STATISTICAL CHARACTERISATION OF THE MAXIMUM EIGENVALUE OF A WISHART DISTRIBUTION WITH APPLICATION TO MULTI-CHANNEL SAR SYSTEM

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ABSTRACT

Multi-channel SAR system characterise the target with multicomponent Gaussian circular vector whose number of components m is equal to the number of polarimetric and/or interferometric channels of the system. In the case of the multivariate (multi-channel) Gaussian system, the second order statistics known as covariance matrix contains all the necessary information to characterise the target vector. In this framework, the eigendecomposition of the covariance matrix have demonstrated as a important analysis in the physical parameter estimation and target detection. Especially, the maximum eigenvalue related to the first eigenvector of the covariance matrix is the most interesting parameter in a wide selection of application, i.e. polarimetry, GMTI (ground moving target indication) and interferometric phase filtering. Related to this, the cornerstone study considering the statistical description of the covariance matrix eigendecomposition in polarimetry has been carried out in [1]. However, the majority of the analysis in [1] was performed on the basis of numerical methods. In this paper we support the results of [1] by addressing analytical solutions. Specifically, we derive new exact closed form expressions for Probability Density Function (PDF), for Cumulative Distribution Function (CDF) and for the Moment Generating Function (MGF) of the multi channel SAR system covariance matrix maximum eigenvalue, thus enabling the exact evaluation of the performance analysis of the estimation and the detection problem considering the number of averaged samples and different correlation scenario. Our results are analysed by means of simulated data.

Key words: multi-baseline (MB), multi-channel SAR systems, ground moving target indication (GMTI), polarimetric matching filtering (PMF).

1. STATISTICAL CHARACTERISTICS OF THE SAMPLE MAXIMUM EIGENVALUE

Multi-channel SAR images are characterised by target vector k including m identically distributed complex zero-mean Gaussian random vectors. The m dimensional target vector follows a complex multivariate normal distribution with mean 0 and covariance matrix Σ, i.e, $\mathcal{N}^C(0,\Sigma)$ [2]. In practical applications the covariance matrix Σ is not known, and hence covariance matrix is estimated by the maximum likelihood technique (MLE) from n samples, known as incoherent averaging by looks. The n-look estimated/sample covariance matrix $Z = \frac{1}{n}\sum_{j=1}^{n} k_j k_j^\dagger$ follows a complex Wishart PDF $\mathcal{W}^C(n,\Sigma)$ with degrees of freedom n and true covariance matrix Σ defined by [2]

$$p_Z(Z) = \frac{n^{mn}|Z|^{n-m}\, etr(n\Sigma^{-1}Z)}{|\Sigma|^n\tilde{\Gamma}_m(n)} \tag{1}$$

with normalisation constant $\tilde{\Gamma}_m(n) = \pi^{m(m-1)/2}\prod_{i=1}^{m}\Gamma(n-i+1)$.

The target decomposition theorem of the sample covariance matrix allows to create a set of orthonormal (independent) group, whereas the corresponding eigenvalue express the individual contribution of decomposed group. Since the true covariance matrix Σ has not known, the target decomposition theorem of the sample covariance matrix can provide a significant improvement in physical parameter estimation.

The decomposition of true covariance matrix and its estimated one are $\Sigma = \sum_{i=1}^{m} l_i(e_i e_i^\dagger)$ and $Z = \sum_{i=1}^{m}\lambda_i(e'_i e'^\dagger_i)$ respectively with their eigenvalues l_i, λ_i, and their eigenvectors e_i, e'_i $i = 1,...m$. Then, the following theorems present PDF, CDF and MGF of the maximum sample eigenvalue of multi-channel covariance matrix. These will be used for deriving the detection and estimation properties of the parameters regarding SAR images projected on the first eigenvector of covariance matrices.

Theorem 1: Let k $\mathcal{N}^C(0,\Sigma)$ a $m\times n$ vector and Σ has $l_m \leq .... \leq l_1$ eigenvalues with the assumption of $m \leq n$. Then the CDF of the maximum eigenvalue λ_{max} of the sample covariance matrix $\langle k^\dagger k\rangle_n$ is given by

$$F_{\lambda_{max}}(x) = \mathcal{S}|\Psi(x)|, \tag{2}$$

with constant term $\mathcal{S}$ [1]

$\mathcal{S} = \frac{\pi^{m(m-1)} n^{n(2n-m+1)/2}}{\tilde{\Gamma}_m(m)\tilde{\Gamma}_m(n)} \frac{\prod_{k=1}^{m-1} k^{m-k}}{\prod_{i=1}^{m} l_i^n \prod_{i<j}^{m}\left(\frac{1}{l_j}-\frac{1}{l_i}\right)}$ where $\Psi(x)$ is an $m \times m$ matrix with (i,j)th element $\Psi(x)_{i,j} = \frac{\gamma(n+1-j, x\frac{n}{l_i})}{(\frac{n}{l_i})^{(n+1-j)}}$. Here, γ is the incomplete Gamma function [3, eq. 2.42]. See [4] for proof.

Theorem 2: Let k $\mathcal{N}^C(0,\Sigma)$ a $m \times n$ vector and Σ has $l_m \leq .... \leq l_1$ eigenvalues with the assumption of $m \leq n$. Then the PDF of the maximum eigenvalue λ_{max} of the sample covariance matrix $\langle k^\dagger k \rangle_n$ is given by

$$p_{\lambda_{max}}(\lambda_{max}) = \mathcal{S}|\Psi(\lambda_{max})| tr\left(\Psi(\lambda_{max})^{-1}\Omega(\lambda_{max})\right) \quad (3)$$

where $\Omega(\lambda_{max})$ is an $m \times m$ matrix with (i,j)th element $\Omega(\lambda_{max})_{i,j} = exp\left(-\frac{n}{l_i}\lambda_{max}\right)\lambda_{max}^{n-j}$, and $\Psi(\lambda_{max})$ and $\mathcal{S}$ are defined in *Theorem 1*.

Proof: Eq. (3) is obtained by differentiating (2) with respect to (x) using the formula of [5, eq. 9]

$$\frac{d}{dt}|X(t)| = |X(t)| tr\left(X(t)^{-1}\frac{d}{dt}X(t)\right). \quad (4)$$

Theorem 3: Let k $\mathcal{N}^C(0,\Sigma)$ be a $m \times n$ vector and Σ has $l_m \leq .... \leq l_1$ eigenvalues with the assumption of $m \leq n$. Then for any positive integers s, the sth moment of the maximum eigenvalue λ_{max} of the sample covariance matrix $\langle k^\dagger k \rangle_n$ is given by

$$\langle x^s \rangle = \mathcal{S}\sum_{i,j=1}^{m}(-1)^{i+j}\sum_{\pi_{sub}\in S_m} sgn(\pi_{sub})\mathcal{G}(k,\pi_{sub}(k)) \quad (5)$$
$$\text{if } k = j \ \wedge \ \pi_k(j) = i, \ \pi_k \in \pi_{sub}$$

where $k, \pi_{sub}(k) = \{\, k, \pi_{sub}(k) \in \{1,2,\ldots,m\},\ k \neq j \ \wedge \ \pi_{sub}(j) \neq i \,\}$, and

$$\mathcal{G}(k,\pi_{sub}(k)) = \prod_{k=1}^{m-1}(n-\pi_{sub}(k))! \times \left(\frac{(n-j+s)!}{(\frac{n}{l_i})^{n-j+s+1}} + \sum_{k_1=1}^{\binom{m-1}{1}}\sum_{t_{k_1}=0}^{n-\pi_{sub}(k_1)}\frac{\left(\frac{n}{l_{k_1}}\right)^{t_{k_1}}}{t_{k_1}!}\frac{(s+n-j+t_{k_1})!}{\left(\frac{n}{l_i}+\frac{n}{l_{k_1}}\right)^{s+n-j+t_{k_1}+1}} + \sum_{k_2=2>k_1=1}^{\binom{m-1}{2}}\sum_{t_{k_1}=0}^{n-\pi_{sub}(k_1)}\frac{\left(\frac{n}{l_{k_1}}\right)^{t_{k_1}}}{t_{k_1}!}\sum_{t_{k_2}=0}^{n-\pi_{sub}(k_2)}\frac{\left(\frac{n}{l_{k_2}}\right)^{t_{k_2}}}{t_{k_2}!}\frac{(s+n-j+t_{k_1}+t_{k_2})!}{\left(\frac{n}{l_i}+\frac{n}{l_{k_1}}+\frac{n}{l_{k_2}}\right)^{s+n-j+t_{k_1}+t_{k_2}+1}} + \ldots + \sum_{t_{k_{m-1}}=0}^{n-\pi_{sub}(k_{m-1})}\frac{\left(\frac{n}{l_{k_{m-1}}}\right)^{t_{k_{m-1}}}}{t_{k_{m-1}}!}\cdots\sum_{t_{k_1}=0}^{n-\pi_{sub}(k_1)}\frac{\left(\frac{n}{l_{k_1}}\right)^{t_{k_1}}}{t_{k_1}!}\frac{(s+n-j+t_{k_1}+\cdots+t_{k_{m-1}})!}{\left(\frac{n}{l_i}+\frac{n}{l_{k_1}}+\cdots+\frac{n}{l_{k_{m-1}}}\right)^{s+n-j+t_{k_1}+\cdots+t_{k_{m-1}}+1}}\right)$$

Here, the sum is computed over $(m-1)!$ permutations π_k of the numbers $\{k = 1,2,...,m\}$. S_m denotes the set of all $m!$ permutations of the set $S = \{1,2,...,m\}$, and $sgn(\pi_{sub})$ denotes the signature of the permutation π_k : $+1$ if π_k is an even permutation and -1 if it is odd. See [4] for proof.

We solve the moments of the sample maximum eigenvalue regarding its PDF. For larger m dimensional systems, such an approach may become complicated due to the need for a large number of calculation of cofactors. However, for applications where only a few eigenvalues, like polarimetry ($m = 3$) and interferometry ($m = 2$) are of interest, the numerical calculations are quite rapid and stable for removing the bias.

2. APPLICATION TO MULTI-CHANNEL SAR SYSTEMS

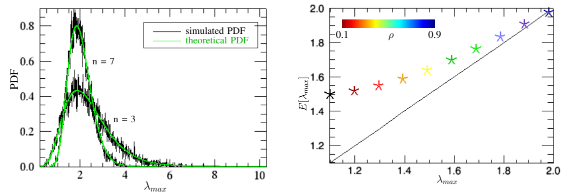
This section focuses on the validation of the theorems including as well a statistical analysis of the maximum sample eigenvalue. In order to completely describe the behaviour of the maximum sample eigenvalue (or any other eigenvalues), an entire function, namely the PDF (estimation analysis) and the CDF (detection analysis), must be given as in the previous section.

2.1. Estimation Theory

In order to validate *Theorem 2*, and hence *Theorem 1*, Figure 1(a) shows the comparison of our exact theoretical (analytical) expressions with simulations. As expected, the theoretical PDF curves based on (3) are clearly agree with the simulated PDFs. As it is well-known [1], and it may also easily be verified that increasing the number of samples improves the signal to noise ratio ($\lambda_{max} \to l_1$), which implies a better parameter estimation.

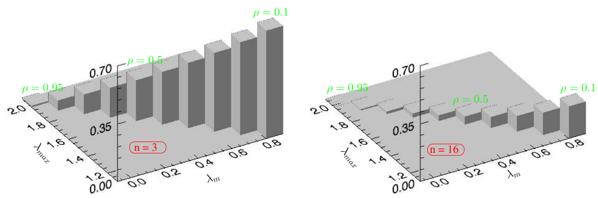
Since in quantitative remote sensing, we are interested in physical parameters summarising the nature of the object, the expectation value of the parameters that it can be used in the removal of the bias of physical parameters can be interesting to know. As numerically already found in [1], it appears that the estimate of eigenvalues are biased towards higher values and/or small number of looks. Here, the analysis of the bias is discussed also regarding the correlation scenario between channels; Figure 1(b). Even with very low number of samples high correlated channels have a very little biased eigenvalue estimator.

Figure 2 shows the bias of the sample maximum eigenvalue, $E[\lambda_{max}] - l_{max}$, regarding different correlation scenario versus different number of samples. We see that the number of samples and high correlation decreases the bias through whole combinations of correlation and the number of samples scenario. However, for high correlated channels, there is no significant decrease in the bias with increasing the number of look.



(a) (b)

Figure 1. (a) Comparison between the theoretical PDFs of the maximum sample eigenvalue and the histograms of the maximum sample eigenvalues obtained from simulated data with the standard deviations of $\sigma_{k_1} = \sigma_{k_2} = \sigma_{k_3} = 1$ and correlation parameters $\rho_{k_1 k_2} = 0$, $\rho_{k_1 k_3} = 0.8$ and $\rho_{k_2 k_3} = 0$. When $n \to \infty$, $\lambda_{max} = l_1 = 1.8$. (b) Effects of the correlation between channels on the expected value of the maximum sample eigenvalue with $n = 3$ and $m = 2$.



(a) n=3 (b) n=16

Figure 2. The bias, $E[\lambda_{max}] - l_{max}$, of the sample maximum eigenvalue with the number of samples 3 and 16 in various correlated channels having a standard deviation of $\sigma_{k_1} = \sigma_{k_2} = 1$.

2.2. Detection Theory

Detection theory is a means to quantify the ability to discriminate the parameter and its noise, and has applications in many fields such as presence of signal in a noise environment, i.e., change detection, polarimetric parameter estimation, noise reduction and etc. The concept is similar to the signal to noise ratio used in the literature, where it is important in presence of noise. Having the closed form of the PDF (3) and/or CDF (2), the probabilities of detection (p_D) and of false alarm (p_{FA}) can be computed which allows a complete detection problem analysis. These analysis can be important especially in the application area of GMTI, target detection, change detection and filtering.

Apart from parameter estimation and change detection, here one example in the context of polarimetric filtering will be given. As mentioned in [6], [7], sample maximum eigenvalue can be also used in recognition of targets in clutter environment. Before analysing the performance of target detection, we shortly review the target detection concept in the case of Polarimetric Matching Filter (PMF). As shown in [6, eq. 59], the optimum weight vector, denoted h, is given by the quadratic form

$$y = \sum_{i=1}^{m} |h^\dagger k_i|^2 > T_D \quad , k = [k_1, k_2, ..., k_m] \quad (6)$$

where observed target vector k is a $i = 1, 2, ..., m$ dimensional vector and we seek the best linear weight vector $y = h^\dagger k$ providing maximum target detection in the presence of clutter. As it is well known and indicated in [8, Theorem 1.4.1], y has a chi-square distribution in the case of Gaussian assumption of k. It means that to determine the detection performance we need only compute $\langle |h^\dagger k|^2 \rangle$. Mathematically, the procedure can be represented as

$$\begin{aligned} |h^\dagger k|^2 &> T_D \\ h^\dagger k^\dagger k h &> T_D \\ h^\dagger \Sigma h &> T_D \,. \end{aligned}$$

Regarding Rayleigh quotient [9, Theorem 15.91], for any $m \times 1$ complex vector x and a given $m \times m$ Hermitian matrix A, $x^\dagger A x \leq \|x\|^2 \lambda_{max}$, where λ_{max} is the maximum eigenvalue of A. Note that the equality is valid if x is along the direction of the eigenvector U_{max} ($\|U_{max}\| = 1$) corresponding to λ_{max}. Therefore, the performance of detection for a given value of the constant multiplier $\alpha = \|h\|^2$ that does not effect the performance of the detection is[1]

$$T_D = \alpha \lambda_{max}. \quad (7)$$

Hence $\lambda_{max} > 0$ with probability 1, the probability density function of $\mathrm{T_D}$ can be easily calculated and the performance analysis of detection can be performed as in the following.

Using the rule of change of variable, the PDF of $\mathrm{T_D}$ is given in closed form by

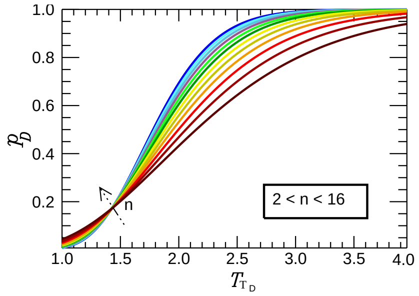
$$p(T_D) = \alpha p_{\lambda_{max}} \left(\frac{T_D}{\alpha} \right) \quad (8)$$

Then, the probability of detection performance for given threshold $\mathrm{T_D}$ can be found directly from CDF function as

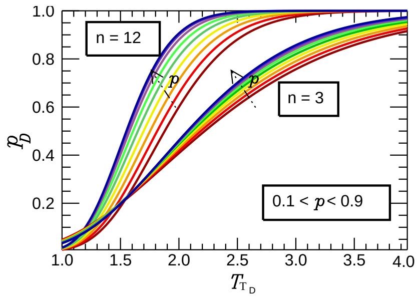
$$p(T_D < \mathcal{T}_{T_D}) = F_{\lambda_{max}} \left(\frac{\mathcal{T}_{T_D}}{\alpha} \right)$$

Figure 3 shows the probability of the detection problem regarding different number of look and correlation scenarios. We see that the number of look increases the probability of detection. Moreover, increasing correlation between channels also improves the target detection performance Figure 3(b).

[1] Here it is worth noting that the aim is to show how the proposed theorems can be implemented in multidimensional SAR application areas. The way to understand the constant parameters and the filtering procedure, the reader is invited to look at [6], [7].



(a)



(b)

Figure 3. The filtering algorithm performance versus the number of look (a) and the correlation (b). The standard deviation of the channels are $\sigma_{k_1} = \sigma_{k_2} = 1$.

Compared to the formulation in [6], it has been show that it easy to make a performance analysis with proposed theorems.

3. CONCLUSION

In this paper, we presented in a depth statistical analysis of the maximum eigenvalue of the eigendecomposition of the sample covariance matrix. The proposed theorems and analysis are supported by simulation results via several examples. Our results are based on a exact closed-form expressions of PDF, CDF and MGF. In this study, we extended and/or implemented existing density functions of the sample maximum eigenvalue into multi-channel SAR system in order to obtain a simple expression of the sample eigenvalues giving a way to fruitful applications. From these closed-form expressions, it has been possible to develop new algorithms to unbiased calculations of parameters extracted form multi-channel SAR covariance matrix. In addition to these implementations, we developed closed-form expressions for MGF of the sample maximum eigenvalue, which can be critical in the area of removing bias and detection performance analysis. This new closed-form expression of MGF can be also interesting for other application areas like, MIMO systems (multiple-input multiple-output). Apart from estimation theory analysis including PDF and MGF, the detection problem of the sample maximum eigenvalue has been also discussed.

The main contribution to the literature to be extracted form the study presented in this paper is that there are new statistical closed-form expressions including PDF, CDF and MGF of the sample maximum eigenvalue extracted from the multi-channel SAR covariance matrix. Investigating the new algorithms using these new expressions to calculate the unbiased physical parameter and to improve the performance of the detection problem are future works of this study.

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