bmatrix} E_{k-1}^T & 0 \\ 0 & -I_{m_k} \end{bmatrix},$$

$$\widetilde{B}_k = \begin{bmatrix} -C_k^T \\ D_k^T \end{bmatrix}, \quad \widetilde{C}_k = \begin{bmatrix} 0 & I_{m_k} \end{bmatrix}, \quad \widetilde{D}_k = 0$$

If the periodic pair (E_k, A_k) has a constant state dimension and A_k is invertible, then an equivalent causal descriptor realization of $\mathcal{S}^\sim$ is given by

$$\mathcal{S}^\sim = (A_k^T, E_{k-1}^T, -C_k^T, B_k^TA_k^{-T}E_{k-1}^T, D_k^T - B_k^TA_k^{-T}C_k^T)$$

E. Lifted representation

We can formulate the factorization problems addressed in this paper and interpret the obtained results in terms of the *transfer-function matrix* (TFM) corresponding to the associated *stacked lifted representation* of [11], which uses the input-state-output behavior of the system over time intervals of length N, rather then 1. The lifted input, output and state vectors are defined as

$$\begin{aligned} \overline{u}_k(h) &= [u^T(hN+k)\cdots u^T(hN+k+N-1)]^T, \\ \overline{y}_k(h) &= [y^T(hN+k)\cdots y^T(hN+k+N-1)]^T, \\ \overline{x}_k(h) &= [x^T(hN+k)\cdots x^T(hN+k+N-1)]^T \end{aligned}$$

and have the dimensions $M := \sum_{i=1}^N m_i$, $P := \sum_{i=1}^N p_i$, and $\sum_{i=1}^N n_i$, respectively. The corresponding lifted system can be represented by a time-invariant discrete-time descriptor system of the form

$$\begin{aligned} \overline{E}_k^S\overline{x}_k(h+1) &= \overline{A}_k^S\overline{x}_k(h) + \overline{B}_k^S\overline{u}_k(h) \\ \overline{y}_k(h) &= \overline{C}_k^S\overline{x}_k(h) + \overline{D}_k^S\overline{u}_k(h) \end{aligned} \quad (3)$$

where

$$\overline{A}_k^S - z\overline{E}_k^S = \begin{bmatrix} A_k & -E_k & O & \cdots & O \\ O & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -E_{k+N-3} & O \\ O & & & \ddots & A_{k+N-2} & -E_{k+N-2} \\ -zE_{k+N-1} & O & \cdots & O & A_{k+N-1} \end{bmatrix} \quad (4)$$

is the *pole pencil* of the periodic pair (A_k, E_k) and

$$\begin{aligned} \overline{B}_k^S &= \text{diag}\{B_k, \ldots, B_{k+N-1}\}, \\ \overline{C}_k^S &= \text{diag}\{C_k, \ldots, C_{k+N-1}\}, \\ \overline{D}_k^S &= \text{diag}\{D_k, \ldots, D_{k+N-1}\}. \end{aligned}$$

We assume throughout the paper that $\overline{A}_k^S - z\overline{E}_k^S$ is a *regular* pencil. The $P \times M$ TFM of the lifted-system corresponding to $\mathcal{S}$ we denote by

$$G_k^{\mathcal{S}}(z) = \overline{C}_k^S (z\overline{E}_k^S - \overline{A}_k^S)^{-1} \overline{B}_k^S + \overline{D}_k^S$$

It can be shown that the TFM $G_k^{\mathcal{S}}(z)$ of a causal system (i.e., E_k invertible) belongs to a *special* type of proper TFMs for which $G_k^{\mathcal{S}}(\infty)$ is finite and has a lower block-triangular structure, with the j-th diagonal block of the form $p_{k+j-1} \times m_{k+j-1}$. Only proper TFMs with this *special* property correspond to *causal* periodic systems.

The previously defined operations with periodic systems: series coupling, parallel coupling, inversion (2), or conjugation, can easily be expressed in terms of TFMs as follows:

$$\begin{aligned} G_k^{\mathcal{S}_1 \star \mathcal{S}_2}(z) &= G_k^{\mathcal{S}_1}(z) G_k^{\mathcal{S}_2}(z), \\ G_k^{\mathcal{S}_1 \oplus \mathcal{S}_2}(z) &= G_k^{\mathcal{S}_1}(z) + G_k^{\mathcal{S}_2}(z), \\ G_k^{\mathcal{S}^{-1}}(z) &= (G_k^{\mathcal{S}}(z))^{-1}, \\ G_k^{\mathcal{S}^{\sim}}(z) &= (G_k^{\mathcal{S}}(1/z))^T. \end{aligned}$$

In what follows we assume that the lifting of input and output concatenated systems is done by preserving the original separation of the inputs and outputs (instead of lifting jointly the inputs and outputs). This has the important consequence that the lifted TFM of an input concatenated system $[\,\mathcal{S}_1 \ \mathcal{S}_2\,]$ can be built by concatenating the individual lifted TFMs column-wise, i.e.

$$G_k^{[\,\mathcal{S}_1 \ \mathcal{S}_2\,]}(z) = [\,G_k^{\mathcal{S}_1}(z) \ \ G_k^{\mathcal{S}_2}(z)\,]$$

Similarly, the lifted TFM of output concatenated systems can be built by stacking the individual lifted TFMs row-wise.

III. Normalized coprime factorizations

In this paper we consider the computation of a right factorized representation of the periodic system $\mathcal{S}$ of the form

$$\mathcal{S} = \mathcal{N} \star \mathcal{M}^{-1} \quad (5)$$

or of a left factorized representation of $\mathcal{S}$ of the form

$$\mathcal{S} = \mathcal{M}^{-1} \star \mathcal{N} \quad (6)$$

where $\mathcal{M}$ and $\mathcal{N}$ are periodic systems with certain desirable properties (e.g., causality, stability). The lifted TFMs corresponding to the factorized representations (5) and (6) can be expressed similarly as

$$G_k^{\mathcal{S}}(z) = G_k^{\mathcal{N}}(z) \left(G_k^{\mathcal{M}}(z)\right)^{-1}$$

or

$$G_k^{\mathcal{S}}(z) = \left(G_k^{\mathcal{M}}(z)\right)^{-1} G_k^{\mathcal{N}}(z),$$

respectively.

A factorized representation $\mathcal{S} = \mathcal{N} \star \mathcal{M}^{-1}$ with $\mathcal{N}$ and $\mathcal{M}$ stable, is called a *periodic right coprime factorization* (PRCF) if there exist stable *special* proper TFMs $U_k(z)$ and $V_k(z)$ such that $U_k(z)G_k^{\mathcal{N}}(z) + V_k(z)G_k^{\mathcal{M}}(z) = I_M$. Analogously, a factorized representation of $\mathcal{S}$ in the form $\mathcal{S} = \mathcal{M}^{-1} \star \mathcal{N}$ with $\mathcal{N}$ and $\mathcal{M}$ stable, is called a *periodic left coprime factorization* (PLCF) if there exist stable *special* proper TFMs $U_k(z)$ and $V_k(z)$ such that $G_k^{\mathcal{N}}(z)U_k(z) + G_k^{\mathcal{M}}(z)V_k(z) = I_P$.

For non-causal periodic systems coprime factorizations with causal factors can be used to turn computational problems for non-causal system into simpler problems for causal systems. In this case, the only requirement on the factors is causality and the corresponding factorizations are called *causal*.

A special class of stable coprime factorizations are the *normalized periodic right coprime factorization* (NPRCF) and the *normalized periodic left coprime factorization* (NPLCF), when the output or input concatenated factors (i.e., $\begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix}$ and $[\,\mathcal{N} \ \mathcal{M}\,]$) form inner or co-inner systems, i.e. satisfy

$$(\mathcal{N}^{\sim} \star \mathcal{N}) \oplus (\mathcal{M}^{\sim} \star \mathcal{M}) = \mathcal{I}_{\mathbf{m}},$$

or

$$(\mathcal{N} \star \mathcal{N}^{\sim}) \oplus (\mathcal{M} \star \mathcal{M}^{\sim}) = \mathcal{I}_{\mathbf{p}},$$

respectively. Here, $\mathcal{I}_{\mathbf{m}}$ and $\mathcal{I}_{\mathbf{p}}$ denote the identity input-output maps with the respective dimensions $(0, \mathbf{m}, \mathbf{m})$ and $(0, \mathbf{p}, \mathbf{p})$. In terms of the corresponding lifted TFMs we can express the above conditions as

$$[G_k^{\mathcal{N}}(1/z)]^T G_k^{\mathcal{N}}(z) + [G_k^{\mathcal{M}}(1/z)]^T G_k^{\mathcal{M}}(z) = I_M$$

and

$$G_k^{\mathcal{N}}(z)[G_k^{\mathcal{N}}(1/z)]^T + G_k^{\mathcal{M}}(z)[G_k^{\mathcal{M}}(1/z)]^T = I_P$$

respectively.

For the computation of a NPRCF we can use the following conceptually simple computational approach (inspired from [4]):

NPRCF Procedure.

1. Compute a *causal* PRCF $\mathcal{S} = \mathcal{N}_1 \star \mathcal{M}_1^{-1}$ such that both $\mathcal{N}_1$ and $\mathcal{M}_1$ are causal periodic systems.
2. Solve for $\mathcal{S}_o$ the causal spectral factorization problem $(\mathcal{N}_1^{\sim} \star \mathcal{N}_1) \oplus (\mathcal{M}_1^{\sim} \star \mathcal{M}_1) = \mathcal{S}_o^{\sim} \star \mathcal{S}_o$.
3. Compute $\mathcal{M} = \mathcal{M}_1 \star \mathcal{S}_o^{-1}$ and $\mathcal{N} = \mathcal{N}_1 \star \mathcal{S}_o^{-1}$.

The same procedure can also be used to compute NPLCFs by applying it to a *dual* system representation defined by the quintuple

$$(\widetilde{E}_k^T, A_{N+1-k}^T, C_{N+1-k}^T, B_{N+1-k}^T, D_{N+1-k}^T),$$

where $\widetilde{E}_N = E_N^T$ and $\widetilde{E}_k = E_{N-k}^T$, for $k = 1, \ldots, N-1$. The factors of the NPLCF can be recovered from their dual right factors, using the above dualization formulas. Therefore, in this paper we will address only the computation of NPRCFs.

For the computation of the NPRCF we can in principle employ general computational methods for standard or descriptor systems [4], [5] to determine the TFMs $G_k^{\mathcal{N}}(z)$ and $G_k^{\mathcal{M}}(z)$ of which we can recover the state-space realizations via existing periodic state space realization algorithms [12]. The method proposed in [4] is essentially the same as the **NPRCF Procedure** and uses at Step 1 proper factorization techniques developed in [13] to determine *right coprime factorizations* (RCFs) with proper factors. The proper factorization algorithm uses pole assignment techniques to move infinite poles to finite locations and rezidualization formulas involving matrix inversions to arrive to a standard state-space realization of the factors. Since these computations are performed as the first step of the overall algorithm, potential accuracy losses can result in the case when ill-conditioned pole assignment or inversion problems are encountered.

The method of [5] is a straightforward specialization of the general algorithm of [14] to the computation of the inner-outer factorization of the TFM of the left-invertible system

$$\begin{bmatrix} G_k^{\mathcal{S}}(z) \\ I_M \end{bmatrix} = G_k^{\mathcal{S}_i}(z) G_k^{\mathcal{S}_o}(z) := \begin{bmatrix} G_k^{\mathcal{N}}(z) \\ G_k^{\mathcal{M}}(z) \end{bmatrix} (G_k^{\mathcal{M}}(z))^{-1}$$

Thus, the factors of the normalized RCF can be simply read out from the ressulting inner factor $G_k^{\mathcal{S}_i}(z)$. Since the overall computation employs exclusively orthogonal transformations, this approach can be considered a satisfactory numerical method for computing normalized coprime factorizations.

There are however strong reasons not to use lifting-based approaches [15]. One reason is the excessive computational effort, which instead of the acceptable effort of $O(N\bar{n}^3)$ floating point operations ($\bar{n} = \max_k n_k$), requires $O(N^3\bar{n}^3)$ operations. The second reason is that using general purpose algorithms, the underlying structure (see for example (4)) is not exploited and not preserved. Therefore, computations done directly on the periodic system matrices are desirable both to keep the computational burden at an acceptable level, as well as to preserve the underlying structures. For the computation of the NPRCF for causal systems with constant dimensions and $E_k = I$ such an algorithm has been already proposed in [3]. This corresponds to Step 2 of **NPRCF Procedure**, where the basic computation is the solution of an appropriate periodic Riccati equation to solve a spectral factorization problem.

In this paper we describe a new algorithm to compute NPRCFs which extends the approach of [3] in two aspects. Firstly, problems with time-varying dimensions are allowed, and secondly, non-causal systems can be handled. In the rest of the paper we present the details of the algorithms used at the successive steps of the **NPRCF Procedure.** At Step 1, a preprocessing is performed to convert the initial problem for possibly non-causal systems into a simpler problem for standard causal periodic systems by means of a causal periodic coprime factorization. The algorithm for this purpose is based on the computation of a periodic Kronecker-like form of the periodic system pencil which allows to obtain right periodic annihilators. The main difference to the spirit of the approach of [4] is the exclusive use in our algorithm of orthogonal transformations to reduce the periodic system pencil to appropriate condensed forms. Therefore the algorithm for the computation of causal factorizations is numerically stable. At Step 2, the solution of a periodic spectral factorization problem involves the computation of the periodic stabilizing solution of a periodic Riccati equation. General algorithms for this purpose, applicable also to problems with time-varying dimensions, have been proposed in [16].

IV. NPRCF OF A CAUSAL SYSTEM

Consider a causal system $\mathcal{S} = (E_k, A_k, B_k, C_k, D_k)$ with E_k square and invertible. We assume that this realization of $\mathcal{S}$ is stabilizable and detectable. When performing the **NPRCF Procedure**, we make at Step 1 the trivial choice $\mathcal{N}_1 = \mathcal{S}$ and $\mathcal{M}_1 = \mathcal{I}_{\mathbf{m}}$. However, as we will show at the end of this section, with minor modifications the obtained results can also be used to compute the spectral factorization at Step 2 of the **NPRCF Procedure** for arbitrary causal PRCFs.

Using straightforward manipulations, the periodic state space realization of $(\mathcal{S}^{\sim} \star \mathcal{S}) \oplus \mathcal{I}_{\mathbf{m}}$ can be represented as $(\widetilde{E}_k, \widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k)$, where

$$\widetilde{E}_k = \begin{bmatrix} E_k & 0 & 0 \\ 0 & A_k^T & 0 \\ 0 & -B_k^T & 0 \end{bmatrix}, \quad \widetilde{A}_k = \begin{bmatrix} A_k & 0 & 0 \\ -C_k^T C_k & E_{k-1}^T & 0 \\ D_k^T C_k & 0 & -I_{m_k} \end{bmatrix}$$

$$\widetilde{B}_k = \begin{bmatrix} B_k \\ -C_k^T D_k \\ R_k \end{bmatrix}, \quad \widetilde{C}_k = \begin{bmatrix} 0 & 0 & I_{m_k} \end{bmatrix}, \quad \widetilde{D}_k = 0,$$

where $R_k = I + D_k^T D_k$. The following result represents the extension of Proposition 2 of [4] to causal periodic discrete-time descriptor systems with time-varying dimensions.

Proposition 1: For the stabilizable and detectable system $\mathcal{S} = (E_k, A_k, B_k, C_k, D_k)$, let X_k be the symmetric periodic stabilizing solution of the *generalized periodic discrete-time algebraic Riccati equation* (GPDARE)

$$\begin{aligned} E_{k-1}^T X_k E_{k-1} = A_k^T X_{k+1} A_k - (A_k^T X_{k+1} B_k + C_k^T D_k) \\ \times (R_k + B_k^T X_{k+1} B_k)^{-1} (B_k^T X_{k+1} A_k + D_k^T C_k) + C_k^T C_k \end{aligned} \quad (7)$$

and let F_k be the corresponding stabilizing periodic feedback matrix

$$F_k = -(R_k + B_k^T X_{k+1} B_k)^{-1} (B_k^T X_{k+1} A_k + D_k^T C_k). \quad (8)$$

Then, the spectral factor $\mathcal{S}_o$ has the periodic state space realization

$$\mathcal{S}_o = (E_k, A_k, B_k, -H_kF_k, H_k)$$

with

$$H_k^TH_k = R_k + B_k^TX_kB_k \qquad (9)$$

and $(\mathcal{S}^{\sim}\star\mathcal{S})\oplus\mathcal{I}_{\mathbf{m}}$ and $\mathcal{S}_o^{\sim}\star\mathcal{S}_o$ have the same input-output map (e.g., the same lifted TFM) . The factors $\mathcal{M} = \mathcal{S}_o^{-1}$ and $\mathcal{N} = \mathcal{S}\star\mathcal{S}_o^{-1}$ of the NPRCF are stable and can be expressed as

$$\begin{aligned}\mathcal{N} &= (E_k, A_k + B_kF_k, B_kH_k^{-1}, C_k + D_kF_k, D_kH_k^{-1})\\ \mathcal{M} &= (E_k, A_k + B_kF_k, B_kH_k^{-1}, F_k, H_k^{-1})\end{aligned}$$

Proof. The assumption on stabilizability and detectability guarantees the existence of a unique stabilizing non-negative definite symmetric periodic solution X_k of the GPDARE (7). It is straightforward to show that with the above choice of F_k and H_k the periodic realization of

$$\mathcal{S}_o^{\sim}\star\mathcal{S}_o = (\widehat{E}_k, \widehat{A}_k, \widehat{B}_k, \widehat{C}_k, 0)$$

with

$$\widetilde{E}_k = \begin{bmatrix} E_k & 0 & 0\\ 0 & A_k^T & 0\\ 0 & -B_k^T & 0\end{bmatrix},\ \widehat{A}_k = \begin{bmatrix} A_k & 0 & 0\\ -F_k^TH_k^TH_kF_k & E_{k-1}^T & 0\\ -H_k^TH_kF_k & 0 & -I_{m_k}\end{bmatrix}$$

$$\widehat{B}_k = \begin{bmatrix} B_k\\ F_k^TH_k^TH_k\\ H_k^TH_k\end{bmatrix},\quad \widehat{C}_k = \begin{bmatrix} 0 & 0 & I_{m_k}\end{bmatrix}$$

is similar to the periodic realization of

$$(\mathcal{S}^{\sim}\star\mathcal{S})\oplus\mathcal{I}_{\mathbf{m}} = (\widetilde{E}_k, \widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, 0)$$

To show this, we choose suitable periodic invertible U_k and V_k to check that

$$\widehat{E}_k = U_k\widetilde{E}_kV_{k+1},\ \widehat{A}_k = U_k\widetilde{A}_kV_k,\ \widehat{B}_k = U_k\widetilde{B}_k,\ \widehat{C}_k = \widetilde{C}_kV_k$$

For X_k satisfying (7), the following choice is appropriate

$$U_k = \begin{bmatrix} I & 0 & 0\\ A_k^TX_{k+1} & I & 0\\ -B_k^TX_{k+1} & 0 & I\end{bmatrix},\quad V_k = \begin{bmatrix} I & 0 & 0\\ -X_kE_{k-1} & I & 0\\ 0 & 0 & I\end{bmatrix}$$

The realizations of the factors $\mathcal{N}$ and $\mathcal{M}$ can be obtained by straightforward manipulations and are stable, F_k being the stabilizing feedback corresponding to X_k. ■

For the numerical solution of the GPDARE (7), consider the periodic matrix pair (L_k, M_k) defined with

$$L_k := \begin{bmatrix} E_k & 0 & 0\\ 0 & A_k^T & 0\\ 0 & -B_k^T & 0\end{bmatrix},\ M_k := \begin{bmatrix} A_k & 0 & B_k\\ -C_k^TC_k & E_{k-1}^T & -C_k^TD_k\\ D_k^TC_k & 0 & R_k\end{bmatrix}$$

It is easy to check that the GPDARE (7) together with (8) allows to write

$$M_k\begin{bmatrix} I_{n_k}\\ X_kE_{k-1}\\ F_k\end{bmatrix} = L_k\begin{bmatrix} I_{n_{k+1}}\\ X_{k+1}E_k\\ F_{k+1}\end{bmatrix}\Theta_k$$

for $\Theta_k := E_k^{-1}(A_k + B_kF_k)$. It follows, that $\begin{bmatrix} I_{n_k} & (X_kE_{k-1})^T & F_k^T\end{bmatrix}^T$ can be interpreted as a stable periodic deflating subspace of dimension n_k of the periodic pair (L_k, M_k). Thus, the solution X_k of the GPDARE and the corresponding stabilizing feedback F_k can be computed by determining first an orthogonal basis V_k for the stable deflating subspace of the periodic pair (L_k, M_k) and then, after partitioning V_k compatibly with the column dimensions of M_k as

$$V_k = \begin{bmatrix} V_{k,1}\\ V_{k,2}\\ V_{k,3}\end{bmatrix},$$

compute X_k and F_k as

$$X_k = V_{k,2}(E_{k-1}V_{k,1})^{-1},\quad F_k = V_{k,3}V_{k,1}^{-1}.$$

Details of suitable computational approaches relying on exclusive use of orthogonal transformations are described in [16].

For the solution of the spectral factorization problem at Step 2, assume that $\mathcal{N}_1$ and $\mathcal{M}_1$ have the realizations

$$\begin{aligned}\mathcal{N}_1 &= (\overline{E}_k, \overline{A}_k, \overline{B}_k, \overline{C}_k, \overline{D}_k)\\ \mathcal{M}_1 &= (\overline{E}_k, \overline{A}_k, \overline{B}_k, \overline{K}_k, \overline{J}_k)\end{aligned}$$

We can solve the spectral factorization problem at Step 2 employing the obvious replacements

$$\begin{aligned}&E_k \leftarrow \overline{E}_k,\quad A_k \leftarrow \overline{A}_k,\quad B_k \leftarrow \overline{B}_k,\\ &C_k \leftarrow \begin{bmatrix}\overline{C}_k\\ \overline{K}_k\end{bmatrix},\quad D_k \leftarrow \begin{bmatrix}\overline{D}_k\\ \overline{J}_k\end{bmatrix},\\ &R_k \leftarrow \overline{D}_k^T\overline{D}_k + \overline{J}_k^T\overline{J}_k\end{aligned} \qquad (10)$$

in the GPDARE (7). The factors $\mathcal{M} = \mathcal{M}_1\star\mathcal{S}_o^{-1}$ and $\mathcal{N} = \mathcal{N}_1\star\mathcal{S}_o^{-1}$ of the NPRCF computed at Step 3 of the **NPRCF Procedure** are given by

$$\begin{aligned}\mathcal{N} &= (\overline{E}_k, \overline{A}_k + \overline{B}_k\overline{F}_k, \overline{B}_k\overline{H}_k^{-1}, \overline{C}_k + \overline{D}_k\overline{F}_k, \overline{D}_k\overline{H}_k^{-1})\\ \mathcal{M} &= (\overline{E}_k, \overline{A}_k + \overline{B}_k\overline{F}_k, \overline{B}_k\overline{H}_k^{-1}, \overline{K}_k + \overline{J}_k\overline{F}_k, \overline{J}_k\overline{H}_k^{-1})\end{aligned}$$

where $\overline{F}_k$ and $\overline{H}_k$ are computed according to (8) and (9), respectively, using the replacements in (10).

V. Computation of causal PRCFs

Consider a non-causal periodic system $\mathcal{S} = (E_k, A_k, B_k, C_k, D_k)$ and we assume it is detectable, i.e., the lifted descriptor pair $(\overline{A}_k^S - z\overline{E}_k^S, \overline{B}_k^S)$ has no unstable unobservable eigenvalues.

At Step 1 of the **NPRCF Procedure** a *causal periodic right coprime factorization* (CPRCF) $\mathcal{S} = \mathcal{N}_1\star\mathcal{M}_1^{-1}$ has to be computed, such that both factors $\mathcal{N}_1$ and $\mathcal{M}_1$ are causal periodic systems. To solve this problem, we can rewrite $\mathcal{S} = \mathcal{N}_1\star\mathcal{M}_1^{-1}$ as

$$[\,\mathcal{S}\ \ -\mathcal{I}_{\mathbf{p}}\,]\star\begin{bmatrix}\mathcal{M}_1\\ \mathcal{N}_1\end{bmatrix} = 0$$

Thus, we can determine the compound system

$$\mathcal{R} := \begin{bmatrix}\mathcal{M}_1\\ \mathcal{N}_1\end{bmatrix}$$

as a *maximal* right annihilator (i.e., with output dimension $\mathbf{p}+\mathbf{m}$ and input dimension $\mathbf{m}$) of the right invertible periodic system $[\,\mathcal{S}\ \ -\mathcal{I}_{\mathbf{p}}\,]$. Equivalently, the lifted TFM $G^{\mathcal{R}}(z)$ is a right rational nullspace basis of $[\,G^{\mathcal{S}}(z)\ \ -\ I_P\,]$. The determination of $\mathcal{R}$ can be done using the method proposed in [17] to compute least order annihilators for periodic systems. The invertibility of $\mathcal{M}_1$, or equivalently of $G^{\mathcal{M}_1}(z)$, is guaranteed by Lemma 2 of [18] by observing that the lifted TFM $G^{\mathcal{R}}(z)$ is a right nullspace basis, and thus has full column rank.

The computation of left annihilators for periodic systems has been addressed in [17] in the context of fault detection. Here we only sketch the corresponding procedure to determine a right annihilator of $[\,\mathcal{S}\ \ -\mathcal{I}_{\mathbf{p}}\,]$. For this we employ the extended periodic pair (S_k, T_k) defined as

$$S_k = \left[\begin{array}{ccc} A_k & B_k & 0 \\ C_k & D_k & -I_{p_k} \end{array}\right], \quad T_k = \left[\begin{array}{ccc} E_k & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{11}$$

Let Q_k and Z_k be orthogonal N-periodic matrices determined using the algorithm proposed in [19] to reduce the N-periodic pair (S_k, T_k) to the Kronecker-like form $(\overline{S}_k, \overline{T}_k) := (Q_k S_k Z_k, Q_k T_k Z_{k+1})$, where

$$\overline{S}_k = \left[\begin{array}{ccc} \overline{B}_k & \overline{A}_k & * \\ 0 & 0 & A_k^{reg} \end{array}\right] \tag{12}$$

$$\overline{T}_k = \left[\begin{array}{ccc} 0 & \overline{E}_k & * \\ 0 & 0 & E_k^{reg} \end{array}\right] \tag{13}$$

where: (a) the periodic system $(\overline{E}_k, \overline{A}_k, \overline{B}_k, *, *)$ is completely reachable and $\overline{E}_k$ is invertible; (b) the pole pencil of the form (4), corresponding to the periodic pair (E_k^{reg}, A_k^{reg}) is regular. Note that, since the extended system $[\,\mathcal{S}\ \ -\mathcal{I}_{\mathbf{p}}\,]$ is right invertible, the periodic pair (S_k, T_k) has no left Kronecker structure.

If we partition $[\,0\ \ I_{m_k+p_k}\,]Z_k$ in accordance with the column partitioning of $\overline{S}_k$ in (12) as

$$[\,0\ \ I_{m_k+p_k}\,]Z_k = \left[\begin{array}{ccc} \overline{K}_k & \overline{J}_k & * \\ \overline{C}_k & \overline{D}_k & * \end{array}\right] \begin{array}{l} \}\, m_k \\ \}\, p_k \end{array}$$

we obtain that

$$\mathcal{R} := \left(\overline{A}_k, \overline{E}_k, \overline{B}_k, \left[\begin{array}{c} \overline{K}_k \\ \overline{C}_k \end{array}\right], \left[\begin{array}{c} \overline{J}_k \\ \overline{D}_k \end{array}\right] \right),$$

is a periodic descriptor system representation for a maximal right annihilator of $[\,\mathcal{S}\ \ -\mathcal{I}_{\mathbf{p}}\,]$. The periodic realizations of the causal factors are

$$\begin{array}{rcl} \mathcal{N}_1 & = & (\overline{E}_k, \overline{A}_k, \overline{B}_k, \overline{C}_k, \overline{D}_k) \\ \mathcal{M}_1 & = & (\overline{E}_k, \overline{A}_k, \overline{B}_k, \overline{K}_k, \overline{J}_k) \end{array} \tag{14}$$

The realization of $\mathcal{R}$ is reachable by construction. It can be also proven that the assumption of detectability of the original system $\mathcal{S}$ guarantees that $\mathcal{R}$ is detectable as well.

This algorithm to compute the causal factors employs only orthogonal similarity transformations, which are performed at the level of component matrices. Therefore, this method is numerically reliable. Since it is possible to prove that the reduced periodic pair $(\overline{S}_k, \overline{T}_k)$ is exact for a slightly perturbed original periodic pair (S_k, T_k), a certain kind of numerical stability can be assessed for this algorithm.

VI. Conclusions

The proposed computational method to determine NPRCFs is completely general, being applicable to standard or descriptor periodic systems with time-varying dimensions. The proposed algorithm is lifting-free and operates exclusively on the matrices of periodic state-space descriptions. The two main computational ingredients for the proposed algorithm are the solution of generalized periodic Riccati equations and computation of maximal periodic right annihilators. For both of these computations, efficient and numerically reliable computational procedures are available. In light of these, the proposed **NPRCF Procedure** can be considered a satisfactory numerical method for periodic systems according to the criteria formulated in [15].

Interestingly, when the causal factorization technique proposed in this paper is applied to the standard case ($N = 1$), it provides a completely satisfactory alternative numerical solution to the proper coprime factorization problem. This can be used to enhance the numerical properties of the procedures described in [4] to compute normalized coprime factorizations of general rational matrices in both continuous-time as well as discrete-time settings.

References

[1] L. Chai and L. Qiu. Multirate periodic systems, ν-gap metric and robust stabilization. In *Proc. of MTNS'04, Leuven, Belgium*, 2004.

[2] A. Varga. Balanced truncation model reduction of periodic systems. In *Proc. CDC'2000, Sydney, Australia*, pages 2379–2384, 2000.

[3] B. Xie and V. Syrmos. Normalized coprime factorizations for discrete-time periodic systems. In *Proc. ACC'97, USA*, 1997.

[4] A. Varga. Computation of normalized coprime factorizations of rational matrices. *Systems & Control Lett.*, 33:37–45, 1998.

[5] C. Oară. On computing normalized-coprime factorizations of general rational matrices. *IEEE Trans. Automat. Control*, 46:286–290, 2001.

[6] V.D. Estruch, V. Hernandez, E. Sanchez, and C. Coll. Forward/backward periodic realizations of nonproper rational matrices. *J. Math. Syst. Estim. Control*, 8:155175, 1998.

[7] A. Varga and P. Van Dooren. Computing the zeros of periodic descriptor systems. *Systems & Control Lett.*, 50:371–381, 2003.

[8] A. Varga. Strongly stable algorithm for computing periodic system zeros. In *Proc. of CDC'2003, Maui, Hawaii*, 2003.

[9] A. Varga. Computation of generalized inverses of periodic systems. In *Proc. of CDC'04, Paradise Island, Bahamas*, 2004.

[10] A. Varga. Computation of $\mathcal{L}_\infty$-norm of linear discrete-time periodic systems. In *Proc. of MTNS'06, Kyoto, Japan*, 2006.

[11] O. M. Grasselli and S. Longhi. Finite zero structure of linear periodic discrete-time systems. *Int. J. Systems Sci.*, 22:1785–1806, 1991.

[12] A. Varga. Computation of minimal periodic realizations of transfer-function matrices. *IEEE Trans. Automat. Control*, 46:146–149, 2004.

[13] A. Varga. Computation of coprime factorizations of rational matrices. *Lin. Alg. & Appl.*, 271:83–115, 1998.

[14] C. Oară and A. Varga. Computation of general inner-outer and spectral factorizations. *IEEE Trans. Automat. Control*, 45:2307–2325, 2000.

[15] A. Varga and P. Van Dooren. Computational methods for periodic systems - an overview. In *Proc. of IFAC Workshop on Periodic Control Systems, Como, Italy*, pages 171–176, 2001.

[16] A. Varga. On solving periodic Riccati equations. *Numerical Linear Algebra with Applications*, 15:809–835, 2008.

[17] A. Varga. Design of fault detection filters for periodic systems. In *Proc. of CDC'04, Paradise Island, Bahamas*, 2004.

[18] G. Verghese, P. Van Dooren, and T. Kailath. Properties of the system matrix of a generalized state-space system. *Int. J. Control*, 30:235–243, 1979.

[19] A. Varga. Computation of Kronecker-like forms of periodic matrix pairs. In *Proc. of MTNS'04, Leuven, Belgium*, 2004.