International Journal of Control
Vol. 83, No. 5, May 2010, 897–914

Optimal periodic output feedback control: a continuous-time approach and a case study

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(*Received 16 June 2009; final version received 14 November 2009*)

This article deals with the problem of optimal static output feedback control of linear periodic systems in continuous time, for which a continuous-time approach, which allows to deal with both stable and unstable open loop systems, is presented. The proposed approach is tested on the problem of designing attitude control laws for a Low-Earth Orbit (LEO) satellite on the basis of feedback from a triaxial magnetometer and a set of high-precision gyros. Simulation results are used to demonstrate the feasibility of the proposed strategy and to evaluate its performance.

Keywords: periodic control; optimal control; static output feedback; spacecraft control; attitude control; magnetic control

1. Introduction

Periodic control theory is now a well-established area, in which a significant body of results is available, covering both the analysis and the design of control systems (see e.g. the recent book by Bittanti and Colaneri (2008)). In particular, in the practice of control engineering there is a lot of interest in linear time-periodic (LTP) systems since many challenging applications can be naturally described as continuous-time LTP systems, particularly in aerospace and mechanical engineering: high-order helicopter rotor dynamics (see e.g. Lovera, Colaneri, Malpica, and Celi (2006)), satellite attitude and orbit dynamics (Wisniewski 2000; Schubert 2001; Silani and Lovera 2005; Pulecchi, Lovera, and Varga 2005), wind turbines (Stol and Balas 2001), etc.

Most of the available control design methods and tools for periodic systems, however, lead to the design of dynamic time-periodic controllers, which are often not implementable due to practical constraints. Static output feedback (SOF), on the other hand, represents a realistic solution for a wide class of practical applications, in particular when full-state measurement is not available and the algorithmic complexity of a high-order or time-varying compensator needs to be avoided. Note that even in the case of full-state measurement, the determination of an optimal constant feedback matrix for an LTP system is a non-trivial problem, since methods based on linear time-invariant (LTI) approximations can guarantee neither closed-loop stability nor the desired level of performance a priori. For example, the solution of the linear-quadratic (LQ) regulation problem obtained by considering the invariant part of a linear periodic system (or the system averaged over one period) does not necessarily stabilise the original LTP system. Similarly, averaging the periodic feedback gain obtained from the solution of the periodic Riccati equation does not insure the stability of the closed-loop system.

SOF stabilisation, however, is known to be a challenging problem, since even when it was shown to be solvable it is generally non-convex even for time-invariant systems (Syrmos, Abdallah, and Dorato 1997; Geromel, de Souza, and Skelton 1998; Astolfi and Colaneri 2005). Time periodicity, of course, makes this problem even harder. A number of approaches to the SOF problem for LTP systems have been proposed in the literature. The assignment of the characteristic exponents of a controllable LTP system has been considered in Aeyels and Willems (1993), where a simple algorithm valid only for second-order, discrete-time, LTP systems has been proposed. In Juan and Kabamba (1989), the concept of *generalised sample and hold functions* (*GSHF*) was introduced, which makes it possible to assign multiple poles arbitrarily using the output measurements only once for each period T. More recently, Chen and Chen (1999) provided more reliable solutions extending the GSHF definition in order to ensure the continuity of the control signal. The related problem of the stabilisation of LTI systems by means of periodic, piece-wise constant output feedback has been studied in Allwright, Astolfi, and Wong (2005).

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ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207170903487439
http://www.informaworld.com

Similarly, the LQ-SOF design problem has been deeply analysed in the periodic framework, but due to the geometrical non-convexity of the problem, only iterative solutions have been proposed so far. These methods require the solution of the nonlinear equations expressing first-order optimality conditions (as in Levine and Athans (1970) for the LTI case) or, alternatively, they exploit general purpose optimisation algorithms. In Calise, Wasikowski, and Schrage (1992), a Floquet-transformed LQ cost-function, penalising the response and control envelopes rather than the actual time histories, was considered. A different approach has been proposed in Varga and Pieters (1998) for discrete-time LTP systems (see also Pulecchi et al. (2005) for an application to magnetic satellite attitude control): the LQ performance index is minimised computing the analytical gradient and exploiting gradient descent optimisation algorithms; reliable numerical functions for Matlab environment have also been developed (Varga 2005a). A similar approach has been proposed in Aliev, Arcasoy, Larin, and Safarova (2005), where the cost function for the discrete-time case is minimised using a gradient-free method which is suitable also for open-loop unstable systems.

Recently, the feasibility of techniques based on linear and bilinear matrix inequalities (LMI, BMI) for the design of periodic controllers has been explored in Farges, Peaucelle, and Arzelier (2006a, b, 2007), where a parameterisation of stabilising output feedback controllers, both constant and periodic, for discrete-time periodic systems is proposed; the characterisation of these controllers, which belong to properly designed convex ellipsoidal sets, relies on the solution of BMIs. The designed ellipsoids offer the interesting property of being *resilient*, i.e. the resulting closed-loop system is robustly stable with respect to the uncertainty of the control law parameters, a property which turns out to be particularly useful in the case of digital controller implementation. While this approach lends itself to the formulation of more general control problems, it suffers from a significant drawback, i.e. it is limited to relatively small-scale problems (both in terms of order and period of the LTP model for the system to be controlled) when compared to techniques based on the solution of periodic Lyapunov and Riccati equations.

In light of the above discussion, the aim of this article is to present a novel approach to the practical design of a constant, stabilising, feedback gain which minimises an LQ performance index for a linear, continuous-time periodic system. In particular, the results presented in this article apply equivalently to the output feedback case and to the state feedback case. The proposed approach is demonstrated both on a simple numerical example and in a detailed case study dealing with the problem of attitude control for a Low-Earth Orbit (LEO) satellite using an innovative sensor configuration.

This article is organised as follows. In Section 2 the LQ-SOF problem is formulated and the proposed optimisation approach is described; in Section 3 the main numerical issues associated with the computation of the solution are outlined. Finally, the results obtained in the application of the proposed approach to a simulation example and to a satellite attitude control case study are presented and discussed in Section 4. Conclusions are given in Section 5, while the Appendix provides the derivation of the main results of this article.

2. Optimal output-feedback control for continuous-time periodic systems

2.1 *Background and notation*

In this section, some basic definitions associated with LTP systems are provided and the relevant notation is established (see e.g. Bittanti and Colaneri (2008) for details). Consider the linear system

$$\dot{x}(t) = A(t)x(t), \tag{1}$$

where $A(t)$ is an $n \times n$ T-periodic matrix. The free motion of the periodic system, i.e. the solution of Equation (1) starting from the initial state $x(\tau)$ at time τ is given by

$$x(t) = \Phi_A(t, \tau)x(\tau),$$

where the *transition matrix* $\Phi_A(t, \tau)$ is given by the solution of the matrix differential equation

$$\frac{\partial \Phi(t, \tau)}{\partial t} = A(t)\Phi_A(t, \tau), \quad \Phi(\tau, \tau) = I_n.$$

In particular, it is easy to see that $\Phi_A(t + T, \tau + T) = \Phi_A(t, \tau)$. The transition matrix over one period

$$\Psi_A(\tau) = \Phi_A(\tau + T, \tau)$$

plays a major role in the analysis of periodic systems and is known as *monodromy matrix* at time τ: it can be shown that its eigenvalues, known as the *characteristic multipliers* of the system, are independent of τ and strongly related to the stability properties of the system. Indeed, the stability margin for the system is defined by the spectral radius $\rho = \max_{1 \le i \le n}(|\lambda_i|)$ of the monodromy matrix.

As is well known, it is possible to find a state-coordinate transformation leading to a periodic system with *constant* dynamic matrix. In continuous-time it can be shown that such a transformation $S(\cdot)$ does exist and the constant dynamic matrix $\hat{A}$ (called the Floquet representation for the system) can be obtained by solving $e^{\hat{A}T} = \Psi_A(\tau)$, where τ is any given time point. The appropriate transformation $S(\cdot)$ is simply given by

$$S(t) = e^{\hat{A}(t-\tau)}\Phi_A(\tau, t).$$

Such a matrix is, in general, periodic of period T and satisfies the linear differential equation

$$\dot{S}(t) = \hat{A}S(t) - S(t)A(t)$$

with initial condition $S(\tau) = I_n$. The eigenvalues of $\hat{A}$ are named *characteristic exponents*; the correspondence between a characteristic multiplier z and a characteristic exponent s is $z = e^{sT}$, so that the above-mentioned asymptotic stability test based on the modulus of the characteristic multipliers can be equivalently formulated in terms of the real part of the characteristic exponents. Unfortunately, the computation of the characteristic exponents/multipliers is beyond analytical treatment for nontrivial problems, so the stability analysis for periodic systems can only be performed numerically (see e.g. Lust (2001)).

The stability condition can be equivalently formulated in terms of the Periodic Lyapunov Differential Equation (PLDE)

$$-\dot{P}(t) = P(t)A^T(t) + A(t)P(t) + Q(t), \tag{2}$$

where the periodic matrix $Q(t)$ is assumed to be positive definite; the condition states that the periodic system having $A(t)$ as dynamic matrix is stable if and only if the Lyapunov equation admits a unique periodic positive definite solution $P(t)$.

In this article, the LTP system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{3}$$

will be considered, where $A(t) \in \mathbf{R}^{n\times n}$, $B(t) \in \mathbf{R}^{n\times m}$, $C(t) \in \mathbf{R}^{p\times n}$ are T-periodic matrices. For this system, the problem of designing stabilising SOF controllers will be studied. The periodic system (3) is output feedback stabilisable if there exists a T-periodic feedback matrix $F(\cdot)$ such that system (3) in feedback with the control law $u(t) = F(t)y(t)$ is asymptotically stable, that is the characteristic multipliers of $\bar{A}(t) = A(t) + B(t)F(t)C(t)$ lie in the open unit disc. A necessary and sufficient condition for output feedback stabilisability by means of a T-periodic matrix (the constant case is excluded) is described in Colaneri, de Souza, and Kucera (1998): assuming that the output distribution matrix $C(t)$ is full row rank for each t, it is possible to define the periodic projection matrix

$$V(t) = I - C^T(t)\big(C(t)C^T(t)\big)^{-1}C(t). \tag{4}$$

Hence, the following theorem holds.

Theorem 1: *The continuous-time periodic system* (3) *is output feedback stabilisable iff the system is stabilisable and detectable and there exist a T-periodic positive semidefinite matrix* $P(t)$ *and a T-periodic matrix* $G(t)$ *such that* $\forall t$

$$\begin{aligned} G(t)V(t) &= B^T(t)P(t)V(t), \\ -\dot{P}(t) &= A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) \\ &\quad + C^T(t)C(t) + G^T(t)G(t). \end{aligned} \tag{5}$$

No such results are available, however, for the case of a constant feedback matrix, i.e. for a given LTP system it is not possible to provide conditions under which it is SOF-stabilisable using a constant-gain feedback.

2.2 *Problem statement*

Consider the LTP system (3) and define the quadratic performance index

$$J = E\left\{\int_0^\infty \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]\mathrm{d}t\right\}, \tag{6}$$

with $Q(t) \geq 0$, $R(t) > 0$ symmetric T-periodic matrices and the expectation operator $E(\cdot)$ is taken over the set of possible initial condition x_0 for the system, assumed to be a random variable with zero mean and known covariance $X_0 = E\{x_0x_0^T\}$. The optimal output feedback control problem consists in finding the feedback matrix $F^*(t)$ of optimal control action

$$u^*(t) = F^*(t)\,y(t), \tag{7}$$

which minimises the performance index J of (6). The expectation operator $E(\cdot)$ used in (6) allows to remove the dependence of the cost function of a particular initial condition, giving more generality to the method. Hence, the resulting $F^*(t)$ matrix may be interpreted as the optimal feedback matrix in an average sense, that is optimal over some set of initial states. Note that, from this point of view, the covariance matrix X_0 is a design parameter (similarly to $Q(t)$ and $R(t)$) which may be used by the designer who has some a priori knowledge of which states of the system are likely to be perturbed (see the classical paper Levine, Johnson, and Athans (1971)). If this information is not available, a common choice consists in assuming an initial state uniformly distributed over a unit hypersphere, that is $X_0 = I$. On the other hand, a fully deterministic approach requires to know exactly the initial states (and therefore setting $X_0 = x_0x_0^T$) and to optimise J according to their values (Hench and Laub 1994).

2.3 *Performance index computation*

With (7), the state equation for the closed-loop system can be written as

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)F(t)\,y(t)\\ &= [A(t) + B(t)F(t)C(t)]x(t) = \bar{A}(t)x(t),\end{aligned} \tag{8}$$

where $\bar{A}(t) = A(t) + B(t)F(t)C(t)$ represents the closed-loop dynamic matrix, which is obviously time-periodic. Assuming that a periodic stabilising matrix $F(t)$ is given, and letting $\bar{Q}(t) = Q(t) + C^T(t)F^T(t)R(t)F(t)C(t)$, then we can express the performance index J as

$$J = E\{x_0^T P(0)x_0\} = \mathrm{tr}(P(0)X_0), \tag{9}$$

where the matrix $P(t)$ satisfies a standard PLDE in the so-called adjoint form, that is

$$-\dot{P}(t) = \bar{A}^T(t)P(t) + P(t)\bar{A}(t) + \bar{Q}(t). \tag{10}$$

See Appendix A.1 for details.

2.4 *Performance index minimisation: the constant gain case*

The minimisation of the performance index given by (9) can be carried out using gradient-free procedures (such as the algorithm of Nelder–Mead, see e.g. Nocedal and Wright (1999)), or using steepest descent methods, provided that an analytical expression for the gradient of the performance index with respect to the coefficients of the F matrix (which should be more properly called a Jacobian matrix) is available. In both cases, a stabilising gain F_0 must be employed to initialise the iterative optimisation process. In the following, necessary conditions for optimality (and therefore the required gradient expression) will be developed. Since the case of invariant feedback matrix is more interesting for applications, the hypothesis $\dot{F} = 0$ will be assumed in this section. The case of a time-periodic gain will be treated in Section 2.5.

The main result is summarised in Proposition 1.

Proposition 1: *Let F be a constant stabilising output feedback gain and assume that the matrices $\bar{A}(t)$, $\bar{Q}(t)$ and $X(t)$ are given, respectively, by $\bar{A}(t) = A(t) + B(t)FC(t)$, $\bar{Q}(t) = Q(t) + C^T(t)F^TR(t)FC(t)$ and $X(t) = \Phi_{\bar{A}}(t,0)X_0\Phi_{\bar{A}}^T(t,0)$; then, the expressions for the performance index (6) and its gradient are*

$$J(F, X_0) = \mathrm{tr}(P(0)X_0), \tag{11}$$

$$\begin{aligned}\nabla_F J(F, X_0) = 2\int_0^T &\left[B^T(t)P(t) + R(t)FC(t)\right]\\ &\times \Phi_{\bar{A}}(t,0)V\Phi_{\bar{A}}^T(t,0)C^T(t)\mathrm{d}t,\end{aligned} \tag{12}$$

where the symmetric matrices $P(t)$ and V satisfy, respectively, the PLDE

$$-\dot{P}(t) = \bar{A}^T(t)P(t) + P(t)\bar{A}(t) + \bar{Q}(t), \tag{13}$$

and the Discrete Lyapunov Equation (DLE)

$$V = \Psi_{\bar{A}} V \Psi_{\bar{A}}^T + X_0. \tag{14}$$

Proof: See Appendix A.2. □

As previously mentioned, the numerical optimisation of (11) requires that the linear periodic system (3) is output stabilisable and thus, at each iteration i, the current approximation F_i of the optimal output feedback must belong to the set $\mathcal{S}_F \subset \mathbf{R}^{m\times p}$ of the stabilising feedback gain matrices. Formally, the optimisation problem can be stated as follows:

$$\min_{F\in\mathcal{S}_F} J(F, X_0). \tag{15}$$

The stopping criterion, indicating the convergence to a global or, at least, a local solution of (15) will be simply $\|\nabla_F J\| < \mathrm{tol}$, where tol is an admissible absolute tolerance on the norm of the gradient. Despite the formal constraint $F \in \mathcal{S}_F$, the numerical optimisation of the performance index can be carried out exploiting general purpose descent algorithms for unconstrained problems, such as conjugate gradient or BFGS quasi-Newton methods, which rely on proven convergence results (to a local minimum) under rather mild conditions. As pointed out in Makila and Toivonen (1987), the minimisation problem can be considered unconstrained since, under certain conditions, the loss function J grows without bound as the boundary of $\mathcal{S}_F$ is approached along any path in the open set $\mathcal{S}_F$; this assumption also implies the compactness of the level set

$$\Pi(d) = \{F \in \mathcal{S}_F | J(F) \le d\}, \tag{16}$$

where $d \ge 0$.

The need to find out a constant stabilising matrix F_0 for the initialisation of the algorithm is generally non-trivial and it should be considered as an independent problem. For that reason, the procedure indicated by Larin (2003) for LTI systems and in Aliev et al. (2005) for periodic discrete-time systems may be followed. The first step consists in considering a modified periodic system having the dynamic matrix $A_\mu(t) = A(t) + \mu I$. Clearly, the characteristic exponents of a given $A(t)$ matrix can be shifted to the left in the complex plane by choosing a suitable value for μ. In particular, it is always possible to choose μ such that the system having the dynamic matrix $A_\mu(t)$ is asymptotically stable. This new system will be associated to a modified optimisation problem expressed by

$$\min_{(F,\mu)\in\mathcal{S}_{F_\mu}\times\mathbf{R}} \tilde{J}(F,\mu,X_0) = \min_{(F,\mu)\in\mathcal{S}_{F_\mu}\times\mathbf{R}} \mathrm{tr}(P(0)X_0) + \sigma\mu^2, \tag{17}$$

where the penalty parameter $\sigma > 0$ typically takes large values. One may expect that, if σ is sufficiently large, the iteration variable μ will rapidly converge to zero, and therefore the optimisation problem (17) will coincide with the nominal problem (15) as $\mu \to 0$. Now the initialisation problem is much simpler, as it consists in searching an initial value for the scalar μ such that, assuming F_0 null for the sake of simplicity, $\rho(\Psi_{A_\mu}) < 1$. In order to proceed as before, the analytical computation of the gradient of $\tilde{J}$ will be performed.

These results are reported in Proposition 2.

Proposition 2: *Let F be a constant stabilising output feedback gain and assume that the matrices $\bar{A}_\mu(t)$, $\bar{Q}(t)$ and $X(t)$ are given, respectively, by $\bar{A}_\mu(t) = A_\mu(t) + B(t)F(t)C(t)$, $\bar{Q}(t) = Q(t) + C^T(t)F^T(t)R(t)F(t)C(t)$ and $X(t) = \Phi_{\bar{A}_\mu}(t,0)X_0\Phi^T_{\bar{A}_\mu}(t,0)$; then, the expressions for the performance index* (17) *and its gradient are*

$$\tilde{J}(F,\mu,X_0) = \mathrm{tr}(P(0)X_0) + \sigma\mu^2, \tag{18}$$

$$\nabla_F J(F,\mu,X_0) = 2\int_0^T \left[B^T(t)P(t) + R(t)FC(t)\right] \times \Phi_{\bar{A}_\mu}(t,0)V\Phi^T_{\bar{A}_\mu}(t,0)C^T(t)\mathrm{d}t, \tag{19}$$

$$\frac{\partial\tilde{J}}{\partial\mu}(F,\mu,X_0) = 2\mu\sigma + 2\int_0^T \mathrm{tr}\left[P(t)\Phi_{\bar{A}_\mu}(t,0)V\Phi^T_{\bar{A}_\mu}(t,0)\right]\mathrm{d}t, \tag{20}$$

where the symmetric matrices $P(t)$ and V satisfy, respectively, the PLDE

$$-\dot{P}(t) = \bar{A}^T_\mu(t)P(t) + P(t)\bar{A}_\mu(t) + \bar{Q}(t) \tag{21}$$

and the DLE

$$V = \Psi_{\bar{A}_\mu}V\Psi^T_{\bar{A}_\mu} + X_0. \tag{22}$$

Proof: See Appendix A.3.

2.5 *Performance index minimisation: the periodic gain case*

The minimisation procedure outlined for the constant feedback case is quite general, since the gradient formula given by (12)–(14) requires only that the Jacobian is computed with respect to a constant matrix F (according to the Hamiltonian-based optimisation). It is rather intuitive that, expanding a generic periodic feedback matrix $F(t)$ in Fourier series

$$F(t) = F_0 + \sum_{j=1}^{\infty} F_{js}\sin(j\omega_0 t) + F_{jc}\cos(j\omega_0 t), \tag{23}$$

(where $\omega_0 = 2\pi/T$) and including all the periodic coefficients up to the k-th harmonic in a constant matrix $\tilde{F} \in \mathbf{R}^{m\times(2k+1)p}$, it is possible to use again the analytical expression for the gradient (12)–(14) for the minimisation of the performance index (11), substituting the original matrix $C(t)$ with an extended output transformation matrix $\tilde{C}(t) \in \mathbf{R}^{(2k+1)p\times n}$. The minimisation procedure allows the computation of the optimal periodic coefficients of $F(t)$. This result leads to Proposition 3, which follows easily from the previous results.

Proposition 3: *Let F be a harmonic stabilising output feedback gain and assume that the matrices $\bar{A}(t)$, $\bar{Q}(t)$ and $X(t)$ are given, respectively, by $\bar{A}(t) = A(t) + B(t)F(t)C(t) = A(t) + B(t)\tilde{F}(t)\tilde{C}(t)$, $\bar{Q}(t) = Q(t) + \tilde{C}^T(t) \times \tilde{F}^T(t)R(t)\tilde{F}(t)\tilde{C}(t)$ and $X(t) = \Phi_{\bar{A}}(t,0)X_0\Phi^T_{\bar{A}}(t,0)$, where the extended matrices $\tilde{F}$ and $\tilde{C}(t)$ are defined as*

$$\tilde{F} = \begin{bmatrix} F_0 & F_{1s} & F_{1c} & F_{2s} & F_{2c} & \dots & F_{ks} & F_{kc} \end{bmatrix}, \tag{24}$$

$$\tilde{C}(t) = \begin{bmatrix} C(t) \\ C(t)\sin(\omega_0 t) \\ C(t)\cos(\omega_0 t) \\ C(t)\sin(2\omega_0 t) \\ C(t)\cos(2\omega_0 t) \\ \dots \\ C(t)\sin(k\omega_0 t) \\ C(t)\cos(k\omega_0 t) \end{bmatrix}. \tag{25}$$

Then, the expressions for the performance index (6) *and its gradient are*

$$J(F,X_0) = \mathrm{tr}(P(0)X_0), \tag{26}$$

$$\nabla_{\tilde{F}} J(F,X_0) = 2\int_0^T \left[B(t)^T P(t) + R(t)\tilde{F}\tilde{C}(t)\right] \times \Phi_{\bar{A}}(t,0)V\Phi^T_{\bar{A}}(t,0)\tilde{C}^T(t)\mathrm{d}t, \tag{27}$$

where the symmetric matrices $P(t)$ and V satisfy, respectively, the PLDE

$$-\dot{P}(t) = \bar{A}^T(t)P(t) + P(t)\bar{A}(t) + \bar{Q}(t), \tag{28}$$

and the DLE

$$V = \Psi_{\bar{A}}V\Psi^T_{\bar{A}} + X_0. \tag{29}$$

3. Numerical issues

The computation of the function and its gradient according to the expressions given in Propositions 1 and 2 involve the numerical solution of the PLDEs in (13) and (21), the DLEs in (14) and (22) and the computation of the integrals (12) and (19)–(20). In what follows, we address these computational problems only for the more general case of Equations (21) and (22), which underly the computation of gradients in (19) and (20). For the solution of the PLDEs (21) we can employ both single- or multiple-shooting methods as described in Varga (2005b, 2008). As we will see, single-shooting-based methods are reasonably efficient and therefore are better suited for serial machines. Multiple-shooting methods are more demanding regarding the computational efforts, but are more accurate than the single-shooting-based approach. The main advantage of the multiple-shooting approach is its ability to exploit the existing inherent parallelism in the solution method and therefore is well suited for parallel machines.

3.1 *Single-shooting-based approach*

Single-shooting methods, also known as *periodic generator* methods, were for a long time the only available approaches to solve several classes of periodic differential equations. The use of these methods is possible provided the stable integration of ODEs can be guaranteed. This is precisely the case with the gradient evaluation problem, which relies on the explicit assumptions that the underlying system is stable or has been stabilised. Besides the solution of the Lyapunov equations (21) and (22), the computation of gradients involves the computation of two integrals in the expression of gradients in (19) and (20). For this purpose, the main computational problem we face is to determine the values of intervening matrices on a sufficiently dense grid of time values which ensures an accurate evaluation of these integrals using standard quadrature formulas.

Let $N \gg 1$ be an integer such that $\Delta := T/N$ represents a meaningful time increment to determine the solution $P(t)$ of (21) in view of computing the integrals intervening in the expression of the gradient. To check the stability and to evaluate the expression of gradients, we also need to determine the values of the transition matrix $\Phi_{\bar{A}_\mu}(t,0)$ for $t = (k-1)\Delta$, $k = 1, \ldots, N+1$. This can be done simply by integrating the ODE

$$\frac{\partial \Phi_{\bar{A}_\mu}(t,\tau)}{\partial t} = \bar{A}_\mu(t)\Phi_{\bar{A}_\mu}(t,\tau), \quad \Phi_{\bar{A}_\mu}(\tau,\tau) = I \tag{30}$$

from $\tau = 0$ to T using appropriate integration methods for ODEs and store the values $\Phi_{\bar{A}_\mu}((k-1)\Delta, 0)$ for $k = 1, \ldots, N+1$. To check the stability, the eigenvalues of the monodromy matrix $\Psi_{\bar{A}_\mu} := \Phi_{\bar{A}_\mu}(T,0)$ must have moduli less than one.

To solve the PLDE (21), we can use the periodic generator method which exploits the periodicity of the solution. By imposing the condition $P(0) = P(T)$, we can determine the initial condition $P(0)$ which satisfies the discrete-time Lyapunov equation

$$P(0) = \Psi^T_{\bar{A}_\mu} P(0) \Psi_{\bar{A}_\mu} + W, \tag{31}$$

where $W := \widetilde{W}(T,0)$ with

$$\widetilde{W}(t_f, t) := \int_t^{t_f} \Phi^T_{\bar{A}_\mu}(\tau, t)\bar{Q}(\tau)\Phi_{\bar{A}_\mu}(\tau, t)\mathrm{d}\tau. \tag{32}$$

To compute W, observe that for given t_f, $Y(t) := \widetilde{W}(t_f, t)$ in (32) satisfies the Lyapunov differential equation

$$-\dot{Y}(t) = \bar{A}^T_\mu(t)Y(t) + Y(t)\bar{A}_\mu(t) + \bar{Q}(t), \quad Y(t_f) = 0. \tag{33}$$

Thus, W can be computed as $W = Y(0)$ by integrating the above ODEs backward in time from $t_f = T$ to $t_0 = 0$. Note that because of the symmetry of $Y(t)$, only $\frac{n(n+1)}{2}$ ODEs need to be integrated. It is important to integrate the above PDLE backward in time, to guarantee the numerical stability of the integration. Forward integration would correspond to unstable dynamics and therefore the integration is likely to fail because of the accumulation of errors. To determine the values of the solution $P((k-1)\Delta, 0)$ for $k = N+1$, $N, \ldots, 1$, the PDLE (21) needs to be integrated backward in time from $t_f = T$ to $t_0 = 0$ using appropriate integration methods for ODEs. Once again, the symmetry of the solution $P(t)$ can be exploited. For the solution of the DLEs (31) and (22), standard solution methods relying on the reduction to the *real Schur form* (RSF) of $\Psi_{\bar{A}_\mu}$ can be employed. Both equations can be solved using a single reduction to RSF (Kitagawa 1977). Note that the periodic generator method needs only three calls of a selected ODE solver to evaluate the function and its gradient. The total number of integrated equations is $n^2 + n(n+1)$ over the time period $[0, T]$.

To evaluate the expressions of the gradients in (19) and (20), we can now compute the values of the integrands on the chosen time grid. From these time values, we can evaluate the integrals by using cubic spline-based interpolation. This approach allows in many cases an accurate evaluation of gradients even for a relatively small number of grid points (e.g. $N \geq 32$). This aspect is discussed in Section 4.1 for one of the numerical examples.

3.2 *Multiple-shooting approach*

Multiple-shooting methods are based on a new algorithmic paradigm (Varga 2007), which, by means of *exact* discretisations, converts continuous-time periodic problems into equivalent discrete-time periodic problems for which reliable computational algorithms are available. By solving the discrete-time problems, the so-called *multi-point periodic generators* are computed simultaneously, representing the values of the solution in the chosen grid points. They also serve to conveniently determine the values in intermediary points by integrating the underlying ordinary matrix differential equations using the nearest known knot as initial or final condition.

We describe shortly the application of the multiple-shooting method to solve the PLDE (21). It is straightforward to check that the solution $P(t)$ at two successive time moments $(k-1)\Delta$ and $k\Delta$ satisfies

$$P_k = \Theta_k^T P_{k+1} \Theta_k + W_k, \tag{34}$$

where

$$\begin{aligned} P_k &:= P((k-1)\Delta), \\ \Theta_k &:= \Phi_{\bar{A}_\mu}(k\Delta, (k-1)\Delta), \\ W_k &:= \widetilde{W}(k\Delta, (k-1)\Delta) \end{aligned}$$

with $\widetilde{W}(t_f, t)$ given in (32). Due to the periodicity of $P(t)$ we have $P_{N+1} = P_1$, and therefore the N-coupled equations in (34) for $k = 1, \ldots, N$ represent a *discrete-time backward periodic Lyapunov equation* (DBPLE) (Bittanti and Colaneri 2008).

For the solution of discrete-time periodic Lyapunov equations, efficient and numerically reliable algorithms have been proposed in Varga (1997). These algorithms rely on the reduction to a *periodic real Schur form* (PRSF) (Bojanczyk, Golub, and Van Dooren 1992; Hench and Laub 1994) of the periodic matrix Θ_k by using periodic orthogonal similarity transformations. The solution of the reduced equation is relatively straightforward and the solution of the unreduced equation can be easily recovered from that of the reduced one. Algorithmic details of this approach are presented in Varga (1997).

By solving the N simultaneous equations (34), we determine simultaneously N values of the solution $P(t)$ at equidistant time instants. The continuous-time solution in intermediary points between $\Delta(k-1)$ and Δk can be easily determined by integrating the ODE (21) in backward-time with $P(t)$ initialised with $P(k\Delta)$. Since the time increment Δ can be chosen arbitrarily small, this *multiple-shooting* approach to evaluate Θ_k and W_k, for $k = 1, \ldots, N$, is well suited for problems with large periods and with weakly damped dynamics.

The computational effort to solve the PLDE (21) is dominated by the discretisation effort to determine Θ_k and W_k, for $k = 1, \ldots, N$. To compute Θ_k, for $k = 1, \ldots, N$, the ODEs (30) must be integrated from $\tau = (k-1)\Delta$ to $k\Delta$ using appropriate methods for ODEs. W_k can be computed as $W_k = Y((k-1)\Delta)$ by integrating the ODEs (33) backward in time from $t_f = k\Delta$ to $t_0 = (k-1)\Delta$. Because of the symmetry of $Y(t)$, only $\frac{n(n+1)}{2}$ ODEs need to be integrated. The computational effort to solve the DBPLE is $O(Nn^3)$, and thus it is usually negligible when compared to the effort required to integrate the above ODEs. However, the numerical integrations of ODEs necessary to determine the matrices intervening in the discretised problems can be trivially parallelised. Thus, on parallel machines, the two computational efforts for large N are better balanced.

Solving the DLE (22) can be recast as the solution of a discrete-time forward periodic Lyapunov equation

$$Z_{k+1} = \Theta_k Z_k \Theta_k^T + G_k, \tag{35}$$

where $G_1 = \cdots = G_{N-1} = 0$, $G_N = X_0$ and the solution of (22) is simply $V = Z_1$. The solution of (35) can be computed using the algorithms of Varga (1997). Solving the PLDE (35) instead of the standard DLE (22) has several advantages. First, we can avoid to form the product $\Psi_{\bar{A}_\mu} = \Theta_N \ldots \Theta_2 \Theta_1$ by working directly on the component matrices Θ_k via the PRSF. Since (34) and (35) share the same matrices Θ_k, only a single reduction to PRSF is necessary to solve both equations (Varga 1997). In this way, the computational effort to solve both equations is practically the same as for solving a single equation. Additionally, by avoiding forming explicitly the above matrix product, the numerical stability of the solution algorithm can be guaranteed (Varga 1997). A second advantage is that the computed solution Z_k represents the value of $V_{\bar{A}_\mu}((k-1)\Delta) := \Phi_{\bar{A}_\mu}((k-1)\Delta, 0) V \Phi_{\bar{A}_\mu}^T((k-1)\Delta, 0)$ intervening in the expression of the gradients (19) and (20). The final values $P(T)$ and $\Psi_{\bar{A}_\mu} V \Phi_{\bar{A}_\mu}$ necessary to apply the cubic spline-based interpolation formulas can be computed as P_1 and $Z_1 - X_0$, respectively.

The multiple-shooting method needs $2N$ calls of a selected ODE solver to evaluate the function and its gradient. The total number of integrated equations is $n^2 + \frac{n(n+1)}{2}$ over time periods $[(k-1)\Delta, k\Delta]$. Although the global integration effort appears to be significantly less than for the periodic generator method, still the large number of calls of the ODE solvers implies automatically a large computational overhead due to the need to initialise each time the computations. Therefore, for small values of n and large values of N the multiple shooting approach is more time demanding than the single-shooting approach. However, for large problems and when the integration of ODEs can be performed in parallel, the multiple-shooting approach should generally be the method of choice due to its better guaranteed numerical properties.

4. Simulation results

In this section the proposed approach to the design of constant-gain controllers for LTP systems will be demonstrated, first (Section 4.1) on a simple numerical example and subsequently (Section 4.2) on the problem of designing a constant gain attitude controller for a satellite with a specific sensor configuration which gives rise to LTP linearised dynamics.

4.1 *A numerical example*

The SISO periodic system proposed in Lovera, Colaneri, Celi, and Bittanti (2002) will be analysed as a simple example. The matrices of this LTP system are given by

$$A(t)=\begin{bmatrix}-1+\sin(t) & 0\\ 1-\cos(t) & -3\end{bmatrix},\quad B(t)=\begin{bmatrix}-1-\cos(t)\\ 2-\sin(t)\end{bmatrix} \tag{36}$$

$$C(t)=\begin{bmatrix}0 & 1\end{bmatrix},\quad D(t)=0. \tag{37}$$

This system is open-loop stable having the characteristic exponents −1 and −3 (Lovera et al. 2002). Therefore the optimisation procedure can be initialised with the trivial choice $F=0$. We used $Q=I_{2\times 2}$ and $R=1$ as weighting matrices in the quadratic criteria.

For this example, the single-shooting approach is remarkably efficient. For the solution of the unconstrained optimisation problem the `fminunc` function from the Optimisation Toolbox for MATLAB has been used. Similarly, for the evaluation of the function and gradient standard MATLAB tools have been used, as for example, the non-stiff solver `ode113` for the integration of the ODEs, the function `spline` for spline-based interpolations, or the function `quadl` for numerical quadratures. For all computations we set the relative and absolute tolerances in both solvers and numerical quadrature function to 10^{-8}.

The optimal constant SOF regulator $F^*=0.681$ was found and eight function and gradient evaluations were performed. In Table 1 we show the effects of choosing different values of N. As can be observed, satisfactory accuracy has been achieved for values of N as small as $N=32$, where a supplementary iteration was necessary to arrive to an even better accuracy than for larger values of N.

To demonstrate the applicability of the multiple-shooting approach on parallel computers we performed several timing measurements on a DELL Precision T5500 desktop with two Intel Xenon X5550 quad-core processors running at 2.66 GHz. We focussed on the parallelisation of solution of the ODEs (30) and (33) on N intervals of the form $[(k-1)\Delta, k\Delta]$ and the determination in parallel of the periodic matrices P_k in (34) and Z_k in (35). For this purpose we used the Parallel Computation Toolbox of MATLAB. In Table 2 we present the total times necessary for $N=64$ and $N=512$ to solve the above problem by using 1–8 processors. As can be observed, the maximum speedup for eight processors of about 4.3 has been obtained for $N=512$. For comparison, the times for solving the same problem using the single-shooting approach was 1.38 s for $N=64$ and 1.47 s for $N=512$. This large discrepancy in performance can be explained by the significant overhead involved when calling many times the ODE solvers.

Table 1. Accuracy versus number of grid points.

N	Number of iterations to convergence	Gradient norm at iteration 6
16	13	8×10^{-4}
32	7	2.3×10^{-5}
64	6	7.1×10^{-8}
128	6	2.6×10^{-8}
256	6	2.5×10^{-8}
512	6	2.4×10^{-8}

Table 2. Timing results using parallel computations.

Number of processors	Time (s) for $N=64$	Time (s) for $N=512$
1	8.8	60.0
2	6.4	43.7
3	5.0	30.6
4	3.9	23.9
5	3.7	19.4
6	3.3	17.1
7	3.2	15.3
8	3.1	13.8

In Table 3 we present results of a comparison study among several controllers designed as follows:

- LQ optimal periodic state feedback controller (*LTP*1).
- LQ constant gain controller designed using the LTP model (*LTP*2).
- LQ constant gain controller obtained averaging the LQ optimal periodic controller over one period T (*LTP*3).

As can be seen from Table 3, the constant controller *LTP*2 provides a performance level which is extremely close to the one achieved by the optimal periodic controller *LTP*1. Note, also, that while the controller *LTP*3 also provides an acceptable performance in this simple example, this comes with no guarantee whatsoever of closed-loop stability for the controlled system.

This solution may be somewhat improved considering a time-varying $F(t)$ and including an increasing number of harmonics; this may be done, for instance, following the simple procedure described in Proposition 3.

Table 3. Comparison of optimal closed-loop performance.

Control	$J^{\star}$
Open loop	1.451
$LTP1$	0.63
$LTP2$	0.643
$LTP3$	0.792
Loss from $LTP1$ to $LTP2$	2.02%

Table 4. Optimal periodic feedback gains for increasing number of harmonics k.

k	$\tilde{F}^{*}$	Difference with $LTP1$ (%)
0	0.6810	2.1
1	[0.18268, 0.70010, 0.27482]	0.05
2	[0.14390, 0.63628, 0.30402, 0.06944, −0.00058]	0.02
3	[0.13546, 0.62382, 0.32978, 0.09989, −0.01020, −0.03783, −0.00035]	0.01

The results, shown in Table 4 and in Figures 1 and 2, demonstrate that the first harmonic contribution is the most relevant, as it could have been reasonably expected. Besides, this numerical analysis underlines the significance of the constant feedback case, particularly in practical applications.

Finally, in order to show the effect of the choice of the covariance matrix of the initial state on the performance index minimisation, the two different approaches proposed in Section 2.2 have been applied to the same simple system: a normal distribution with zero mean and unitary standard deviation of initial conditions x_0 (1000 samples) has been generated using the function randn of Matlab. Then, for each sample, the corresponding (deterministic, not expected value) performance index has been computed according to two different scenarios, using the SOF controllers optimised, respectively, for $X_0 = I_2$ and $X_0 = x_0 x_0^T$. The results reported in Table 5 show that the assumption $X_0 = I_2$ produces a closed-loop system that behaves better in an average sense, i.e. it is characterised by a smaller expected cost μ_J and variance σ_J. On the other hand, the hypothesis $X_0 = x_0 x_0^T$ not only slightly reduces the stability degree of the closed-loop system but also it turns out to be indeed the best choice when the initial state is known exactly (smaller J_{x0}).

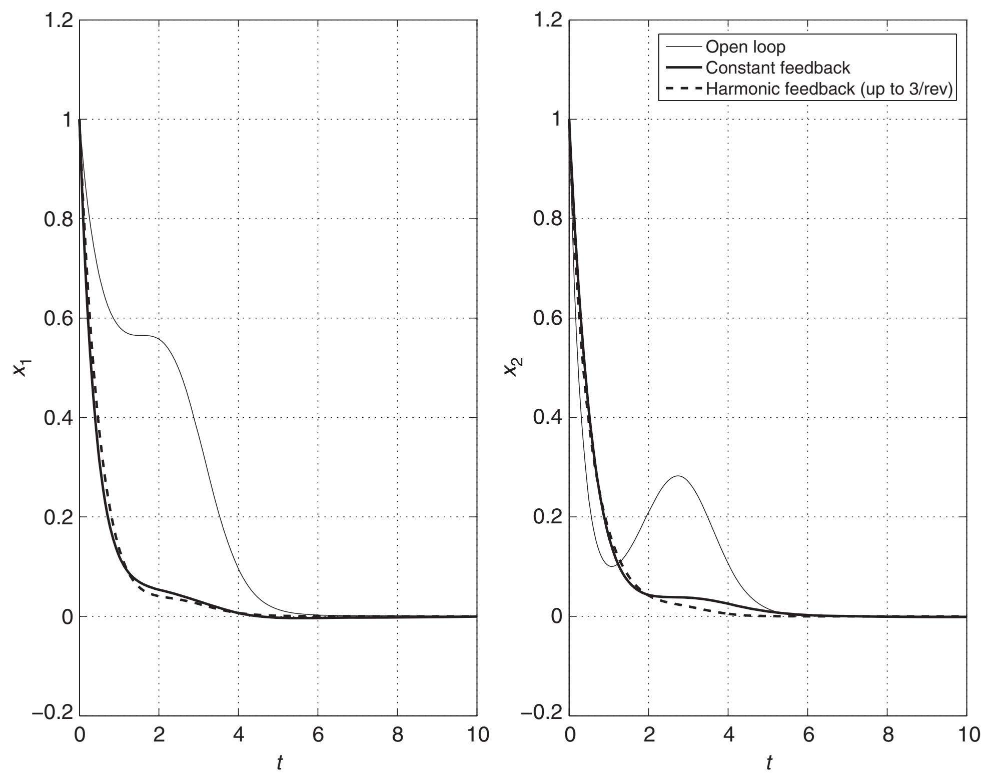


Figure 1. State vector trajectory from $x_0 = [1, 1]^T$.

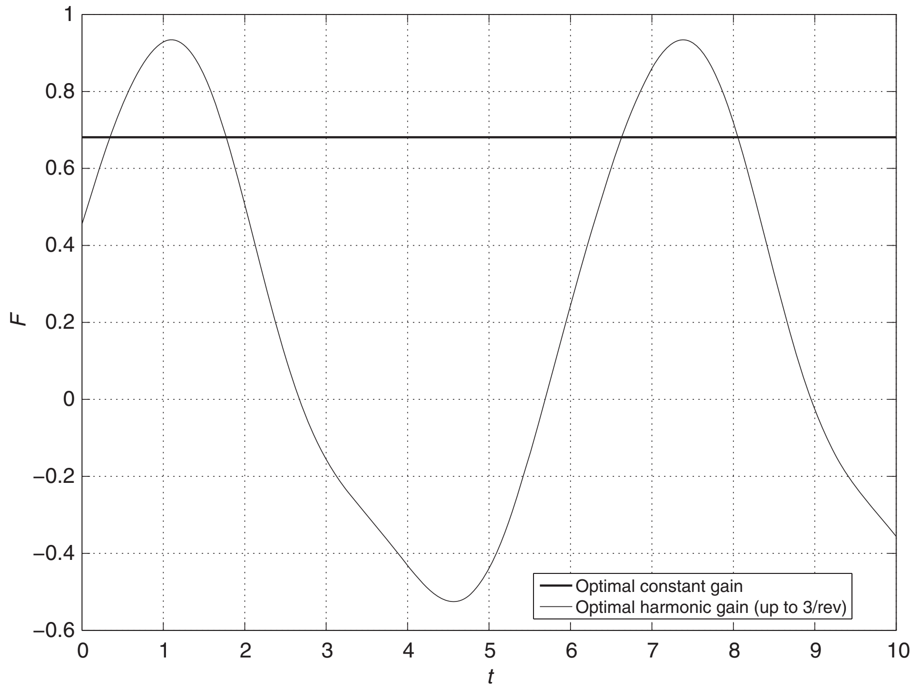


Figure 2. Constant versus harmonic optimal SOF controller.

Table 5. Performance evaluation with random initial conditions.

X_0	F^*	μ_J	σ_J	J_{x0}
I_2	0.06813	1.3913	3.0905	1.33026
$x_0x_0^T$	0.68104	2.0758	6.5960	0.64271

4.2 *A case study: attitude control of an LEO satellite*

Angular rate sensors are frequently used in space missions in order to provide either an accurate alternative to pseudo-derivatives of attitude measurements in the implementation of derivative feedback or as a source of accurate attitude measurements in rate integration mode, for e.g. high-accuracy attitude manoeuvres. Unfortunately it is well known that the main issue associated with rate gyro feedback is the presence of bias and drift, which makes such sensors unreliable over long time spans and introduces the need for the on-line estimation of calibration parameters.

The availability of new generation angular rate sensors with significantly improved characteristics in terms of bias and bias stability, however, might lead to a very different scenario as far as attitude control system (ACS) design is concerned since they would make it possible to design and implement control laws with highly accurate derivative action with limited or no concern for calibration issues. In particular, the availability of accurate angular rate information might lead to more relaxed requirements as far as attitude sensors are concerned, so it would be conceivable to operate the ACS loop using only feedback from simple and low-cost sensors such as magnetometers. In view of this discussion, in the following sections the problem of designing a linear attitude controller for a satellite on the basis of feedback provided by a triaxial magnetometer and a set of angular rate gyros will be analysed and solved using the methods presented in Section 2.

4.2.1 *Mathematical model*

Reference frames In order to represent the attitude motion of an Earth-pointing spacecraft on a circular orbit, the following reference systems are adopted:

- Earth centred inertial (ECI) reference axes. The Earth's centre is the origin of these axes. The positive X-axis points in the vernal equinox direction. The Z-axis points in the direction of the North Pole. The Y-axis completes the right-handed orthogonal triad.
- Orbital axes (X_0, Y_0, Z_0). The origin of these axes is at the satellite centre of mass. The X-axis points to the Earth's centre; the Y-axis points in

the direction of the orbital velocity vector. The Z-axis is normal to the satellite orbit plane.
- Satellite body axes. The origin of these axes is at the satellite centre of mass; in nominal Earth-pointing conditions, the X_b (yaw), Y_b (roll) and Z_b (pitch) axes are aligned with the corresponding orbital axes.

Attitude dynamics For the purpose of the present study, we consider as state variables the quaternion vector $q_{BR} = \left[q_r^T, q_4\right]^T \in \mathbf{R}^4$ representing the relative attitude of the satellite with respect to the orbital axes and the components $\omega_{BI} \in \mathbf{R}^3$ of the inertial angular rate vector of the satellite with respect to the body axes. In view of the above definitions for attitude kinematics and dynamics, we have that the quaternion, when the body system is coincident with the orbital system, is equal to the unit quaternion defined as $1_q = [0\ 0\ 0\ 1]^T$. Letting

$$x(t) = \begin{bmatrix} q_{BR}(t) \\ \omega_{BI}(t) \end{bmatrix}, \tag{38}$$

the considered nominal state is therefore given by

$$x_{\text{Nom}} = \begin{bmatrix} 1_q^T & 0 & 0 & \Omega_{\text{orb}} \end{bmatrix}^T, \tag{39}$$

where Ω_{orb} is the orbital angular frequency.

Measurement models As previously discussed, the goal of this study is to demonstrate the feasibility of an attitude control design approach based solely on static feedback of magnetometer and gyro measurements. To this purpose, suitable models for such measurements will be defined.

The measurement provided by an (ideal) triaxial magnetometer can be simply defined as the vector of body-frame components of the geomagnetic field of the Earth. Therefore, letting b be the onboard measured components of the Earth's magnetic field and b_0 the Earth's magnetic field vector in orbital frame it holds that

$$b(t) = C_{BR}(t) b_0(t), \tag{40}$$

$C_{BR}(t)$ being the attitude matrix corresponding to the quaternion $q_{BR}(t)$. For the purpose of deriving an analytical expression of the linearised measurement model, the dipole model for the geomagnetic field in orbital frame (Wertz 1978; Lovera and Astolfi 2006) can be considered, i.e.

$$b_0(t) = \frac{\mu_F}{(R_E + a)^3} \begin{bmatrix} 2\sin(\Omega_{\text{orb}} t)\sin(i_m) \\ \cos(\Omega_{\text{orb}} t)\sin(i_m) \\ \cos(i_m) \end{bmatrix}, \tag{41}$$

where μ_F is the strength of the dipole of the Earth's magnetic field, i_m is the orbit's inclination with respect to the geomagnetic equator and R_E, a are, respectively, the Earth radius and the orbit altitude. The simplified model of the magnetic field is considered reliable enough for control purposes, though the impact on the closed-loop stability and performance of the approximations implied by the use of such a simplified model should be investigated a posteriori.

As far as gyros are concerned, in modelling the measurements available for control design it will be assumed that ideal access to the true components of the absolute angular rate is available.

Linearised model In the following, the linearised model describing the attitude motion near the nominal state x_{nom} is presented. As expected, the output equations associated with the selected measured variables turn out to be time periodic.

For the linearised state equations it can be assumed that $q_4 \simeq 1$, so that the state vector of the linearised model reduces to

$$\delta x(t) = \begin{bmatrix} \delta q_r(t) \\ \delta\omega(t) \end{bmatrix} = \begin{bmatrix} q_r(t) \\ \omega_{BI}(t) \end{bmatrix} - \begin{bmatrix} 0_{5\times 1} \\ \Omega_{\text{orb}} \end{bmatrix}. \tag{42}$$

In view of these definitions, the linearised model is given by

$$\delta\dot{x}(t) = \begin{bmatrix} \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial \omega} \\ \frac{\partial \dot{\omega}}{\partial q} & \frac{\partial \dot{\omega}}{\partial \omega} \end{bmatrix}_{\text{Nom}} \delta x(t) + \begin{bmatrix} 0 \\ I^{-1} \end{bmatrix} \delta u(t), \tag{43}$$

where

$$\frac{\partial \dot{q}}{\partial q}|_{\text{Nom}} = \begin{bmatrix} 0 & \Omega_{\text{orb}} & 0 \\ -\Omega_{\text{orb}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \dot{q}}{\partial \omega}|_{\text{Nom}} = \frac{1}{2} I_3, \tag{44}$$

$$\frac{\partial \dot{\omega}}{\partial q}|_{\text{Nom}} = 6\Omega_{\text{orb}}^2 I^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & (I_{xx} - I_{zz}) & 0 \\ 0 & 0 & (I_{xx} - I_{yy}) \end{bmatrix},$$
$$\frac{\partial \dot{\omega}}{\partial \omega}|_{\text{Nom}} = \Omega_{\text{orb}} \begin{bmatrix} 0 & \frac{I_{yy} - I_{zz}}{I_{xx}} & 0 \\ \frac{I_{zz} - I_{xx}}{I_{yy}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{45}$$

The expression $\frac{\partial \dot{\omega}}{\partial \omega}$ shown in (45) is valid only if the inertia matrix is diagonal, otherwise it assumes a more complex form. Substituting (44) and (45) in (43) the state equation of the linearised model is obtained.

Linearising (40) one gets

$$b(t) \simeq b_0(t) + 2S^T(b_0(t))\delta q(t), \tag{46}$$

where

$$S(b_0(t)) = \begin{bmatrix} 0 & b_{0z}(t) & -b_{0y}(t) \\ -b_{0z}(t) & 0 & b_{0x}(t) \\ b_{0y}(t) & -b_{0x}(t) & 0 \end{bmatrix}. \tag{47}$$

In view of the problem of designing an SOF controller, it is useful to modify the output equation so as to get a (time-varying) gain between the linearised vector part of the quaternion and the output which is: (i) positive semidefinite and (ii) as close as possible to an identity matrix. This can be achieved by defining the output associated with magnetometer measurements as

$$\begin{aligned} y_1(t) &= \frac{1}{2\|b_0(t)\|^2} S(b_0(t))b(t) \\ &\simeq \frac{1}{\|b_0(t)\|^2} S(b_0(t))S^T(b_0(t))\delta q(t). \end{aligned} \tag{48}$$

For the angular rate measurements one can simply define the output as $y_2(t) = \omega(t)$, so the overall output equation for the linearised model reads

$$\begin{aligned} \delta y(t) &= \begin{bmatrix} \delta y_1(t) \\ \delta y_2(t) \end{bmatrix} = C(t)\delta x(t) \\ &= \begin{bmatrix} \frac{1}{\|b_0(t)\|^2} S(b_0(t))S^T(b_0(t)) & \\ & I_3 \end{bmatrix} \begin{bmatrix} \delta q(t) \\ \delta\omega(t) \end{bmatrix}. \end{aligned} \tag{49}$$

As is well known from the literature (see e.g. Wen and Kreutz-Delgado 1991), if full attitude and angular rate feedback were available (i.e. if we were to replace the periodic matrix gain $\frac{1}{\|b_0\|^2} S(b_0)S^T(b_0)$ with an identity matrix) then closed-loop stability would be guaranteed for any positive value of the proportional and derivative controller gains. In the case of attitude feedback provided by a magnetometer, however, closed-loop stability may be significantly affected by the choice of such parameters. This is due to the fact that feedback from the magnetometers renders the closed-loop dynamics time-periodic, so that stability of the closed-loop system depends on the controller parameters in a fundamentally different way. In particular, this implies that analysis and design have to be carried out using tools from periodic systems theory, as will be discussed in detail in the following section.

4.2.2 *Controller design*

The design approach presented in Section 2 has been applied to the linearised model derived in Section 4.2.1, with specific reference to a case study loosely based on the spacecraft for the ESA SWARM mission (Haagmans 2004), the goal of which is to provide the best ever survey of the geomagnetic field and the first global representation of its variation on time scales from an hour to several days. The Swarm concept consists of a constellation of three satellites in three different polar orbits between 400 and 550 km altitude. For the purpose of this study the following assumptions have been made:

(1) The considered satellite is operating on a circular, near polar orbit ($i = 86.9°$ inclination) with an altitude of 450 km (and a corresponding orbital period of 5614.8 s).
(2) The satellite inertial properties are:
   - Satellite mass $m = 496$ kg
   - Satellite inertia matrix:

$$I = \begin{bmatrix} 465.8 & -15 & -1 \\ -15 & 48.5 & -2.8 \\ -1 & -2.8 & 439.9 \end{bmatrix} \quad (\text{kg m}^2). \tag{50}$$

More precisely, as far as the LQP design approach is concerned, the weighting matrices in (6) have been chosen as $Q = I_6$ and $R = 10^3 I_3$; the computed output-feedback gain leads to a stable closed-loop system with the following values for the characteristic multipliers:

$$\lambda_{LQP} = \begin{bmatrix} 0.0339 & 0.0000 & 0.0000 & 0 & 0 & 0 \end{bmatrix}. \tag{51}$$

In order to be able to quantify the benefits of taking the periodicity of the linearised model into account in the design of the control law, a second controller has been designed using a time-invariant approximation of the linearised model. The time-invariant approximation has been obtained by computing the average over one orbit period T of matrix $C(t)$ in (49)

$$\bar{C} = \frac{1}{T}\int_0^T C(t)\mathrm{d}t. \tag{52}$$

As $\bar{C}$ turns out to be nonsingular, for the averaged linearised model it is possible to design a constant gain controller by solving a state feedback rather than an output feedback problem. The control law is given by

$$\delta u(t) = F\delta y(t), \tag{53}$$

with $F = -K\bar{C}^{-1}$, where K is the LQ state-feedback gain computed using the same Q and R weighting matrices as in the periodic design case. In order to check the closed-loop stability of the original linearised periodic system under the feedback (53), the characteristic multipliers of the closed-loop system have been computed:

$$\lambda_{LQ} = \begin{bmatrix} 0.5369 & 0.0673 & 0.0003 & 0 & 0 & 0 \end{bmatrix}. \tag{54}$$

The linearised model and the gains computed with the two design approaches have been implemented and the performance of these control laws have been assessed. In the simulations, the initial state

$$x_0 = \begin{bmatrix} 0.05 & 0.05 & 0.05 & 0.001 & 0.001 & 0.001 \end{bmatrix}^T, \tag{55}$$

has been considered, which corresponds to an error of about 5.7° between the body frame and the orbital frame for each axis and to an error close to Ω_{orb} on the body components of the angular rate,

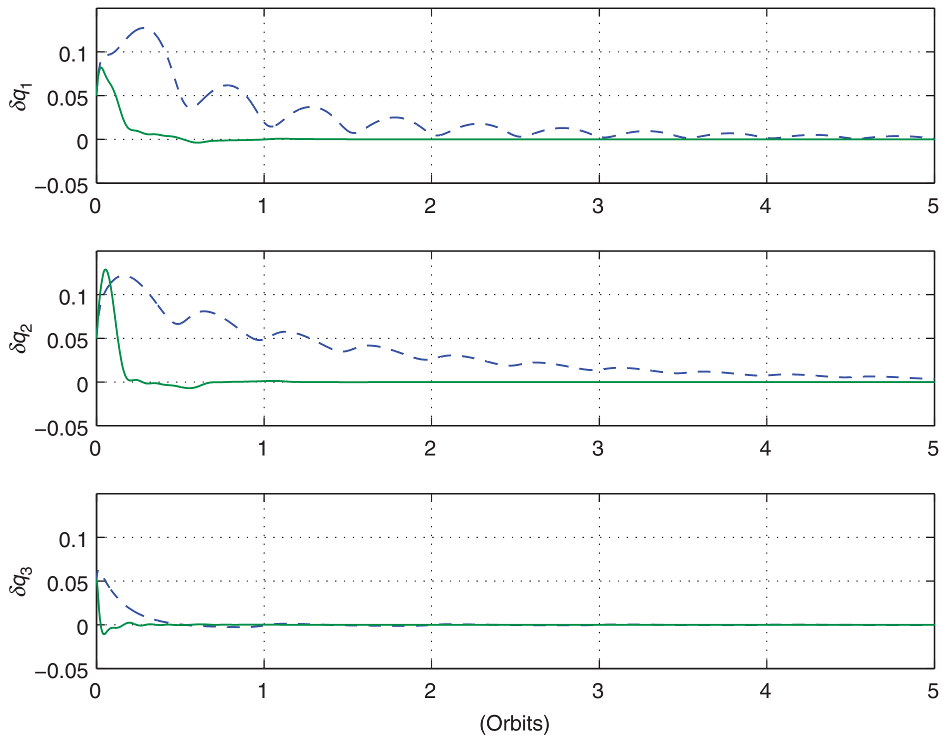
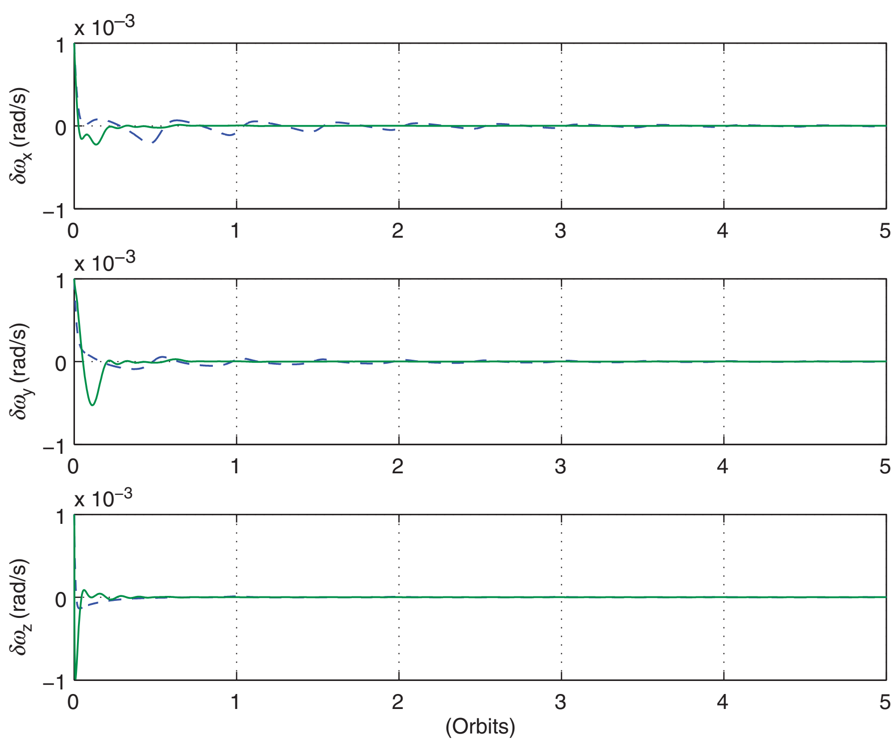


Figure 3. Closed-loop time histories of δq_r using the LQ (dashed lines) and LQP (solid lines) controllers.



$\delta\omega$

Figure 4. Closed-loop time histories of $\delta\omega$ using the LQ (dashed lines) and LQP (solid lines) controllers.

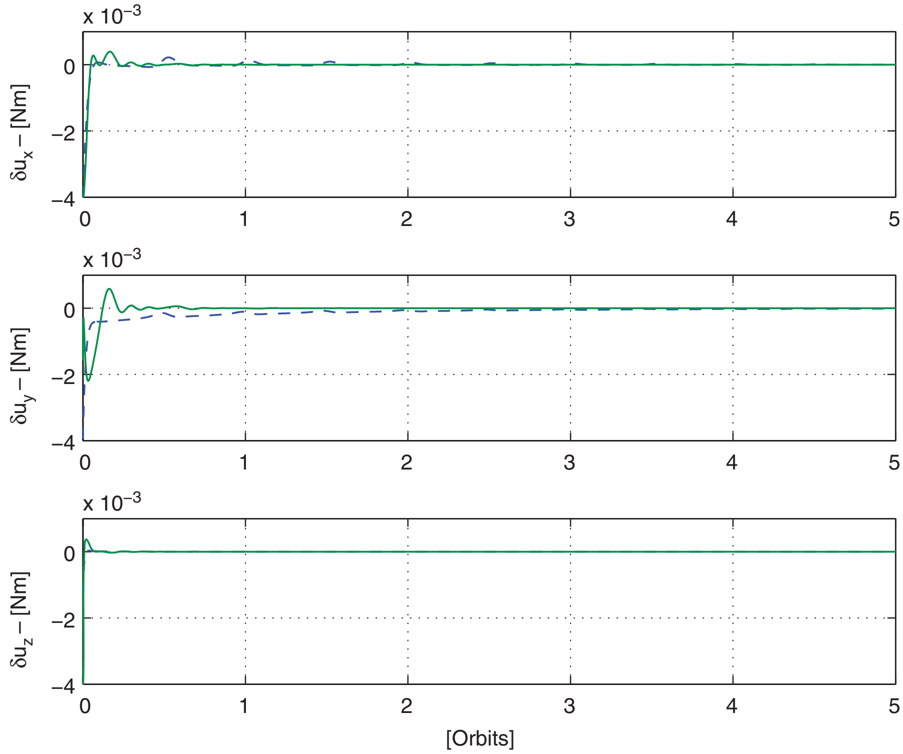


Figure 5. Control torques, using the LQ (dashed lines) and LQP (solid lines) controllers.

i.e. a representative initial condition for a nominal attitude controller.

The results of the simulations are shown in Figures 3–5. As can be seen, both control laws bring the satellite back to its nominal attitude, removing the initial attitude and angular rate error by applying control torques of acceptable values.

A comparison of the closed-loop dynamics obtained using the two controllers, however, show clearly that using the design approach capable of taking the periodicity of the output equation into account a better result can be obtained, namely a faster and smoother transient for the attitude parameters. Indeed, with similar control torques the LQ controller brings the satellite to its nominal attitude in about 5 orbits (≈30,000 s), while the LQP controller achieves the same results much more effectively, i.e. in less than one orbit.

5. Conclusions

The problem of optimal SOF control of linear periodic systems has been considered and a novel, continuous-time approach has been proposed, which allows to deal with both stable and unstable open loop systems. Simulation results demonstrate that excellent performance can be obtained, when comparing the novel design approach with optimal periodic control techniques, with the additional, significant benefit of providing a constant gain controller instead of a time-periodic one. The proposed approach has been applied to the problem of designing attitude control laws for an LEO spacecraft relying on measurements provided by a triaxial magnetometer and a set of high-precision gyros. Simulation results show the feasibility of the proposed strategy and the performance achievable by means of the optimal, constant gain controller.

Acknowledgement

This work was supported in part by the Italian National Research Project 'New Techniques of Identification and Adaptive Control for Industrial Systems'.

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Appendix A. Optimal output-feedback control for continuous-time periodic systems

The aim of this Appendix is to provide a complete technical treatment of the optimisation-based design approaches which have been outlined in Section 2. In particular, the iterative optimization method proposed in Varga and Pieters (1998) is first extended to the continuous-time case; then, a penalty method for open-loop unstable systems is introduced.

A.1. Performance index computation

The closed-loop dynamic matrix $\bar{A}(t)$ defined in (8) is associated with a transition matrix $\Phi_{\bar{A}}(t,0)$ satisfying the well-known equation

$$\dot{\Phi}_{\bar{A}}(t,0) = \bar{A}(t)\Phi_{\bar{A}}(t,0), \quad \Phi_{\bar{A}}(0,0) = I. \tag{56}$$

Moreover, the transition matrix is given by the Peano–Baker series (Brockett 1970)

$$\Phi_{\bar{A}}(t,0) = I + \int_0^t A(\tau_1)\mathrm{d}\tau_1 + \int_0^t A(\tau_1)\int_0^t A(\tau_2)\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \cdots. \tag{57}$$

Suppose to know a periodic stabilising matrix $F(t)$, i.e. such that $|\lambda_i(\Psi_{\bar{A}})| < 1\ \forall i$, where $\Psi_{\bar{A}}$ is the monodromy matrix of the closed-loop system; the performance index J defined in (6) may be written as

$$\begin{aligned} J &= E\left\{\int_0^\infty \left[x^T(t)Q(t)x(t) + x^T(t)C^T(t)F^T(t)R(t)F(t)C(t)x(t)\right]\mathrm{d}t\right\} \\ &= E\left\{\int_0^\infty \left[x^T(t)\bar{Q}(t)x(t)\right]\mathrm{d}t\right\}, \end{aligned} \tag{58}$$

where $\bar{Q}(t) = Q(t) + C^T(t)F^T(t)R(t)F(t)C(t)$. Observing that $x(t) = \Phi_{\bar{A}}(t,0)x_0$, the expression (58) yields

$$J = E\left\{\int_0^\infty x_0^T\Phi_{\bar{A}}^T(t,0)\bar{Q}(t)\Phi_{\bar{A}}(t,0)x_0\mathrm{d}t\right\}. \tag{59}$$

If we define $P(t) = \int_t^\infty \Phi_{\bar{A}}^T(\tau,t)\bar{Q}(\tau)\Phi_{\bar{A}}(\tau,t)\mathrm{d}\tau$, we can express the performance index J as shown in Section 2 (Equation (9))

$$\begin{aligned} J &= E\left\{x_0^T\int_0^\infty \Phi_{\bar{A}}^T(t,0)\bar{Q}(t)\Phi_{\bar{A}}(t,0)\mathrm{d}t\,x_0\right\} \\ &= E\left\{x_0^TP(0)x_0\right\} = \mathrm{tr}(P(0)X_0). \end{aligned}$$

In particular, in order to verify (10), the two following properties can be used:

(1) Consider a generic integral function of the form: $P(t) = \int_t^\infty f(t,\tau)\mathrm{d}\tau$; according to the fundamental theorem of calculus, we can write: $\dot{P}(t) = -f(t,\tau)|_{\tau=t} + \int_t^\infty \frac{\partial}{\partial t}f(t,\tau)\mathrm{d}\tau$.
(2) If $\Phi_{\bar{A}}(t,\tau)\Phi_{\bar{A}}(\tau,t) = I$ then $\dot{\Phi}_{\bar{A}}(t,\tau)\Phi_{\bar{A}}(\tau,t) + \Phi_{\bar{A}}(t,\tau)\dot{\Phi}_{\bar{A}}(\tau,t) = 0$; so it follows: $\bar{A}(t) + \Phi_{\bar{A}}(t,\tau)\times\dot{\Phi}_{\bar{A}}(\tau,t) = 0$, that is $\dot{\Phi}_{\bar{A}}(\tau,t) = -\Phi_{\bar{A}}(\tau,t)A(t)$ (being $\Phi_{\bar{A}}^{-1}(t,\tau) = \Phi_{\bar{A}}(\tau,t)$).

Properties 1 and 2 allow us to write

$$\begin{aligned} \dot{P}(t) = -\Phi_{\bar{A}}^T(t,t)\bar{Q}(t)\Phi_{\bar{A}}(t,t) - \int_t^\infty \Big[&\Phi_{\bar{A}}^T(\tau,t)\bar{Q}(\tau)\Phi_{\bar{A}}(\tau,t)\bar{A}(t) \\ &+ \bar{A}^T(t)\Phi_{\bar{A}}^T(\tau,t)\bar{Q}(\tau)\Phi_{\bar{A}}(\tau,t)\Big]\mathrm{d}\tau, \end{aligned} \tag{60}$$

which verifies (10).

A.2. Performance index minimisation: the constant gain case

In order to derive the gradient expression presented in Proposition 1 (see (12) in Section 2) we proceed as follows. Following the approach described in Knapp and Basuthakur (1972), let us introduce the Hamiltonian function

$$\begin{aligned} H &= x^T(t)\bar{Q}(t)x(t) + \lambda_x^T(t)\dot{x}(t) + \Lambda_F^T(t)\dot{F} \\ &= x^T(t)\bar{Q}(t)x(t) + \lambda_x^T\bar{A}(t)x(t) \\ &= \mathrm{tr}\left[\bar{Q}(t)x(t)x^T(t) + \bar{A}(t)x(t)\lambda_x^T(t)\right], \end{aligned} \tag{61}$$

with

$$\begin{aligned} \dot{\lambda}_x(t) &= -\frac{\partial H}{\partial x} = -2\left[Q(t) + C^T(t)F^TR(t)FC(t)\right]x(t) \\ &\quad - [A(t) + B(t)FC(t)]^T\lambda_x, \quad \lambda_x(\infty) = 0, \\ \dot{\Lambda}_F(t) &= -\frac{\partial H}{\partial F} = -\frac{\partial}{\partial F}\mathrm{tr}\big[(Q(t) + C^T(t)F^TR(t)FC(t))x(t)x^T(t) \\ &\quad + (A(t) + B(t)FC(t))x(t)\lambda_x^T(t)\big]. \end{aligned}$$

Making use of the derivation rules for trace operator employed by Varga and Pieters (1998), the equation for the dynamics of Λ_F can be written as

$$\begin{aligned} \dot{\Lambda}_F(t) &= -R(t)FC(t)x(t)x^T(t)C^T(t) - R(t)FC(t)x(t)x^T(t)C^T(t) \\ &\quad - B^T(t)\lambda_x(t)x^T(t)C^T(t) \\ &= -2R(t)FC(t)x(t)x^T(t)C^T(t) - B^T(t)\lambda_x(t)x^T(t)C^T(t), \\ &\quad \Lambda_F(\infty) = 0. \end{aligned} \tag{62}$$

Necessary conditions for optimality state that

$$\frac{\partial}{\partial F}E\{J(F,x(0)\} = E\left\{\frac{\partial}{\partial F}J(F,x(0))\right\} = 0, \tag{63}$$

which implies, according to (62)

$$
\begin{aligned}
0 &= E\left\{\int_0^\infty -\dot{\Lambda}_F \mathrm{d}t\right\} \\
&= E\left\{\int_0^\infty 2R(t)FC(t)x(t)x^T(t)C^T(t) + B^T(t)\lambda_x(t)x^T(t)C^T(t)\mathrm{d}t\right\} \\
&= \int_0^\infty \left[2R(t)FC(t)E\left\{x(t)x^T(t)\right\}C^T(t)\right. \\
&\quad \left. + B^T(t)E\left\{\lambda_x(t)x^T(t)\right\}C^T(t)\right]\mathrm{d}t.
\end{aligned} \tag{64}
$$

Performing the substitution $\lambda_x(t) = 2P(t)x(t)$, where $P(t)$ is a symmetric positive definite matrix, Equation (62) for $\dot{\lambda}_x$ allows to write

$$
2\dot{P}(t)x(t) + 2P(t)\bar{A}(t)x(t) = -2\bar{Q}(t)x(t) - 2\bar{A}^T(t)P(t)x(t), \tag{65}
$$

which is the same equation represented by the PLDE (10), having the periodic solution $P(t)$. Moreover, the above substitution also implies that

$$
\nabla_F J = 2\int_0^\infty \left(R(t)FC(t) + B^T(t)P(t)\right)E\left\{x(t)x^T(t)\right\}C^T(t)\mathrm{d}t = 0. \tag{66}
$$

It is worth noticing that the state covariance matrix $X(t) = E\{x(t)x^T(t)\}$ satisfies

$$
X(t) = \Phi_{\bar{A}}(t,0)E\left\{x(0)x^T(0)\right\}\Phi_{\bar{A}}^T(t,0) = \Phi_{\bar{A}}(t,0)X_0\Phi_{\bar{A}}^T(t,0), \tag{67}
$$

or, equivalently, the linear matrix homogeneous differential equation

$$
\dot{X}(t) = \bar{A}(t)X(t) + X(t)\bar{A}^T(t). \tag{68}
$$

It is possible to obtain the gradient expression in (69) in an alternative way, which avoids the (somewhat arbitrary) substitution $\lambda_x(t) = 2P(t)x(t)$. Note that the minimisation of the performance index $J = \mathrm{tr}(P(0)X_0)$ corresponds to an optimisation problem applied to the functional

$$
J = \mathrm{tr}\left(\int_{-\infty}^\infty P(t)X(t)\delta(t)\mathrm{d}t\right) = \mathrm{tr}\left(\int_0^\infty P(t)X(t)\delta(t)\mathrm{d}t\right), \tag{69}
$$

where $\delta(t)$ represents the Dirac function. Observe that the quantities $P(t)$ and $X(t)$ are different from zero only for $t \geq 0$. Introduce now the new Hamiltonian function

$$
H = \mathrm{tr}\left[P(t)X(t)\delta(t) - \Lambda_P^T(t)\left(\bar{A}^T(t)P(t) + P(t)\bar{A}(t) + \bar{Q}(t)\right)\right], \tag{70}
$$

and, correspondingly, let

$$
\begin{aligned}
\dot{\Lambda}_F(t) &= -\frac{\partial H}{\partial F} = 2B^T(t)P(t)\Lambda_P(t)C^T(t) \\
&\quad + 2R(t)FC(t)\Lambda_P(t)C^T(t) \\
&= 2\left(B^T(t)P(t) + R(t)FC(t)\right)\Lambda_P(t)C^T(t), \quad \Lambda_F(\infty) = 0
\end{aligned} \tag{71}
$$

$$
\dot{\Lambda}_P(t) = -\frac{\partial H}{\partial P} = \bar{A}(t)\Lambda_P(t) + \Lambda_P(t)\bar{A}^T(t) - X(t)\delta(t), \quad \Lambda_P(\infty) = 0. \tag{72}
$$

The equation describing the evolution of the co-state matrix Λ_P can be equivalently expressed as

$$
\dot{\Lambda}_P(t) = \bar{A}(t)\Lambda_P(t) + \Lambda_P(t)\bar{A}^T(t), \quad \Lambda_P(0) = -X_0 \tag{73}
$$

suggesting that $\Lambda_P(t) = -X(t)$. Therefore, first-order optimality conditions can be expressed again as (66).

Finally, in order to write the analytical expression of the gradient in a form suitable for numerical optimisation algorithms, the integral present in (66) should be computed only over a finite horizon (reasonably one period). For that reason, the following analytical manipulations are suggested:

$$
\begin{aligned}
\nabla_F J &= 2\int_0^\infty \left(R(t)FC(t) + B^T(t)P(t)\right)X(t)C^T(t)\mathrm{d}t \\
&= 2\sum_{n=0}^\infty \int_0^T \left[\left(R(t+nT)FC(t+nT)\right.\right. \\
&\quad \left.\left. + B^T(t+nT)P(t+nT)\right)X(t+nT)C^T(t+nT)\right]\mathrm{d}t \\
&= 2\sum_{n=0}^\infty \int_0^T \left[\left(R(t)FC(t) + B^T(t)P(t)\right)X(t+nT)C^T(t)\right]\mathrm{d}t.
\end{aligned} \tag{74}
$$

Since $x(t+nT) = \Phi_{\bar{A}}(t+nT,0)x_0 = \Phi_{\bar{A}}(t,0)\Psi_{\bar{A}}^n x_0$, the term $X(t+nT)$ may be written as

$$
\begin{aligned}
X(t+nT) &= E\left\{x(t+nT)x^T(t+nT)\right\} \\
&= \Phi_{\bar{A}}(t,0)\Psi_{\bar{A}}^n X_0\left(\Psi_{\bar{A}}^n\right)^T\Phi_{\bar{A}}^T(t,0).
\end{aligned} \tag{75}
$$

Substituting (75) into (74) we have

$$
\begin{aligned}
\nabla_F J &= 2\int_0^T \left(R(t)FC(t) + B^T(t)P(t)\right)\Phi_{\bar{A}}(t,0) \\
&\quad \times \left[\sum_{n=0}^\infty \Psi_{\bar{A}}^n X_0\left(\Psi_{\bar{A}}^n\right)^T\right]\Phi_{\bar{A}}^T(t,0)C^T(t)\mathrm{d}t \\
&= 2\int_0^T \left(R(t)FC(t) + B^T(t)P(t)\right)\Phi_{\bar{A}}(t,0)V\Phi_{\bar{A}}^T(t,0)C^T(t)\mathrm{d}t,
\end{aligned} \tag{76}
$$

where the matrix V is symmetric positive definite and it satisfies the discrete Lyapunov equation

$$
V = \Psi_{\bar{A}}V\Psi_{\bar{A}}^T + X_0. \tag{77}
$$

Moreover, it is easy to show that the overall term $S = \Phi_{\bar{A}}(t,0)V\Phi_{\bar{A}}^T(t,0)$ satisfies the differential Lyapunov equation

$$
\dot{S}(t) = \bar{A}(t)S(t) + S(t)\bar{A}^T(t), \quad S(0) = V. \tag{78}
$$

A.3. Performance index minimisation: penalty method

Along the lines of the derivation in A.2, the results given in Proposition 2 can be obtained by considering the Hamiltonian function

$$
\begin{aligned}
H = \mathrm{tr}\big[&\left(P(t)X(t) + \sigma\mu^2 I\right)\delta(t) \\
&- \Lambda_P^T(t)\left(\bar{A}_\mu^T(t)P(t) + P(t)\bar{A}_\mu(t) + \bar{Q}(t)\right)\big],
\end{aligned} \tag{79}
$$

where the closed-loop dynamic matrix $\bar{A}_\mu = A_\mu(t) + B(t)FC(t)$, the extremality conditions are given by

$$\dot{\Lambda}_F(t) = -\frac{\partial H}{\partial F} = 2\big(B^T(t)P(t) + R(t)FC(t)\big)\Lambda_P C^T(t), \quad \Lambda_F(\infty) = 0, \tag{80}$$

$$\dot{\Lambda}_P(t) = -\frac{\partial H}{\partial P} = \bar{A}_\mu(t)\Lambda_P(t) + \Lambda_P(t)\bar{A}_\mu^T(t) - X(t)\delta(t-0), \quad \Lambda_P(\infty) = 0, \tag{81}$$

$$\dot{\Lambda}_\mu(t) = -\frac{\partial H}{\partial \mu} = -2\big[\mu\sigma\delta(t) + \mathrm{tr}\big(\Lambda_P^T(t)P(t)\big)\big], \quad \Lambda_\mu(\infty) = 0. \tag{82}$$

Observing that $\nabla_F \tilde{J}$ is analogous to (76) (except that we are now considering the modified system having $A_\mu(t)$ as dynamic matrix), the sensitivity of the cost function $\tilde{J}$ with respect to μ can be expressed as

$$\begin{aligned}\frac{\partial \tilde{J}}{\partial \mu} &= 2\mu\sigma \int_{-\infty}^{\infty} \delta(t-0)\mathrm{d}t - 2\int_0^\infty \mathrm{tr}\big(\Lambda_P^T(t)P(t)\big)\\ &= 2\mu\sigma + 2\int_0^\infty \mathrm{tr}(P(t)X(t))\mathrm{d}t\\ &= 2\mu\sigma + \sum_{n=0}^{\infty}\int_0^T \mathrm{tr}[P(t+nT)X(t+nT)]\mathrm{d}t\\ &= 2\mu\sigma + \int_0^T \mathrm{tr}\left[P(t)\Phi_{\bar{A}_\mu}(t,0)\sum_{n=0}^{\infty}\Psi_{\bar{A}_\mu}^n X_0(\Psi_{\bar{A}_\mu}^n)^T\Phi_{\bar{A}_\mu}^T(t,0)\right]\mathrm{d}t.\end{aligned} \tag{83}$$

It should be noted that the stationary solution of (83) does not imply, in general, $\mu = 0$; nevertheless, as said before, a sufficiently large value of parameter σ can steer the stationary value of μ to a negligible level.