Probabilities of multiple quantum teleportation

Short title: Multiple quantum teleportation

Author: Dipl.-Phys. Dr. Richard Woesler

German Aerospace Centre, Institute of Transportation Research
Rutherfordstr. 2, 12489 Berlin, Germany

E-mail: Richard.Woesler@dlr.de

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Abstract

Using quantum teleportation a quantum state can be teleported with a certain probability. Here the probabilities for multiple teleportation are derived, *i.e.* for the case that a teleported quantum state is teleported again or even more than two times, for the two-dimensional case, *e.g.*, for the two orthogonal directions of the polarization of photons. It is shown that the probability for an exact teleportation, except for an irrelevant phase factor, is 25%, *i.e.*, surprisingly, this result holds for the case of a single teleportation as well as for an arbitrary number of a sequence of teleportations. In the remaining 75% of the cases, unitary transformations occur, which are equivalent to those occurring for a single teleportation except for an irrelevant phase factor.

Introduction

It is known that the quantum state of a particle can be transferred onto another particle, the latter may in principle be, e.g., several kilometres away from the initial one [1,2]. The process is called quantum teleportation. For the two-dimensional case the original quantum state can be written as

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \ , \tag{1}$$

where $|0\rangle$ may denote horizontal polarization, and $|1\rangle$ vertical polarization. a and b are two complex numbers with $|a|^2 + |b|^2 = 1$. This state is in the hand of the sender (Alice). Via an experimental setup called quantum teleportation, as experimentally shown first by Bouwmeester *et al.* [2], this state can be teleported to a receiver (Bob). Except for an irrelevant phase factor, Bob obtains one of the following four possible states with equal, *i.e.*, 25% probability

$$|\varphi_1\rangle = M_1\,|\psi\rangle \, , \, |\varphi_2\rangle = M_2\,|\psi\rangle \ , \, |\varphi_3\rangle = M_3\,|\psi\rangle \ , \, |\varphi_4\rangle = M_4\,|\psi\rangle \ , \tag{2}$$

where

$$M_1 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ M_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ M_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

i.e., in the first case, an exact teleportation of the state is achieved, except for an irrelevant phase factor. In the three other cases, simple unitary transformations do occur.

In the following text, first, the equations for the states for double teleportation are described and the probabilities for different outcomes are derived, second, the corresponding results for an arbitrary number of teleportations are obtained, and, finally, it is concluded that multiple teleportation can be realised experimentally with available technology.

Equations for double quantum teleportation

Bob can teleport the state he received from Alice to a further receiver (Charlie). In the following, the probability that Charlie receives the original state $|\psi\rangle$ is computed. At first one could suggest that the probability would be only $\frac{1}{16} = 6.25\%$, because Bob received the state with 25% probability, however, detailed computation will show that this is not the case. Charlie will receive with equal probability one of the following 16 states

$$|\phi_{nm}\rangle = M_m\,|\varphi_n\rangle = M_m M_n\,|\psi\rangle \text{, with } m,n \in \{1,2,3,4\}. \tag{4}$$

Computation of $M_m M_n$ yields

$$M_1 M_1 = M_2 M_2 = M_3 M_3 = -M_4 M_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -M_1 \text{,} \tag{5}$$

$$M_1 M_2 = M_2 M_1 = M_3 M_4 = -M_4 M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -M_2 \text{,} \tag{6}$$

$$M_1 M_3 = M_3 M_1 = -M_2 M_4 = M_4 M_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -M_3 \text{,} \tag{7}$$

and

$$M_1 M_4 = M_4 M_1 = -M_2 M_3 = M_3 M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -M_4 \text{.} \tag{8}$$

From here it follows that in all cases, four possible results essentially correspond to one result known from the single teleportation. From (5) it follows that, except for an irrelevant phase factor, with probability of 25% the final state will be the same as the original state, as it is known for the single teleportation, and not just with probability of 6.25%. According to (6-8) also the other three possible states are, except for a phase factor, the same as for the single teleportation.

Equations for multiple quantum teleportation

The case of an arbitrary number of sequential teleportations is considered. Using complete induction it is clear that the final states will again be the same as for the single teleportation except for an irrelevant phase factor. To determine the states exactly, consider the case that the original state is teleported p times. The final states can be written as

$$|\varphi_{k_1 \cdots k_p}\rangle = M_{k_p} \cdots M_{k_1} |\psi\rangle \, . \qquad (9)$$

Here, $k_1=q$ in case the first teleportation is the one corresponding to the transformation M_q (3), where q can be 1, 2, 3, or 4 respectively. Analogously, k_2 , k_3, …, k_p indicate the results of the second, third till to the pth teleportation. In order to evaluate (9) it is efficient to replace each product of two neighboring matrices in (9) by a single matrix, continuing this p-1 times till only one matrix is left. This takes just p-1 computational steps in total, and it can be done using (5-8). An operator R for replacement is defined via the following equations

$$RM_m M_n = \begin{cases} -M_1, & \text{if } m = n \text{ and } m,n \in \{2,3\}, \\ +M_1, & \text{if } m = n = 4, \\ -M_n, & \text{if } m = 1, \\ -M_m, & \text{if } n = 1, \\ (-1)^m M_q, & \text{if } m+1 = n, \text{or if } m-1 = n, \text{or if } m+2 = n \\ & \text{with } m,n,q \in \{2,3,4\}, q \neq m, q \neq n, \\ -M_3, & \text{if } m = 4 \text{ and } n = 2. \end{cases} \qquad (10)$$

The following results hold:

$$RM_m M_n = M_m M_n \, , \; m,n \in \{1,2,3,4\} \, , \text{and } \exists\, q \in \{1,2,3,4\} : RM_m M_n \propto M_q \, . \qquad (11)$$

The two-matrices operator R can be applied to (9) iteratively

$$|\varphi_{k_1 \cdots k_p}\rangle = M_{k_p} \cdots M_{k_1} |\psi\rangle = R^{p-1} M_{k_p} \cdots M_{k_1} |\psi\rangle \, . \qquad (12)$$

After doing so, this equation contains just one single matrix. Only p-1 times two neighboring indices have to be replaced by a single index, and eventually up to p-1 occurring minus signs on the right side of eq. (10) have to be taken into account.

Conclusion

Applying exactly the experimental scheme of Bouwmeester *et al.* [2] only two out of the four Bell states can be discriminated [3]. It is known that it is impossible to conduct a complete Bell measurement on two-mode polarization states using only linear passive elements [4], unless the two photons are entangled in a further degree of freedom [5]. Schemes involving non-linearities have been developed, *e.g.*, applying resonant atomic interactions [6], or by Fock filtering using the Kerr effect [7]. Vitali *et al.* [8] have described a scheme to detect all four Bell states and to conduct a complete quantum teleportation using a Kerr non-linearity which is feasible using available technology. Further, rather different versions of quantum teleportation have been performed successfully yet [9-11].

Realising, e.g., the scheme in [8] for single teleportation, complete multiple teleportation as described in the present text can be undertaken as follows. Each sender (Alice, Bob, ...) determines which of the four possible unitary transformations (M_1, M_2, M_3, or M_4) occurred from her/him to the corresponding next receiver (Bob, Charlie, ..., Fred). Each sender can encode this information within two classical bits. All bits, together with the information from which sender the bits are, can be sent to the final receiver (Fred) who then simply has to apply (12) as described using (10), determining a single matrix (M_1, M_2, M_3, or M_4) which describes the transformation that occurred to the original state during multiple teleportation. According to (5), Fred can then apply the same single unitary transformation (M_1, M_2, M_3, or M_4) to the state he received. Of course, in case that the single matrix is M_1, no transformation has to be made at all. Doing so he will, according to (5), obtain in all cases the original quantum state, except for an irrelevant phase factor.

The described scheme is minimal in the sense that a minimal number of transformations, *i.e.* no one or one, has to be conducted by the final receiver to recover the original state.

Realising multiple teleportation of quantum states it will be possible to chain quantum states over large distances, and, further, multiple teleportation can be used within quantum computers.

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