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A Note on the Spatial Jet-Instability of the Compressible Cylindrical Vortex Sheet

von

A. Michalke

Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt

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Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt E.V.
Institut für Turbulenzforschung
Abteilung: Aeroakustik und Instabilität
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Institutsleiter:
Prof. Dr.-Ing. R. WILLE
Abteilungsleiter:
Dr.-Ing. A. MICHALKE

Bearbeiter:
Dr.-Ing. A. MICHALKE

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Summary

The instability of compressible circular jets with respect to spatially growing disturbances is the subject of this note. The linearized inviscid disturbance equations have been derived for a compressible circular jet with variable temperature and density profile. The eigenvalues of the instability problem for the cylindrical vortex sheet have been computed and discussed for various Strouhal numbers, Mach numbers and temperature ratios. It is found that contrary to temporal amplification the phase velocity of spatially growing disturbances can exceed the jet velocity. Furthermore, additional disturbance modes can exist in the spatial case.

Eine Bemerkung zur Freistrahl-Instabilität einer kompressiblen zylindrischen Wirbelschicht bei räumlicher Anfachung

Übersicht

Dieser Bericht beschäftigt sich mit der Instabilität des kompressiblen, runden Freistrahls bei räumlich angefachten Störungen. Die reibungslose, linearisierte Störungsgleichung wird für einen kompressiblen runden Freistrahl mit variabler Temperatur- und Dichteverteilung hergeleitet. Die Eigenwerte des Instabilitätsproblems für eine zylindrische Wirbelschicht bei verschiedenen Strouhalzahlen, Machzahlen und Temperaturverhältnissen werden berechnet und diskutiert. Es zeigt sich, daß im Gegensatz zur zeitlichen Anfachung die Phasengeschwindigkeit räumlich angefachter Störungen größer als die Freistrahlgeschwindigkeit sein kann. Außerdem können zusätzliche Störungsmoden auftreten.

Contents

1. Introduction

The instability of circular jets to small wavy disturbances was firstly investigated by Lord RAYLEIGH [9]. He found that the simplest case of a circular jet - a cylindrical vortex sheet - is unstable to inviscid axisymmetric disturbances. Further theoretical contributions to instability of circular jets were made by SCHADE [10], while BATCHELOR and GILL [1] and MICHALKE and SCHADE [7] calculated the eigenvalues of incompressible circular jets with simple velocity profiles. LESSEN, FOX and ZIEN [6] and GILL [3] treated the influence of the Mach number on the instability of the cylindrical vortex sheet. KAMBE [5] dealt with the viscous instability of a parabolic jet. All these investigations were based on temporally growing disturbances. It is, however, known from the papers of GASTER [2] and MICHALKE and FREYMUTH [8] that disturbances growing spatially in basic flow direction seem to be more significant for the realistic jet instability than temporally growing disturbances.

For this reason, spatially growing disturbances will be applied to circular jets in the following. For compressible circular jets with variable basic temperature and density profiles the linearized disturbance equations will be derived. For the special case of the cylindrical vortex sheet the eigenvalues of the spatial instability problem will be calculated and discussed.

2. The linearized disturbance equations

Since we are dealing with circular jets, we use cylindrical coordinates (x,r,φ) where the x-axis coincides with the jet axis. The velocity components are (c_x, c_r, c_φ). The undisturbed basic jet flow consists of only one velocity component $U(r)$ in x-direction with $U(0) = U_1$ and $U(\infty) = 0$. The basic temperature distribution is $T(r)$ with $T(0) = T_1$ and $T(\infty) = T_0$. Hence the local sound speed $a(r)$ is given by

$$(1) \qquad a(r) = a_1 \sqrt{T(r)/T_1}$$

The pressure p_0 is constant in the undisturbed jet. Hence it follows from the equation of state that the density distribution is

$$(2) \qquad \bar{\rho}(r) = \rho_1 \, T_1/T(r)$$

The temperature ratio will be denoted by

$$(3) \qquad T_0/T_1 = \rho_1/\rho_0 = (a_0/a_1)^2 = T^*$$

We superimpose small disturbances c'_x, c'_r, c'_φ, p', ρ' upon the basic flow and assume that the Reynolds number is large and the Prandtl number is unity. Then the Euler equation can be used, and the entropy of a fluid particle has to be constant. Hence the following linearized disturbance equations are obtained

$$\bar{\rho}\left[\frac{\partial c'_x}{\partial t} + U\frac{\partial c'_x}{\partial x} + \frac{dU}{dr}c'_r\right] = -\frac{dp'}{\partial x}$$

$$\bar{\rho}\left[\frac{\partial c'_r}{\partial t} + U\frac{\partial c'_r}{\partial x}\right] = -\frac{\partial p'}{\partial r}$$

$$(4) \qquad \bar{\rho}\left[\frac{\partial c'_\varphi}{\partial t} + U\frac{\partial c'_\varphi}{\partial x}\right] = -\frac{1}{r}\frac{\partial p'}{\partial \varphi}$$

$$\frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial x} + c'_r\frac{d\bar{\rho}}{dr} + \bar{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\,c'_r) + \frac{1}{r}\frac{\partial c'_\varphi}{\partial \varphi} + \frac{\partial c'_x}{\partial x}\right] = 0$$

$$\frac{\partial p'}{\partial t} + U\frac{\partial p'}{\partial x} = a^2\left[\frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial x} + c'_r\frac{d\bar{\rho}}{dr}\right].$$

Furthermore we assume wavy disturbances of the type

$$(5) \qquad [c'_x,\, c'_r,\, c'_\varphi,\, p',\, \rho'] = [\tilde{u}(r),\, \tilde{v}(r),\, \tilde{w}(r),\, \tilde{p}(r),\, \tilde{\rho}(r)]\, e^{i(\alpha x + m\varphi - \beta t)}$$

For spatially growing disturbances the cyclic frequency β and the integer azimuthal wave-number m have to be real, while $\alpha = \alpha_r + i\alpha_i$ is generally complex. α_r is the axial wave-number and $-\alpha_i > 0$ is the spatial growth rate. With (5) the equations (4) can be reduced to the following system for the amplitude functions

(6) $$i\alpha\bar{\rho}\left[(U-\beta/\alpha)\frac{1}{r}\frac{d}{dr}(r\,\tilde{v})-\frac{dU}{dr}\,\tilde{v}\right]=-\left[\lambda^2+\frac{m^2}{r^2}\right]\tilde{p}$$

(7) $$i\alpha\bar{\rho}(U-\beta/\alpha)\,\tilde{v}=-\frac{d\tilde{p}}{dr}$$

$$\bar{\rho}\tilde{u}=\frac{i}{\alpha}\,\bar{\rho}\,\frac{dU}{dr}\,\frac{\tilde{v}}{U-\beta/\alpha}-\frac{\tilde{p}}{U-\beta/\alpha}$$

(8) $$\bar{\rho}\tilde{w}=-\frac{m}{\alpha}\frac{1}{r}\frac{\tilde{p}}{U-\beta/\alpha}$$

$$\tilde{\rho}=\frac{\tilde{p}}{a^2}+\frac{i}{\alpha}\frac{d\bar{\rho}}{dr}\frac{\tilde{v}}{U-\beta/\alpha}$$

where

(9) $$\lambda^2=\alpha^2\left[1-\left(\frac{U-\beta/\alpha}{a}\right)^2\right]\,.$$

Equations (6) and (7) can be resolved into a single, second-order differential equation for either the pressure amplitude function $\tilde{p}(r)$ or the amplitude function $\tilde{v}(r)$ of the radial velocity component. The boundary conditions of the instability problem require that the solution has to be regular at $r = 0$ and has to vanish at $r = \infty$.

In a flow region with constant basic velocity U_v, density ρ_v and sound speed a_v equations (6) and (7) yield

(10) $$\frac{d^2\tilde{p}}{dr^2}+\frac{1}{r}\frac{d\tilde{p}}{dr}-\left[\lambda_v^2+\frac{m^2}{r^2}\right]\tilde{p}=0$$

which has the general solution

(11) $$\tilde{p}(r)=C_1\,I_m(\lambda_v r)+C_2\,K_m(\lambda_v r)$$

and

(12) $$\tilde{v}(r)=\frac{i\lambda_v}{\alpha\,\rho_v(U_v-\beta/\alpha)}\left[C_1\,I'_m(\lambda_v r)+C_2\,K'_m(\lambda_v r)\right]$$

Here I_m and K_m are the modified Bessel functions of order m and I'_m and K'_m its derivatives with respect to the argument. C_1 and C_2 are arbitrary constants.

3. The spatial instability of the cylindrical vortex sheet

A circular jet flow, produced by a cylindrical vortex sheet at $r = R$, consists of a uniform velocity $U(r) \equiv U_1$ for $r < R$ and $U(r) \equiv 0$ for $r > R$. Furthermore, we assume that $\bar{\rho}(r) \equiv \rho_1$ and $a(r) \equiv a_1$ for $r < R$, while $\bar{\rho}(r) \equiv \rho_o$ and $a(r) \equiv a_o$ for $r > R$. Then the solution of the disturbance equation (6) and (7) is given by (11) and (12). The pressure amplitude function satisfying the boundary conditions is

$$p(r) = C_1 I_m(\lambda_1 r) \qquad \text{for } 0 \leq r < R,$$

(13)

$$p(r) = C_0 K_m(\lambda_0 r) \qquad \text{for } R < r$$

with

(14) $$\lambda_1^2 = \alpha^2 \left[1 - \left(\frac{U_1 - \beta/\alpha}{a_1}\right)^2\right] \; ; \qquad \Re(\lambda_1) > 0$$

(15) $$\lambda_o^2 = \alpha^2 \left[1 - \left(\frac{\beta/\alpha}{a_o}\right)^2\right] \; ; \qquad \Re(\lambda_o) > 0 .$$

As matching conditions at $r = R$ we have to require that the pressure and the radial displacement are steady functions. The amplitude of the latter is known to be proportional to $\tilde{v}(U - \beta/\alpha)^{-1}$. These two conditions yield two linear homogeneous equations for C_1 and C_o. Hence an eigenvalue equation exists for non-trivial solutions. This was derived and evaluated by LESSEN, FOX and ZIEN [6] for temporally growing disturbances, while BATCHELOR and GILL [1] discussed the incompressible case.

In the following the eigenvalue equation shall be evaluated for spatially growing disturbances i.e. for complex α. Introducing the Mach number $M = U_1/a_1$ and the abbreviations for the non-dimensional quantities

(16) $$z = \alpha U_1/\beta$$

and

(17) $\sigma = \beta R / U_1$

which is a Strouhal number, (14) and (15) become

(18) $\lambda_1 R = \sigma f_1(z) = \sigma \sqrt{z^2 - M^2 (1-z)^2}$

(19) $\lambda_o R = \sigma f_o(z) = \sigma \sqrt{z^2 - M^2/T^*}$.

The general eigenvalue equation due to LESSEN, FOX and ZIEN [6] can then be written

(20) $$(1-z)^2 + \frac{1}{T^*} \frac{K_m(\sigma f_o)[\sigma f_1 I_{m-1}(\sigma f_1) - m I_m(\sigma f_1)]}{[\sigma f_o K_{m-1}(\sigma f_o) + m K_m(\sigma f_o)] I_m(\sigma f_1)} = 0 .$$

This complex equation has been solved numerically for axisymmetric disturbances ($m = 0$) and first azimuthal disturbances ($m = 1$) by means of a subroutine for computing the modified Bessel functions of complex argument.

Let us first discuss the solution of (20) for $M = 0$ and $m = 0$. Then we have $\sigma f_1 = \sigma f_o = \sigma z = \alpha R$, and (20) becomes

(21) $$(1-z)^2 + \frac{1}{T^*} \frac{K_o(\sigma z) I_1(\sigma z)}{K_1(\sigma z) I_o(\sigma z)} = 0 .$$

For $\sigma z = \alpha R \rightarrow \infty$ equ. (21) yields the solution

(22) $$z = \alpha U_1/\beta = 1 - i \frac{1}{\sqrt{T^*}}$$

which agrees with the solution for the plane vortex sheet. We note that the spatial growth rate becomes larger, if for fixed T_o the jet temperature T_1 is increased. For $\sigma z = \alpha R \rightarrow 0$ one can expand the modified Bessel functions and obtain from (21) to $O(\sigma^2 z^2)$

(23) $$(1-z)^2 + \frac{\sigma^2 z^2}{2T^*} [f(\sigma) - \log z] = 0$$

with

(24) $$f(\sigma) = \ln 2 - C - \ln \sigma$$

Here C is the Euler constant. An approximate solution of (23) for $\sigma \rightarrow 0$ is

(25) $$z = 1 - \frac{\sigma^2}{2T^*}\left[f(\sigma) - \frac{1}{2}\right] - i\,\frac{\sigma}{\sqrt{2T^*}}\,\sqrt{f(\sigma)}\ .$$

From the real part of (25) it follows that for $\sigma \longrightarrow 0$ the phase velocity is approximately

(26) $$\frac{c_{ph}}{U_1} = \frac{\beta/\alpha_r}{U_1} = 1 + \frac{\sigma^2}{2T^*}\left[-\ln\sigma\right] \geqq 1\ .$$

This result is quite different from that for temporally growing disturbances where the condition $c_{ph}/U_1 \leqq 1$ is valid. The proof of this necessary condition is, however, restricted to temporally growing disturbances and does not hold for spatially growing ones. A phase velocity greater than the maximum jet velocity has, in fact, been found experimentally in circular jets+).

The phase velocity c_{ph} and the imaginary part of z vs. the Strouhal number σ are shown in Figure 1 for $M = 0$, $T^* = 1$ and $m = 0;\ 1$. We see that the axisymmetry becomes important for Strouhal numbers $\sigma < 6$ approximately. For the axisymmetric disturbances with $m = 0$ the phase velocity is $c_{ph} \geqq U_1$ and has a maximum at nearly $\sigma = 1.5$, while the spatial growth rate is always smaller than that of the plane vortex sheet. The first azimuthal disturbance with $m = 1$ is more unstable than the axisymmetric one only for Strouhal numbers $\sigma < 2$. For $m \geqq 1$ and $\sigma \longrightarrow 0$ or $\sigma \longrightarrow \infty$ the limit of z is given by (22).

Besides of this disturbance mode I, plotted in Figure 1, for complex α there exist additional disturbance modes for small Strouhal numbers with non-vanishing α_i. Equation (21) can be written in the form

(27) $$(\alpha R - \sigma)^2 + \frac{1}{T^*}\,\frac{K_o(\alpha R)\,I_1(\alpha R)}{K_1(\alpha R)}\,\frac{\sigma^2}{I_o(\alpha R)} = 0\ .$$

If now $b > 0$ is real and is a zero of the Bessel function $J_o(b)$, then

(28) $$\alpha R = -ib + \sigma^2 g(\sigma)$$

is a solution of (27) for $\sigma \longrightarrow 0$ with

(29) $$g(0) = \frac{1}{T^* b^2}\,\frac{K_o(-ib)}{K_1(-ib)} = -\frac{i}{T^* b^2}\,\frac{H_o^{(1)}(b)}{H_1^{(1)}(b)}$$

+) private communication by E. PFIZENMAIER where $H_o^{(1)}$ and $H_1^{(1)}$ are the Hankel functions. The first zero of $J_o(b)$ is at $b = 2.405$. The eigenvalues z of this mode II are shown in Figures 2 and 3 together with that of mode I vs. Strouhal number σ for $m = 0$. The parameter is the temperature ratio T^*. The behaviour of the imaginary part of z for mode II is of $O(b\,\sigma^{-1})$ for $\sigma \rightarrow 0$. At Strouhal numbers $\sigma \approx 2$ mode II disappears, since its wave-number α_r vanishes. The maximum of the phase velocity $c_{ph} = \beta/\alpha_r$ of mode I depends on the temperature ratio T^* and is for $T^* = 0.7$ nearly $1.75\ U_1$, while for mode II $c_{ph} \rightarrow \infty$ for $\sigma \rightarrow 0$ and approximately for $\sigma \rightarrow 2$. For temperature ratios $T^* \rightarrow 0.7$ mode I and II interchange the stability characteristics about $\sigma = 1.25$.

The physical meaning of mode II is not quite clear. But some insight is gained by comparing the pressure distribution of both modes+). For this reason, the radial distribution of the amount of pressure amplitude and the curve of constant pressure phase in the (x,r)-plane have been calculated and plotted in Figure 4 for both modes at a Strouhal number $\sigma = 1$ for $M = 0$, $T^* = 1$ and $m = 0$. Mode I shows the well-known behaviour with a maximum of pressure amplitude at the jet border and of a slight phase variation across the jet. Contrary to this, mode II has a maximum of pressure amplitude only at the jet axis. Furthermore, the pressure fluctuations have a rapid phase variation across the jet. Up till now the mode II has never been found experimentally. It may be supposed that the pressure distribution of mode II is only due to the assumption of an infinitely extended parallel flow and is not compatible with a realistic jet escaping out of a nozzle.

Finally, the influence of the Mach number on the instability of the cylindrical vortex sheet will be discussed. For the Strouhal numbers $\sigma = 0.5;\ 1;\ 2$ the eigenvalues have been calculated from (20) for $0 \leq M \leq 2$, $m = 0;\ 1$ and $T^* = 1$. The phase velocity and the ratio of spatial growth rate to wave-number are plotted in Figure 5 vs. Mach number. It is evident that for sufficiently high Mach number the cylindrical vortex sheet becomes less unstable which was also found by LESSEN, FOX and ZIEN [6] for temporally growing disturbances. The stabilizing influence of the Mach number is larger for a temperature ratio $T^* < 1$ as shown in Figure 6 for $T^* = 0.6$ and $m = 0$. This is in agreement with results found by GROPENGIESSER [4] for a plane free shear layer.

+) This suggestion was made by D. BECHERT who found additional spatially growing modes even for the plane jet (private communication).

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Dr.-Ing. Alfons MICHALKE, Berlin 12

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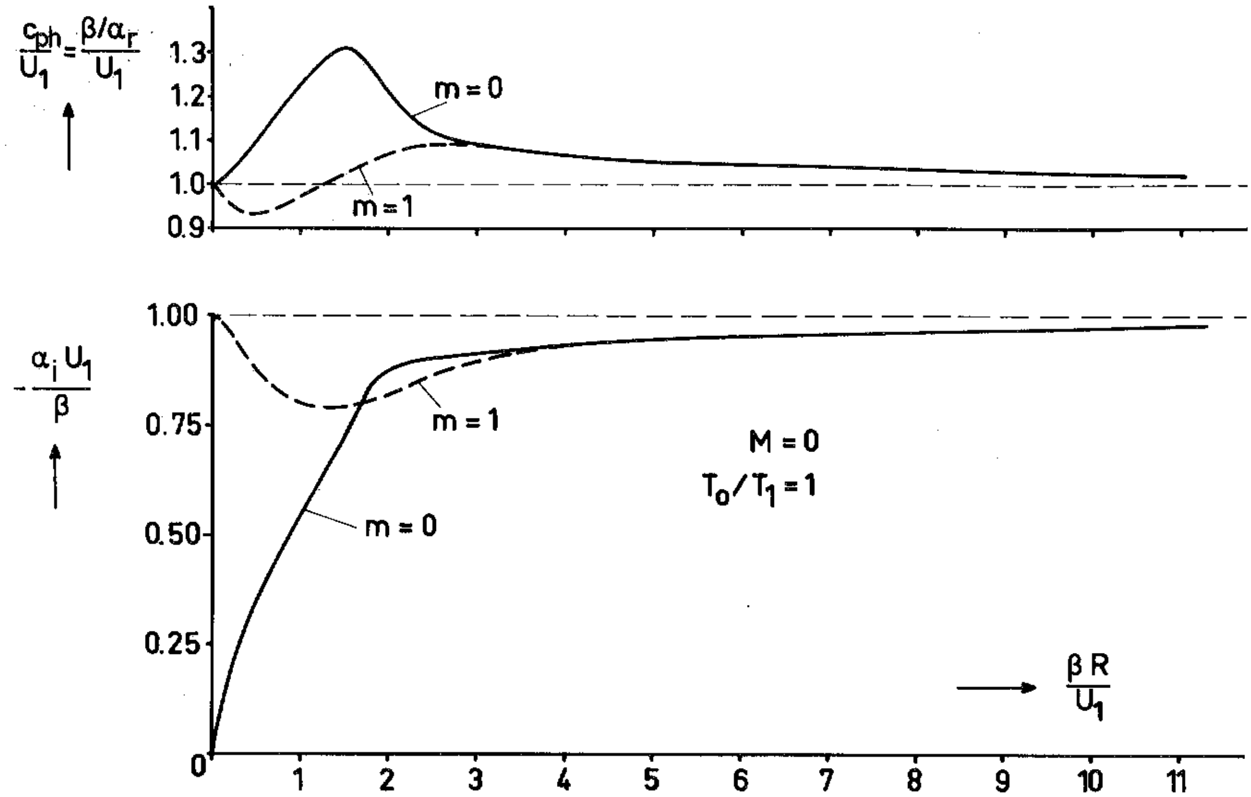

Figure 1 The phase velocity and the spatial amplification ratio of axisymmetric ($m = 0$) and first azimuthal ($m = 1$) disturbances vs. Strouhal number for the cylindrical vortex sheet at Mach number $M = 0$ and temperature ratio $T^* = 1$.

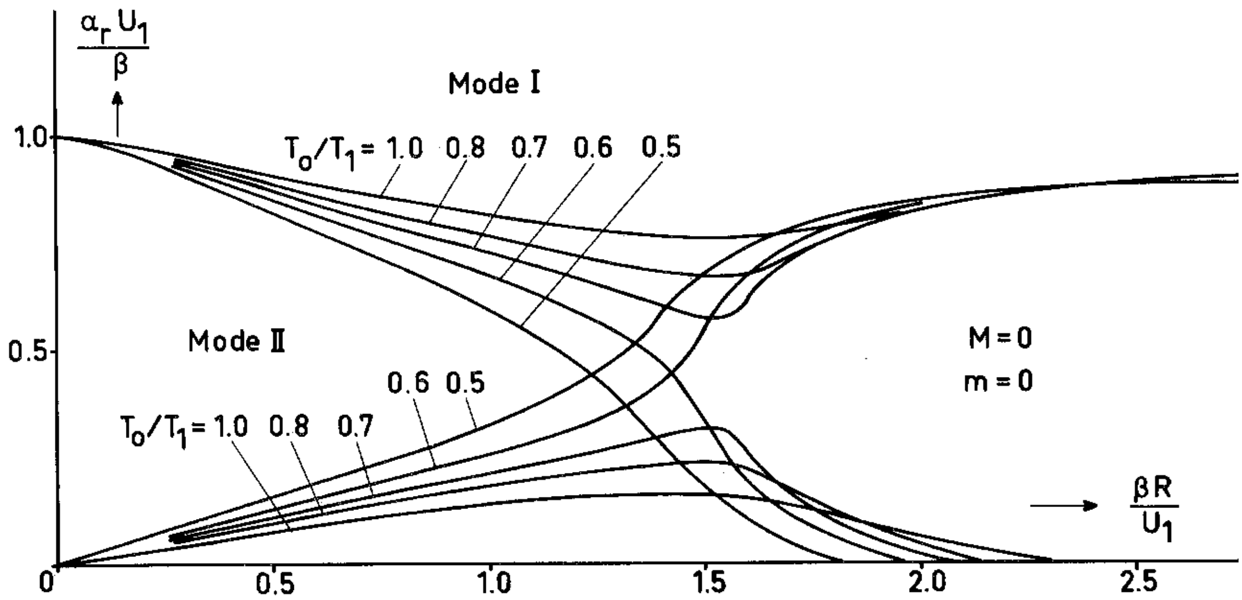

Figure 2 Real part of the eigenvalue of the axisymmetric mode I and II vs. Strouhal number for various temperature ratios at Mach number $M = 0$.

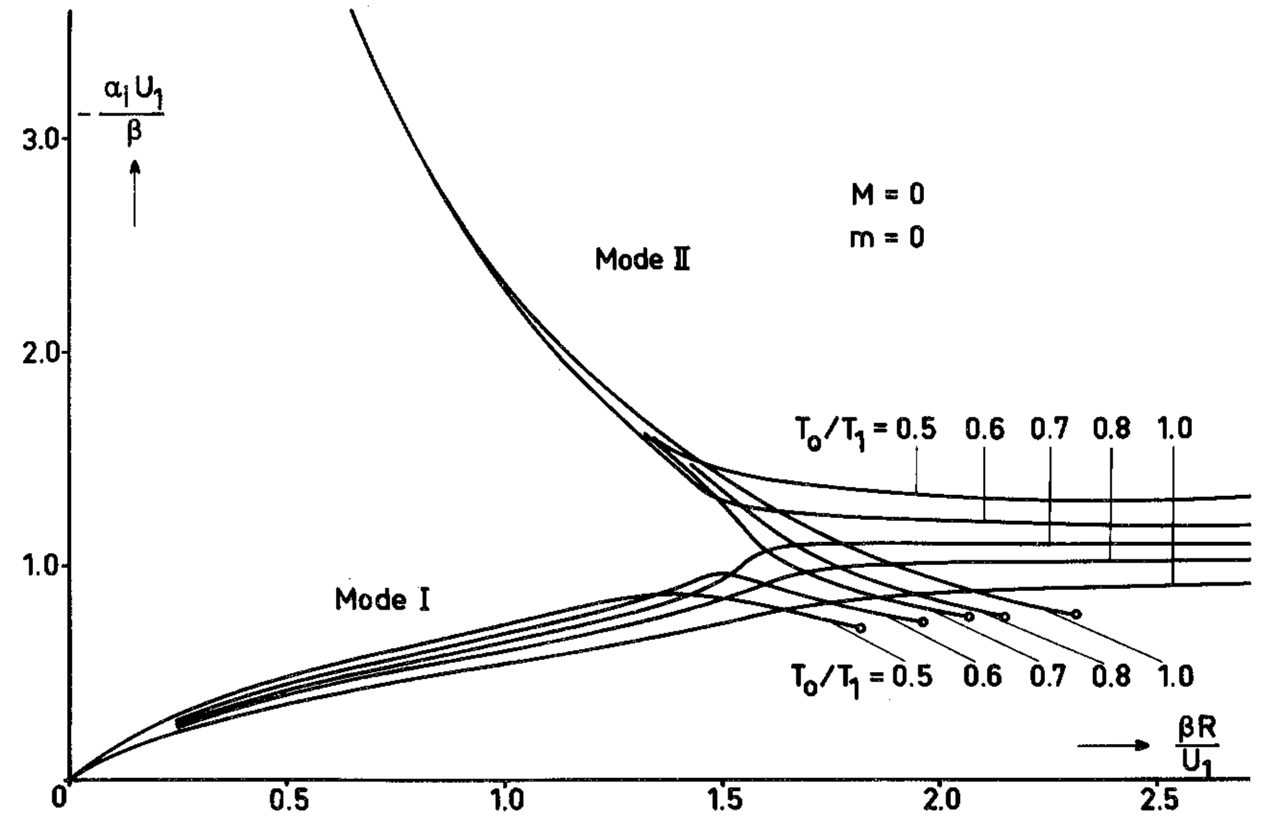

Figure 3 Imaginary part of the eigenvalue of the axisymmetric mode I and II vs. Strouhal number for various temperature ratios at Mach number M = 0.

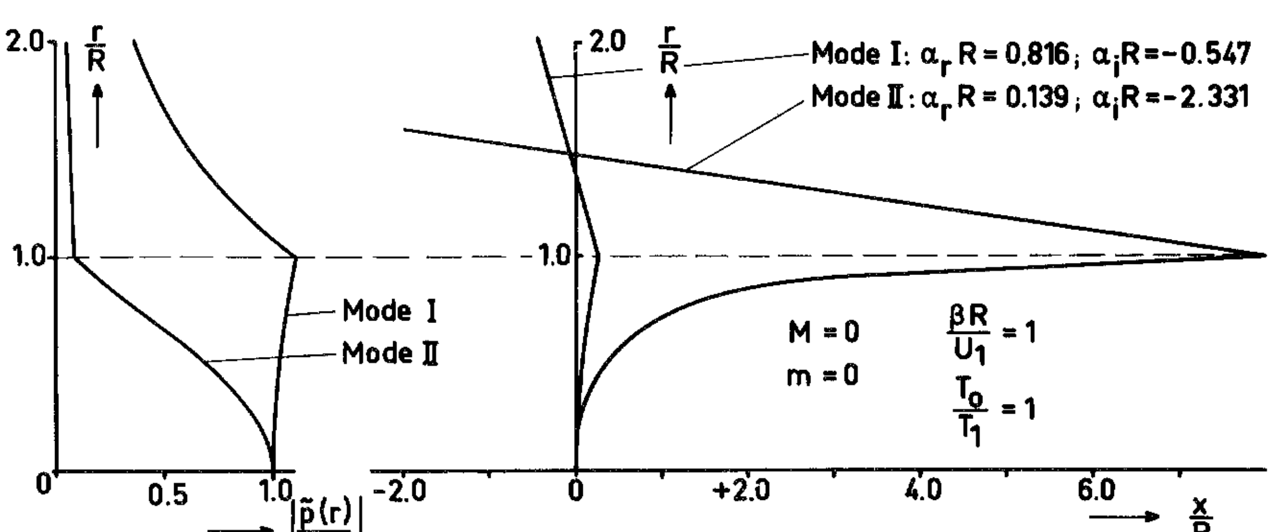

Figure 4 The radial distribution of the amount of pressure amplitude and the curve of constant pressure phase for the axisymmetric mode I and II at a Strouhal number $\sigma = 1$, Mach number M = 0 and temperature ratio $T^* = 1$.

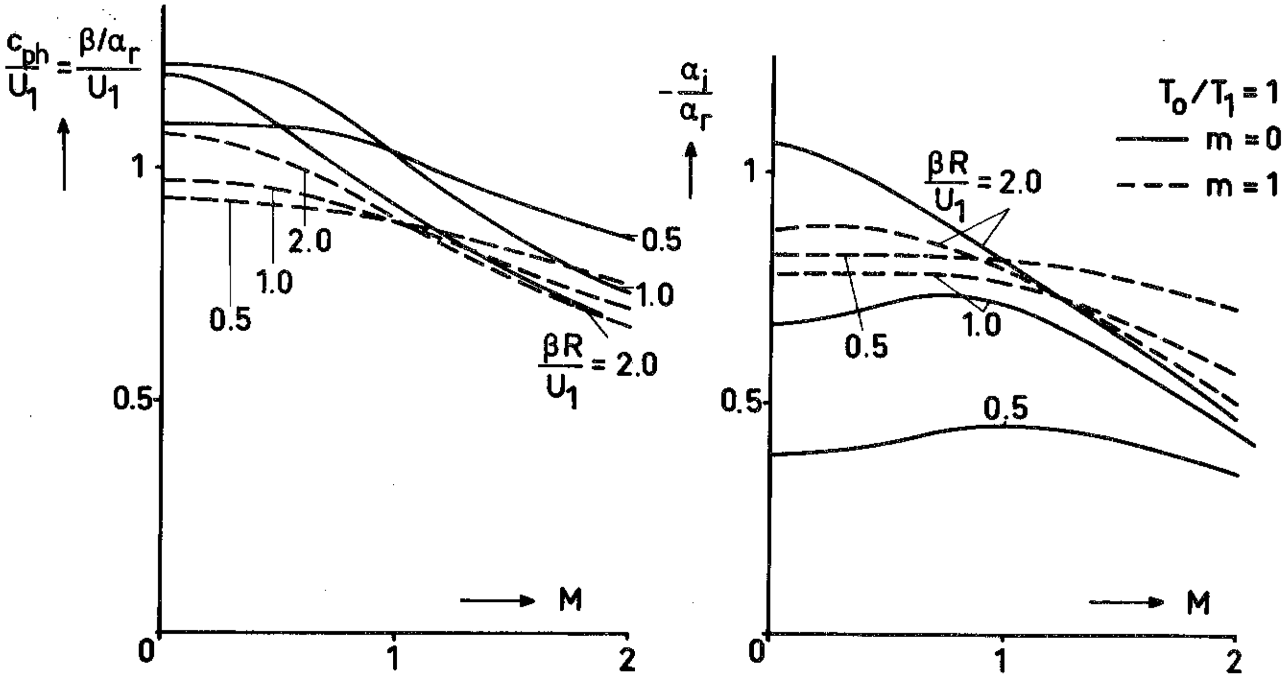

Figure 5 The phase velocity and the ratio of spatial growth rate to wave-number of the axisymmetric and first azimuthal mode I vs. Mach number for various Strouhal numbers and a temperature ratio $T^* = 1$.

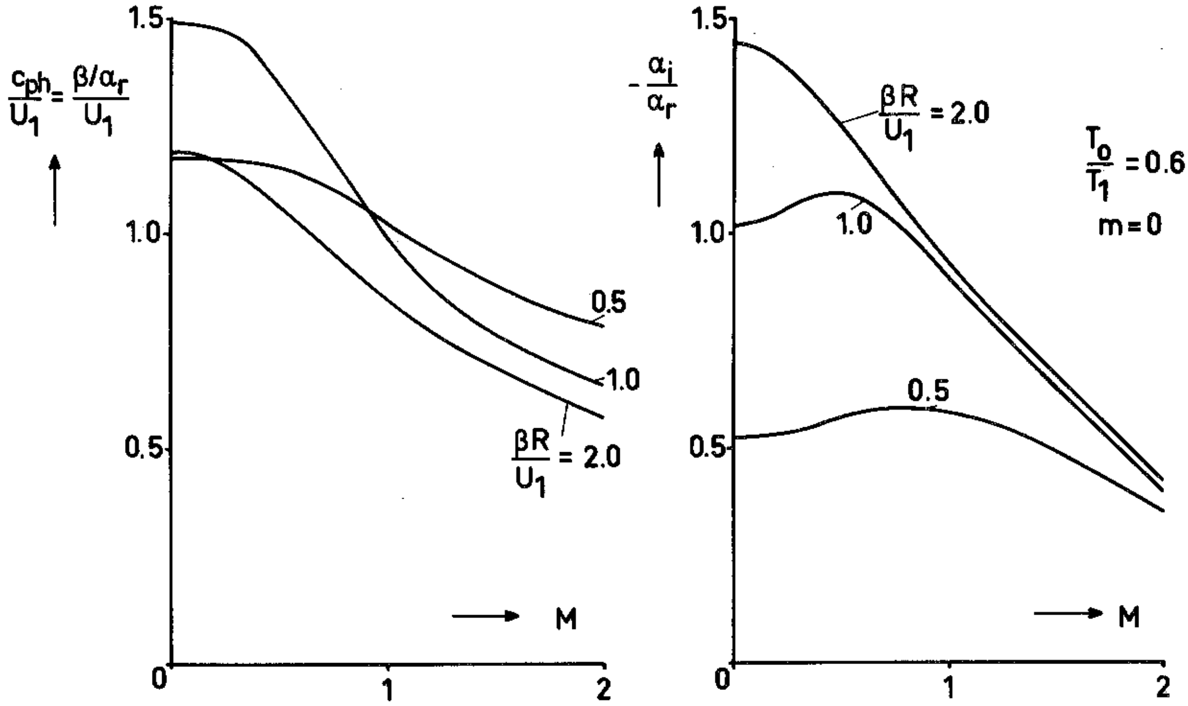

Figure 6 The phase velocity and the ratio of spatial growth rate to wave-number of the axisymmetric disturbance vs. Mach number for various Strouhal numbers and a temperature ratio $T^* = 0.6$.