Numerical Analysis of Higher Order Discontinuous Galerkin Finite Element methods

Ralf Hartmann

Institute of Aerodynamic and Flow Technology
DLR (German Aerospace Center)

14. Oct. 2008

Outline

Adjoint consistency: Overview

- Optimal order error estimates in the L^2-norm only for **adjoint consistent** discretizations

Adjoint consistency: Overview

- Optimal order error estimates in the L^2-norm only for **adjoint consistent** discretizations

We will see:

- Optimal order error estimates in target quantities $J(\cdot)$ only for **adjoint consistent** discretizations

Adjoint consistency: Overview

- Optimal order error estimates in the L^2-norm
  only for **adjoint consistent** discretizations

We will see:

- Optimal order error estimates in target quantities $J(\cdot)$
  only for **adjoint consistent** discretizations

Up to now:

- Adjoint consistency analysis for DG discretizations of
  the **homogeneous Dirichlet problem** of **Poisson's equation**

Adjoint consistency: Overview

- Optimal order error estimates in the L^2-norm only for **adjoint consistent** discretizations

We will see:

- Optimal order error estimates in target quantities $J(\cdot)$ only for **adjoint consistent** discretizations

Up to now:

- Adjoint consistency analysis for DG discretizations of the **homogeneous Dirichlet problem** of **Poisson's equation**

In the following:

- Adjoint consistency analysis for DG discretizations of **linear problems** with **inhomogeneous** boundary conditions (e.g. Dirichlet-Neumann) in connection with **target quantities** $J(\cdot)$

Adjoint consistency: Overview

- Optimal order error estimates in the L^2-norm
  only for **adjoint consistent** discretizations

We will see:

- Optimal order error estimates in target quantities $J(\cdot)$
  only for **adjoint consistent** discretizations

Up to now:

- Adjoint consistency analysis for DG discretizations of
  the **homogeneous Dirichlet problem** of **Poisson's equation**

In the following:

- Adjoint consistency analysis for DG discretizations of **linear problems**
  with **inhomogeneous** boundary conditions (e.g. Dirichlet-Neumann)
  in connection with **target quantities** $J(\cdot)$

Later:

- Adjoint consistency analysis for DG discretizations of **nonlinear problems**
  in connection with **target quantities** $J(\cdot)$

Adjoint consistency, Preview: We will see that ...

- Adjoint consistency involves the discretization
  - of element terms
  - of interior faces terms
  - of **boundary conditions**
  - and of the **target functionals** $J(\cdot)$

Adjoint consistency, Preview: We will see that ...

- Adjoint consistency involves the discretization
  - of element terms
  - of interior faces terms
  - of **boundary conditions**
  - and of the **target functionals** $J(\cdot)$
- Adjoint consistency and thus optimal order estimates can be obtained only for target functionals which are **compatible** with the primal equations.

Adjoint consistency, Preview: We will see that ...

- Adjoint consistency involves the discretization
    - of element terms
    - of interior faces terms
    - of **boundary conditions**
    - and of the **target functionals** $J(\cdot)$
- Adjoint consistency and thus optimal order estimates can be obtained only for target functionals which are **compatible** with the primal equations.
- An adjoint consistent DG(p) discretization of the linear advection equ.
    - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p+1})$

Adjoint consistency, Preview: We will see that ...

- Adjoint consistency involves the discretization
  - of element terms
  - of interior faces terms
  - of **boundary conditions**
  - and of the **target functionals** $J(\cdot)$
- Adjoint consistency and thus optimal order estimates can be obtained only for target functionals which are **compatible** with the primal equations.
- An adjoint consistent DG(p) discretization of the linear advection equ.
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p+1})$
- An adjoint consistent DG(p) discretization of Poisson's equation
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p})$

Adjoint consistency, Preview: We will see that ...

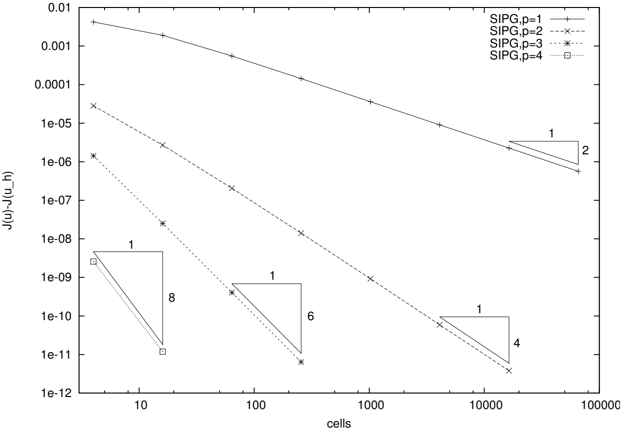
- Adjoint consistency involves the discretization
  - of element terms
  - of interior faces terms
  - of **boundary conditions**
  - and of the **target functionals** $J(\cdot)$
- Adjoint consistency and thus optimal order estimates can be obtained only for target functionals which are **compatible** with the primal equations.
- An adjoint consistent DG(p) discretization of the linear advection equ.
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p+1})$
- An adjoint consistent DG(p) discretization of Poisson's equation
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p})$
- An adjoint **in**consistent DG(p) discretization of Poisson's equation
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{p})$

Preview example 1: Model problem

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$J_1(u_h) = \int_\Omega j_\Omega \, u_h \, \mathrm{d}\mathbf{x}, \qquad \text{with} \quad j_\Omega(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2) \quad \text{on } \Omega$$

This target quantity is **compatible** with the model problem.



SIPG discretization of Poisson's equation:

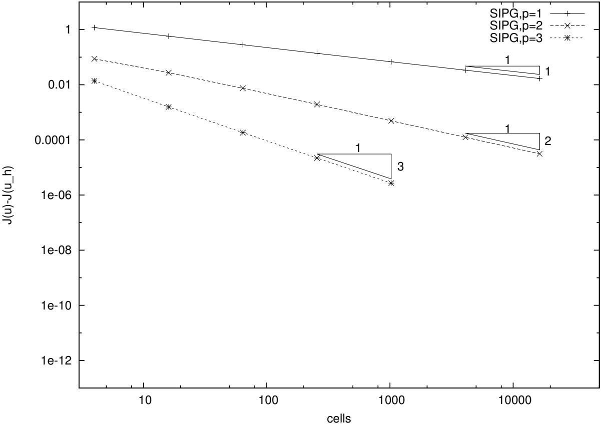
The error $|J_1(u) - J_1(u_h)|$ of the DG(p), $p = 1, \ldots, 4$, discretization behaves like $\mathcal{O}(h^{2p})$

Preview example 2: Model problem

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$J_2(u_h) = \int_\Gamma j_D \, \mathbf{n} \cdot \nabla_h u_h \, \mathrm{d}s, \qquad \text{with} \quad j_D \equiv 1 \quad \text{on } \Gamma_D = \Gamma$$

This target quantity is also **compatible** with the model problem.



SIPG discretization of Poisson's equation:

The error $|J_2(u) - J_2(u_h)|$ of the DG(p), $p = 1, \ldots, 3$, discretization behaves like $\mathcal{O}(h^p)$

Outline

Definition of consistency and adjoint consistency for linear problems

Primal problem: $Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma,$

Target quantity:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma$$

Definition of consistency and adjoint consistency for linear problems

Primal problem: $Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \Gamma,$

Target quantity:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma$$

Compatibility condition: $J(\cdot)$ is compatible to the primal problem if

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

Definition of consistency and adjoint consistency for linear problems

Primal problem: $Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \Gamma,$

Target quantity:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma$$

Compatibility condition: $J(\cdot)$ is compatible to the primal problem if

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

Adjoint problem: $L^* z = j_\Omega \quad \text{in } \Omega, \qquad B^* z = j_\Gamma \quad \text{on } \Gamma.$

Definition of consistency and adjoint consistency for linear problems

Primal problem: $Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \Gamma,$

Target quantity:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma$$

Compatibility condition: $J(\cdot)$ is compatible to the primal problem if

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

Adjoint problem: $L^* z = j_\Omega \quad \text{in } \Omega, \qquad B^* z = j_\Gamma \quad \text{on } \Gamma.$

Let the primal problem be discretized: Find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v \in V_h$$

Consistency: The exact solution u to the primal problem satisfies:

$$B_h(u, v) = F_h(v) \quad \forall v \in V$$

Adjoint consistency: The exact solution z to the adjoint problem satisfies:

$$B_h(w, z) = J(w) \quad \forall w \in V$$

Outline

Theorem 7a) A priori error estimates for target functionals $J(\cdot)$

Given a discretization which in combination with a **compatible** target functional $J(\cdot)$ is **consistent** and **adjoint consistent**. Assume that

$$B_h(w,v) \le C_B |||w|||\ |||v||| \quad \forall w,v \in V.$$

Furthermore, assume that there are constants $C>0$ and $r=r(p)>0$ such that

$$|||u-u_h||| \le Ch^r |u|_{H^{p+1}(\Omega)} \quad \forall u \in H^{p+1}(\Omega).$$

and there are constants $C>0$ and $\tilde{r}=\tilde{r}(p)>0$ such that

$$|||v-P^d_{h,p}v||| \le Ch^{\tilde{r}} |v|_{H^{p+1}(\Omega)} \qquad \forall v \in H^{p+1}(\Omega).$$

Let $z\in V$ be the solution to the adjoint problem. Due to adjoint consistency we have $B_h(w,z)=J(w)$ for all $w\in V$. Thus, for $|J(u)-J(u_h)|=|J(e)|$ we have

$$\begin{aligned}|J(e)| = |B_h(e,z)| = |B_h(u-u_h, z-P_hz)| &\le C|||u-u_h|||\ |||z-P_hz||| \\ &\le Ch^r|u|_{H^{p+1}(\Omega)}Ch^{\tilde{r}}|z|_{H^{p+1}(\Omega)} = Ch^{r+\tilde{r}}|u|_{H^{p+1}(\Omega)}|z|_{H^{p+1}(\Omega)} \qquad \forall u \in H^{p+1}(\Omega)\end{aligned}$$

I.e. the error $|J(u)-J(u_h)|$ is of order $\mathcal{O}(h^{r+\tilde{r}})$.

Theorem 7b) A priori error estimates for target functionals $J(\cdot)$

Same situation as before. But now consider a discretization which in combination with a specific target functional $J(\cdot)$ is **adjoint inconsistent**.

Then the solution z to the adjoint problem does **not** satisfy

$$B_h(w, z) = J(w) \qquad \forall w \in V.$$

Theorem 7b) A priori error estimates for target functionals $J(\cdot)$

Same situation as before. But now consider a discretization which in combination with a specific target functional $J(\cdot)$ is **adjoint inconsistent**.

Then the solution z to the adjoint problem does **not** satisfy

$$B_h(w,z) = J(w) \qquad \forall w \in V.$$

Instead define the solution ψ to following **mesh-dependent adjoint problem**:

$$B_h(w,\psi) = J(w) \quad \forall w \in V.$$

ψ is mesh-dependent and not smooth. We obtain

$$\begin{aligned} |J(e)| = |B_h(e,\psi)| = |B_h(u-u_h,\psi - P_h\psi)| &\leq C|||u-u_h|||\,|||\psi - P_h\psi||| \\ &\leq Ch^r|u|_{H^{p+1}(\Omega)} \end{aligned}$$

I.e. the error $|J(u)-J(u_h)|$ is of order $\mathcal{O}(h^r)$.

Example: A priori error estimates for target functionals $J(\cdot)$

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

For the NIPG and the SIPG discretization we have continuity of B_h:

$$B_h(w, v) \leq C_B |\|w\||_\delta | \, \|v\||_\delta \qquad \forall w, v \in V,$$

the *a priori* error estimate: $|\|u - u_h\||_\delta \leq C h^p |u|_{H^{p+1}(\Omega)} \quad \forall u \in H^{p+1}(\Omega),$

and the approximation estimate:

$$|\|v - P^d_{h,p} v\||_\delta \leq C h^p |v|_{H^{p+1}(\Omega)} \quad \forall v \in H^{p+1}(\Omega),$$

Thus $r = p$ and $\tilde{r} = p$.

Adjoint consistent discretization: $|J(u) - J(u_h)|$ is of order $\mathcal{O}(h^{r+\tilde{r}}) = \mathcal{O}(h^{2p})$
Adjoint inconsistent discretization: $|J(u) - J(u_h)|$ is of order $\mathcal{O}(h^r) = \mathcal{O}(h^p)$

Outline

Derivation of the adjoint problem

Given the primal problem

$$Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma,$$

and the target quantity

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma.$$

Find the adjoint operators L^*, B^* and C^* via the compatibility condition

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

Then the adjoint problem is given by

$$L^* z = j_\Omega \quad \text{in } \Omega, \qquad\qquad B^* z = j_\Gamma \quad \text{on } \Gamma.$$

Consistency analysis of the discrete primal problem

Rewrite the discrete problem: Find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v \in V_h$$

in following element-based **primal residual form**: Find $u_h \in V_h$ such that

$$\int_\Omega R(u_h) v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} r(u_h) v_h + \boldsymbol{\rho}(u_h) \cdot \nabla_h v_h \,\mathrm{d}s \\ + \int_\Gamma r_\Gamma(u_h) v_h + \boldsymbol{\rho}_\Gamma(u_h) \cdot \nabla_h v_h \,\mathrm{d}s = 0 \quad \forall v_h \in V_h,$$

The discretization is **consistent**
if the exact solution u to the primal problem satisfies

$$\begin{aligned} R(u) &= 0 & & \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ r(u) &= 0, & \boldsymbol{\rho}(u) &= 0 \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_\Gamma(u) &= 0, & \boldsymbol{\rho}_\Gamma(u) &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Adjoint consistency of element, interior face and boundary terms

Rewrite the discrete adjoint problem: find $z_h \in V_h$ such that

$$B_h(w_h, z_h) = J(w_h) \quad \forall w_h \in V_h,$$

in following element-based **adjoint residual form**: find $z_h \in V_h$ such that

$$\int_\Omega w_h\, R^*(z_h)\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} w_h\, r^*(z_h) + \nabla w_h \cdot \boldsymbol{\rho}^*(z_h)\,\mathrm{d}s$$
$$+ \int_\Gamma w_h\, r_\Gamma^*(z_h) + \nabla w_h \cdot \boldsymbol{\rho}_\Gamma^*(z_h)\,\mathrm{d}s = 0 \quad \forall w_h \in V_h.$$

The discrete adjoint problem is a **consistent** discretization of the adjoint problem if the exact solution z to the adjoint problem satisfies

$$\begin{aligned} &R^*(z) = 0 && && \text{in } \kappa, \kappa \in \mathcal{T}_h,\\ &r^*(z) = 0, && \boldsymbol{\rho}^*(z) = 0 && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h,\\ &r_\Gamma^*(z) = 0, && \boldsymbol{\rho}_\Gamma^*(z) = 0 && \text{on } \Gamma. \end{aligned}$$

Then we say: The primal discrete problem is an **adjoint consistent** discretization.

Target functional modifications

Sometimes the target functional must be modified in order to obtain an adjoint consistent discretization. Example:

$$\widetilde{J}(u_h) = J(i(u_h)) + \int_\Gamma r_J(u_h)\,\mathrm{d}s, \tag{1}$$

Definition: $\widetilde{J}(u_h)$ is a **consistent** modification of the target functional $J(u_h)$ if the true (exact) value is unchanged, i.e. if

$$\widetilde{J}(u) = J(u)$$

holds for the exact solution u.

In particular, $\widetilde{J}(u_h)$ in (1) is a consistent modification of $J(u_h)$ if

$$i(u) = u \qquad \text{and} \qquad r_J(u) = 0$$

Outline

The continuous adjoint problem to Poisson's equation

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

Multiply left hand side by z and integrate by parts twice

$$(-\Delta u, z)_\Omega = (\nabla u, \nabla z)_\Omega - (\mathbf{n} \cdot \nabla u, z)_\Gamma = (u, -\Delta z)_\Omega + (u, \mathbf{n} \cdot \nabla z)_\Gamma - (\mathbf{n} \cdot \nabla u, z)_\Gamma.$$

The continuous adjoint problem to Poisson's equation

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

Multiply left hand side by z and integrate by parts twice

$$(-\Delta u, z)_\Omega = (\nabla u, \nabla z)_\Omega - (\mathbf{n} \cdot \nabla u, z)_\Gamma = (u, -\Delta z)_\Omega + (u, \mathbf{n} \cdot \nabla z)_\Gamma - (\mathbf{n} \cdot \nabla u, z)_\Gamma.$$

After splitting the boundary terms according to $\Gamma = \Gamma_D \cup \Gamma_N$ and shuffling terms

$$(-\Delta u, z)_\Omega + (u, -\mathbf{n} \cdot \nabla z)_{\Gamma_D} + (\mathbf{n} \cdot \nabla u, z)_{\Gamma_N} = (u, -\Delta z)_\Omega + (\mathbf{n} \cdot \nabla u, -z)_{\Gamma_D} + (u, \mathbf{n} \cdot \nabla z)_{\Gamma_N}.$$

The continuous adjoint problem to Poisson's equation

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

Multiply left hand side by z and integrate by parts twice

$$(-\Delta u, z)_\Omega = (\nabla u, \nabla z)_\Omega - (\mathbf{n} \cdot \nabla u, z)_\Gamma = (u, -\Delta z)_\Omega + (u, \mathbf{n} \cdot \nabla z)_\Gamma - (\mathbf{n} \cdot \nabla u, z)_\Gamma.$$

After splitting the boundary terms according to $\Gamma = \Gamma_D \cup \Gamma_N$ and shuffling terms

$$(-\Delta u, z)_\Omega + (u, -\mathbf{n}\cdot\nabla z)_{\Gamma_D} + (\mathbf{n}\cdot\nabla u, z)_{\Gamma_N} = (u, -\Delta z)_\Omega + (\mathbf{n}\cdot\nabla u, -z)_{\Gamma_D} + (u, \mathbf{n}\cdot\nabla z)_{\Gamma_N}.$$

Comparing with the compatibility condition

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

we see that for $Lu = -\Delta u$ in Ω and

$$\begin{aligned} Bu &= u, & Cu &= \mathbf{n} \cdot \nabla u & &\text{on } \Gamma_D, \\ Bu &= \mathbf{n} \cdot \nabla u, & Cu &= u & &\text{on } \Gamma_N, \end{aligned}$$

the adjoint operators are given by $L^* z = -\Delta z$ on Ω and

$$\begin{aligned} B^* z &= -z, & C^* z &= -\mathbf{n} \cdot \nabla z & &\text{on } \Gamma_D, \\ B^* z &= \mathbf{n} \cdot \nabla z, & C^* z &= z & &\text{on } \Gamma_N. \end{aligned}$$

The continuous adjoint problem to Poisson's equation

Primal problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla u = g_N \quad \text{on } \Gamma_N,$$

For the operators $Lu = -\Delta u$ in Ω and

$$\begin{aligned} Bu &= u, & Cu &= \mathbf{n}\cdot\nabla u & &\text{on } \Gamma_D,\\ Bu &= \mathbf{n}\cdot\nabla u, & Cu &= u & &\text{on } \Gamma_N, \end{aligned}$$

the adjoint operators are given by $L^*z = -\Delta z$ on Ω and

$$\begin{aligned} B^*z &= -z, & C^*z &= -\mathbf{n}\cdot\nabla z & &\text{on } \Gamma_D,\\ B^*z &= \mathbf{n}\cdot\nabla z, & C^*z &= z & &\text{on } \Gamma_N. \end{aligned}$$

The continuous adjoint problem to Poisson's equation

Primal problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla u = g_N \quad \text{on } \Gamma_N,$$

For the operators $Lu = -\Delta u$ in Ω and

$$\begin{aligned} Bu &= u, & Cu &= \mathbf{n}\cdot\nabla u & &\text{on } \Gamma_D,\\ Bu &= \mathbf{n}\cdot\nabla u, & Cu &= u & &\text{on } \Gamma_N, \end{aligned}$$

the adjoint operators are given by $L^*z = -\Delta z$ on Ω and

$$\begin{aligned} B^*z &= -z, & C^*z &= -\mathbf{n}\cdot\nabla z & &\text{on } \Gamma_D,\\ B^*z &= \mathbf{n}\cdot\nabla z, & C^*z &= z & &\text{on } \Gamma_N. \end{aligned}$$

In particular,

$$\begin{aligned} J(u) &= \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma\, Cu \,\mathrm{d}s\\ &= \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D\, \mathbf{n}\cdot\nabla u \,\mathrm{d}s + \int_{\Gamma_N} j_N\, u \,\mathrm{d}s, \end{aligned}$$

is **compatible** and the continuous **adjoint problem** is given by

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad -z = j_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla z = j_N \quad \text{on } \Gamma_N.$$

Primal residual form of the interior penalty DG discretization

We rewrite the **discrete primal problem:** find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

in element-based **primal residual form:** find $u_h \in V_h$ such that

$$\int_\Omega R(u_h) v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} r(u_h) v_h + \boldsymbol{\rho}(u_h) \cdot \nabla_h v_h \,\mathrm{d}s \\ + \int_\Gamma r_\Gamma(u_h) v_h + \boldsymbol{\rho}_\Gamma(u_h) \cdot \nabla_h v_h \,\mathrm{d}s = 0 \quad \forall v_h \in V_h,$$

where the **primal residuals** are given by $R(u_h) = f + \Delta_h u_h$ on Ω, and

$$\begin{aligned} r(u_h) &= -\tfrac{1}{2}[\![\nabla_h u_h]\!] - \delta[u_h], & \boldsymbol{\rho}(u_h) &= -\tfrac{1}{2}\theta[\![u_h]\!] & &\text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_\Gamma(u_h) &= \delta(g_D - u_h), & \boldsymbol{\rho}_\Gamma(u_h) &= \theta(g_D - u_h)\mathbf{n} & &\text{on } \Gamma_D, \\ r_\Gamma(u_h) &= g_N - \mathbf{n} \cdot \nabla_h u_h, & \boldsymbol{\rho}_\Gamma(u_h) &= 0 & &\text{on } \Gamma_N. \end{aligned}$$

Consistency of the interior penalty DG discretization

The **primal residuals** are given by $R(u_h) = f + \Delta_h u_h$ on Ω, and

$$
\begin{array}{lll}
r(u_h) = -\frac{1}{2}[\![\nabla_h u_h]\!] - \delta[u_h], & \boldsymbol{\rho}(u_h) = -\frac{1}{2}\theta[\![u_h]\!] & \text{on } \partial\kappa\setminus\Gamma, \kappa\in\mathcal{T}_h,\\
r_\Gamma(u_h) = \delta(g_D - u_h), & \boldsymbol{\rho}_\Gamma(u_h) = \theta(g_D - u_h)\mathbf{n} & \text{on } \Gamma_D,\\
r_\Gamma(u_h) = g_N - \mathbf{n}\cdot\nabla_h u_h, & \boldsymbol{\rho}_\Gamma(u_h) = 0 & \text{on } \Gamma_N.
\end{array}
$$

The exact solution $u \in H^2(\Omega)$ to the **primal problem:**

$$
-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla u = g_N \quad \text{on } \Gamma_N,
$$

satisfies

$$
\begin{array}{lll}
R(u) = 0 & & \text{in } \kappa, \kappa\in\mathcal{T}_h,\\
r(u) = 0, & \boldsymbol{\rho}(u) = 0 & \text{on } \partial\kappa\setminus\Gamma, \kappa\in\mathcal{T}_h,\\
r_\Gamma(u) = 0, & \boldsymbol{\rho}_\Gamma(u) = 0 & \text{on } \Gamma.
\end{array}
$$

Thereby, the interior penalty DG discretization (NIPG and SIPG) are **consistent**.

Adjoint residual form of the interior penalty DG discretization

We rewrite the **discrete adjoint problem**: find $z_h \in V_h$ such that

$$B_h(w_h, z_h) = J(w_h) \quad \forall w_h \in V_h,$$

in following element-based **adjoint residual form**: find $z_h \in V_h$ such that

$$\int_\Omega w_h\, R^*(z_h)\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma} w_h\, r^*(z_h) + \nabla w_h\cdot\boldsymbol{\rho}^*(z_h)\,\mathrm{d}s$$
$$+ \int_\Gamma w_h\, r_\Gamma^*(z_h) + \nabla w_h\cdot\boldsymbol{\rho}_\Gamma^*(z_h)\,\mathrm{d}s = 0 \quad \forall w_h \in V_h.$$

where the **adjoint residuals** are given by $R^*(z_h) = j_\Omega + \Delta_h z_h$ on Ω, by

$$r^*(z_h) = -\tfrac{1}{2}[\![\nabla_h z_h]\!] - (1+\theta)\mathbf{n}\cdot\{\!\{\nabla_h z\}\!\} - \delta[z_h], \qquad \boldsymbol{\rho}^*(z_h) = \tfrac{1}{2}[\![z_h]\!],$$

on interior faces $\partial\kappa\setminus\Gamma, \kappa\in\mathcal{T}_h$, and by

$$r_\Gamma^*(z_h) = -(1+\theta)\mathbf{n}\cdot\nabla_h z_h - \delta z_h, \qquad \boldsymbol{\rho}_\Gamma^*(z_h) = (j_D + z_h)\mathbf{n} \qquad \text{on } \Gamma_D,$$
$$r_\Gamma^*(z_h) = j_N - \mathbf{n}\cdot\nabla_h z_h, \qquad \boldsymbol{\rho}_\Gamma^*(z_h) = 0 \qquad \text{on } \Gamma_N.$$

Adjoint consistency of the interior penalty DG discretization

The **adjoint residuals** are given by $R^*(z_h) = j_\Omega + \Delta_h z_h$ on Ω, by

$$r^*(z_h) = -\tfrac{1}{2}[\![\nabla_h z_h]\!] - (1+\theta)\mathbf{n}\cdot\{\!\{\nabla_h z\}\!\} - \delta[z_h], \qquad \boldsymbol{\rho}^*(z_h) = \tfrac{1}{2}[\![z_h]\!],$$

on interior faces $\partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h$, and by

$$\begin{aligned} r^*_\Gamma(z_h) &= -(1+\theta)\mathbf{n}\cdot\nabla_h z_h - \delta z_h, &\quad \boldsymbol{\rho}^*_\Gamma(z_h) &= (j_D + z_h)\mathbf{n} &\quad &\text{on } \Gamma_D,\\ r^*_\Gamma(z_h) &= j_N - \mathbf{n}\cdot\nabla_h z_h, & \boldsymbol{\rho}^*_\Gamma(z_h) &= 0 & &\text{on } \Gamma_N. \end{aligned}$$

The exact solution $z \in H^2(\Omega)$ to the continuous **adjoint problem:**

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad -z = j_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla z = j_N \quad \text{on } \Gamma_N.$$

satisfies $R^*(z) = 0$ on Ω, $r^*(z) = -2\mathbf{n}\cdot\nabla z \not\equiv 0$ for $\theta = 1$ on $\partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h$,

$$\begin{aligned} r^*(z) &= 0, \text{ provided } \theta = -1, &\quad \boldsymbol{\rho}^*(z) &= 0 \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h\\ r^*_\Gamma(z) &= 0, & \boldsymbol{\rho}^*_\Gamma(z) &= 0 \quad \text{on } \Gamma_N\\ r^*_\Gamma(z) &= \delta j_D, \text{ provided } \theta = -1 & \boldsymbol{\rho}^*_\Gamma(z) &= 0 \quad \text{on } \Gamma_D \end{aligned}$$

Adjoint consistency of the interior penalty DG discretization

The exact solution $z \in H^2(\Omega)$ to the adjoint problem satisfies $R^*(z) = 0$ on Ω,

$$\begin{aligned}
&r^*(z) = 0, \text{ provided } \theta = -1, && \boldsymbol{\rho}^*(z) = 0 \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h\\
&r_\Gamma^*(z) = 0, && \boldsymbol{\rho}_\Gamma^*(z) = 0 \quad \text{on } \Gamma_N\\
&r_\Gamma^*(z) = \delta j_D, \text{ provided } \theta = -1 && \boldsymbol{\rho}_\Gamma^*(z) = 0 \quad \text{on } \Gamma_D
\end{aligned}$$

- From $r^*(z) = -2\mathbf{n} \cdot \nabla z \not\equiv 0$ for $\theta = 1$: NIPG is **adjoint inconsistent**.
- SIPG is **adjoint consistent** on interior faces $\partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h$
- SIPG is **adjoint consistent** on the Neumann boundary Γ_N
- SIPG in combination with $J(\cdot)$ and $j_D \neq 0$ is **adjoint inconsistent**

Modification of the target functional

SIPG in combination with $J(u_h) = \int_{\Gamma_D} j_D \, \mathbf{n} \cdot \nabla_h u_h \, \mathrm{d}s$

and $j_D \neq 0$ is **adjoint inconsistent**. Modify $J(u_h)$ as follows:

$$\widetilde{J}(u_h) = J(u_h) - \int_{\Gamma_D} \delta(u_h - g_D) j_D \, \mathrm{d}s$$

Then the corresponding discrete adjoint problem is: find $z_h \in V_h$ such that

$$B_h(w_h, z_h) = \widetilde{J}'[u_h](w_h) \quad \forall w_h \in V_h,$$

where $\widetilde{J}'[u_h](w_h) = J'[u_h](w_h) - \int_{\Gamma_D} w_h \, \delta j_D \, \mathrm{d}s = J(w_h) - \int_{\Gamma_D} w_h \, \delta j_D \, \mathrm{d}s.$

Thereby, $r_\Gamma^*(z_h) = -(1+\theta)\mathbf{n} \cdot \nabla_h z_h - \delta z_h \boxed{-\delta j_D} \quad \text{on } \Gamma_D$

and the solution z to the **adjoint problem**:

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad -z = j_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla z = j_N \quad \text{on } \Gamma_N.$$

satisfies $r^*(z) = 0$ provided $\theta = -1$.

Thereby, SIPG in combination with $\widetilde{J}(u_h)$ is **adjoint consistent**.

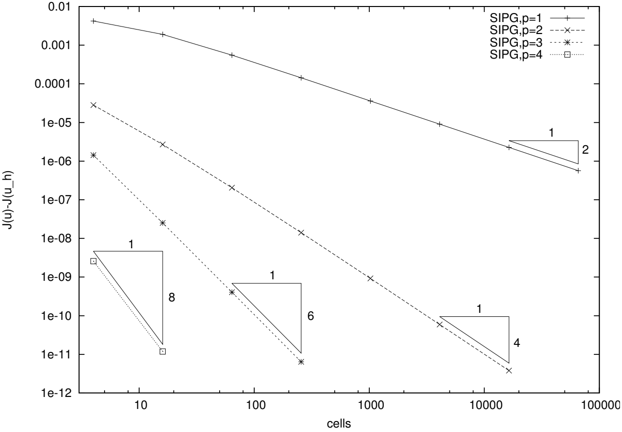
Outline

Example 1: Model problem with SIPG

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$J_1(u_h) = \int_\Omega j_\Omega\, u_h\, \mathrm{d}\mathbf{x}, \qquad \text{with} \quad j_\Omega(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2) \quad \text{on } \Omega$$

This target quantity is **compatible** with the model problem.



SIPG discretization of Poisson's equation:

The error $|J_1(u) - J_1(u_h)|$ of the DG(p), $p = 1, \ldots, 4$, discretization is of $\mathcal{O}(h^{2p})$

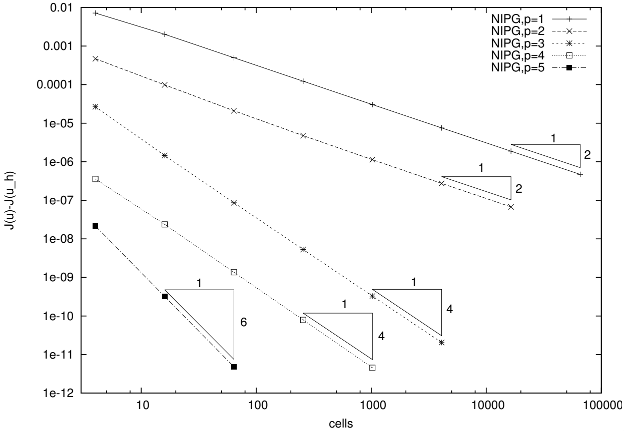
adjoint consistent

Example 1: Model problem with NIPG

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$J_1(u_h) = \int_\Omega j_\Omega\, u_h \,\mathrm{d}\mathbf{x}, \qquad \text{with} \quad j_\Omega(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2) \quad \text{on } \Omega$$

This target quantity is **compatible** with the model problem.



NIPG discretization of Poisson's equation:

The error $|J_1(u) - J_1(u_h)|$ of the DG(p), $p = 1, \ldots, 5$, discretization is of $\mathcal{O}(h^{p+1})$ for odd p and of $\mathcal{O}(h^p)$ for even p

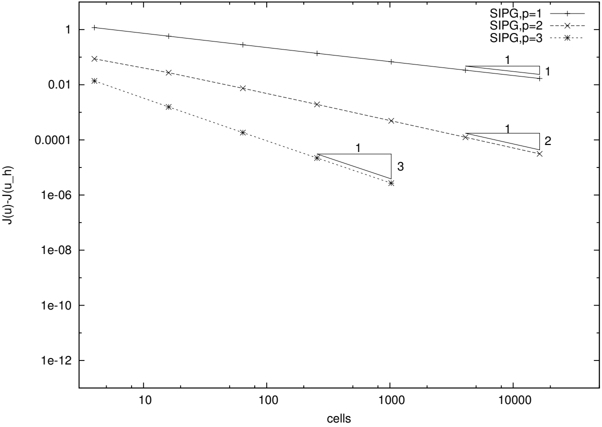
adjoint inconsistent

Example 2: Model problem with SIPG but adjoint inconsistent

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$J_2(u_h) = \int_\Gamma j_D \, \mathbf{n} \cdot \nabla_h u_h \, \mathrm{d}s, \qquad \text{with} \quad j_D \equiv 1 \quad \text{on } \Gamma_D = \Gamma$$

This target quantity is also **compatible** with the model problem.



SIPG discretization of Poisson's equation:

The error $|J_2(u) - J_2(u_h)|$ of the DG(p), $p = 1, \ldots, 3$, discretization is of $\mathcal{O}(h^p)$

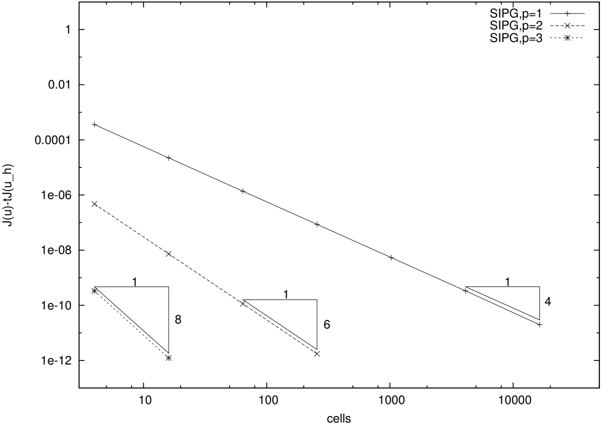
adjoint inconsistent

Example 2: Model problem with SIPG and adjoint consistent

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$\widetilde{J}_2(u_h) = \int_\Gamma j_D\, \mathbf{n}\cdot\nabla_h u_h\, ds - \int_{\Gamma_D} \delta(u_h - g_D) j_D\, ds \quad \text{with} \quad j_D \equiv 1 \quad \text{on } \Gamma_D = \Gamma$$

is a consistent modification of $J_2(u_h)$.



SIPG discretization of Poisson's equation:

The error $|J_2(u) - \widetilde{J}_2(u_h)|$ of the DG(p), $p = 1, \ldots, 3$, discretization behaves like $\mathcal{O}(h^{2(p+1)})$

adjoint consistent

of even higher order than the expected $\mathcal{O}(h^{2p})$

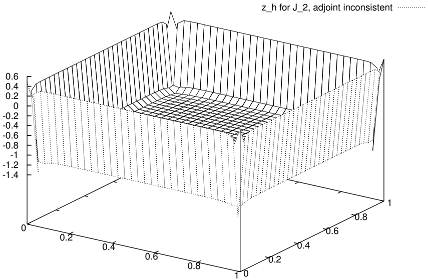
Example 2: Smoothness of the discrete adjoint solution

The exact solution to the adjoint problem

$$-\Delta z = 0 \quad \text{in } \Omega, \qquad\qquad -z = j_D \quad \text{on } \Gamma_D$$

with $j_D \equiv 1$ is given by $z \equiv -1$ on Ω.

Using the SIPG discretization in combination with $J_2(u_h)$ and $\widetilde{J}_2(u_h)$:



discrete adjoint solution z_h
connected to $J_2(u_h)$
adjoint inconsistent

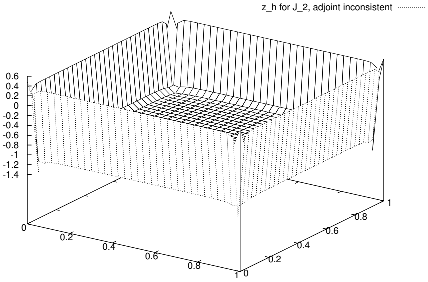
Example 2: Smoothness of the discrete adjoint solution

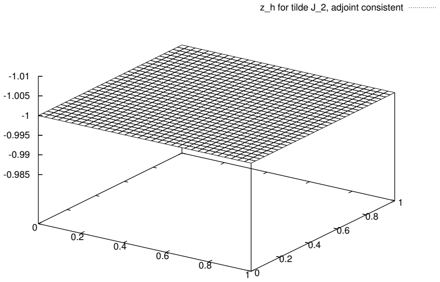
The exact solution to the adjoint problem

$$-\Delta z = 0 \quad \text{in } \Omega, \qquad\qquad -z = j_D \quad \text{on } \Gamma_D$$

with $j_D \equiv 1$ is given by $z \equiv -1$ on Ω.

Using the SIPG discretization in combination with $J_2(u_h)$ and $\widetilde{J}_2(u_h)$:





discrete adjoint solution z_h connected to $J_2(u_h)$ **adjoint inconsistent**

discrete adjoint solution z_h connected to $\widetilde{J}_2(u_h)$ **adjoint consistent**

Example 3: Another Dirichlet problem

Consider $\Omega = (0,1) \times (0.1,1)$ and Poisson's equation with forcing function f such that

$$u(\mathbf{x}) = \tfrac{1}{4}(1+x_1)^2 \sin(2\pi x_1 x_2).$$

Dirichlet boundary conditions are based on the exact solution u.
Consider the target quantity $J_3(u_h)$ and its consistent modification $\widetilde{J}_3(u_h)$:

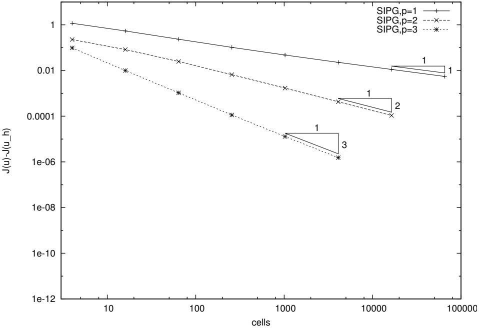
$$\begin{aligned} J_3(u_h) &= \int_\Gamma j_D \, \mathbf{n} \cdot \nabla_h u_h \, \mathrm{d}s, \\ \widetilde{J}_3(u_h) &= J_3(u_h) - \int_\Gamma \delta(u_h - g_D) j_D \, \mathrm{d}s. \end{aligned}$$

and choose $j_D \in L^2(\Gamma)$ to be given by

$$j_D(\mathbf{x}) = \begin{cases} \exp\left(4 - \frac{1}{16}((x_1 - \frac{1}{4})^2 - \frac{1}{8})^{-2}\right) & \text{for } \mathbf{x} \in (0, \frac{1}{4}) \times (0.1, 1), \\ \exp\left(4 - \frac{1}{16}((x_1 - \frac{3}{4})^2 - \frac{1}{8})^{-2}\right) & \text{for } \mathbf{x} \in (\frac{3}{4}, 1) \times (0.1, 1), \\ 1 & \text{for } \mathbf{x} \in (\frac{1}{4}, \frac{3}{4}) \times (0.1, 1), \\ 0 & \text{elsewhere on } \Gamma. \end{cases}$$

Example 3: Another Dirichlet problem

Using the SIPG discretization in combination with $J_3(u_h)$ and $\widetilde{J}_3(u_h)$:

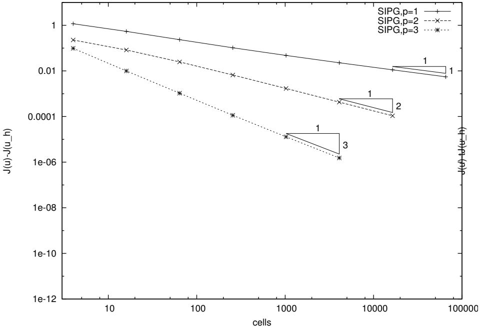


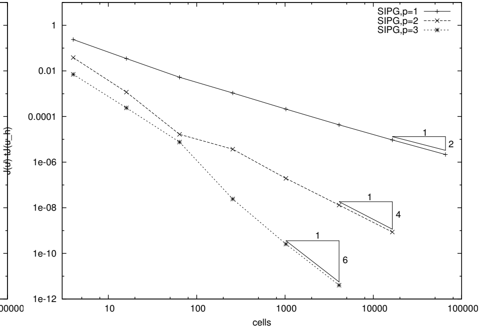
The error $|J_3(u) - J_3(u_h)|$
of the DG(p), $p = 1, \dots, 3$,
discretization
behaves like $\mathcal{O}(h^p)$

adjoint inconsistent

Example 3: Another Dirichlet problem

Using the SIPG discretization in combination with $J_3(u_h)$ and $\widetilde{J}_3(u_h)$:





The error $|J_3(u) - J_3(u_h)|$
of the DG(p), $p = 1, \ldots, 3$,
discretization
behaves like $\mathcal{O}(h^p)$

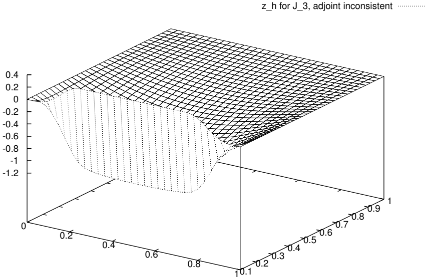
adjoint inconsistent

The error $|J_3(u) - \widetilde{J}_3(u_h)|$
of the DG(p), $p = 1, \ldots, 3$,
discretization
behaves like $\mathcal{O}(h^{2p})$

adjoint consistent

Example 3: Smoothness of the discrete adjoint solution

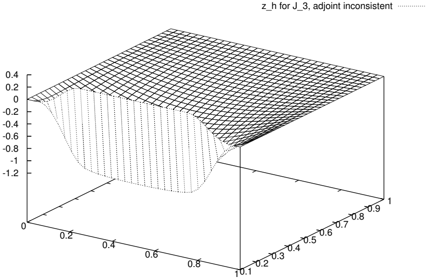
Using the SIPG discretization in combination with $J_2(u_h)$ and $\widetilde{J}_2(u_h)$:



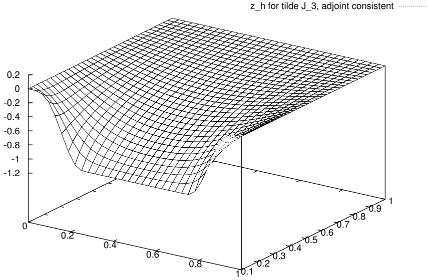
discrete adjoint solution z_h
connected to $J_3(u_h)$
adjoint inconsistent

Example 3: Smoothness of the discrete adjoint solution

Using the SIPG discretization in combination with $J_2(u_h)$ and $\widetilde{J}_2(u_h)$:



discrete adjoint solution z_h
connected to $J_3(u_h)$
adjoint inconsistent



discrete adjoint solution z_h
connected to $\widetilde{J}_3(u_h)$
adjoint consistent

Outline

The continuous adjoint problem to the linear advection equation

Consider the linear advection equation

$$Lu := \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_- = \{\mathbf{x} \in \Gamma, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}.$$

The continuous adjoint problem to the linear advection equation

Consider the linear advection equation

$$Lu := \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_- = \{\mathbf{x} \in \Gamma, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}.$$

Multiply by $z \in H^{1,\mathbf{b}}(\mathcal{T}_h)$, integrate over Ω and integrate by parts

$$\int_\Omega (\nabla \cdot (\mathbf{b}u) + cu)\, z \, \mathrm{d}\mathbf{x} = -\int_\Omega (\mathbf{b}u) \cdot \nabla z \, \mathrm{d}\mathbf{x} + \int_\Omega cuz \, \mathrm{d}\mathbf{x} + \int_\Gamma \mathbf{b} \cdot \mathbf{n} \, uz \, \mathrm{d}s.$$

The continuous adjoint problem to the linear advection equation

Consider the linear advection equation

$$Lu := \nabla\cdot(\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_- = \{\mathbf{x}\in\Gamma, \mathbf{b}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\}.$$

Multiply by $z \in H^{1,\mathbf{b}}(\mathcal{T}_h)$, integrate over Ω and integrate by parts

$\int_\Omega (\nabla\cdot(\mathbf{b}u) + cu)\, z \,\mathrm{d}\mathbf{x} = -\int_\Omega (\mathbf{b}u)\cdot\nabla z\,\mathrm{d}\mathbf{x} + \int_\Omega cuz\,\mathrm{d}\mathbf{x} + \int_\Gamma \mathbf{b}\cdot\mathbf{n}\, uz\,\mathrm{d}s.$

After splitting the boundary $\Gamma = \Gamma_-\cup\Gamma_+$ we obtain:

$$(\nabla\cdot(\mathbf{b}u) + cu, z)_\Omega + (u, -\mathbf{b}\cdot\mathbf{n}\, z)_{\Gamma_-} = (u, -\mathbf{b}\cdot\nabla z + cz)_\Omega + (u, \mathbf{b}\cdot\mathbf{n}\, z)_{\Gamma_+}.$$

The continuous adjoint problem to the linear advection equation

Consider the linear advection equation

$$Lu := \nabla\cdot(\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_- = \{\mathbf{x}\in\Gamma, \mathbf{b}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\}.$$

Multiply by $z \in H^{1,\mathbf{b}}(\mathcal{T}_h)$, integrate over Ω and integrate by parts

$\int_\Omega (\nabla\cdot(\mathbf{b}u) + cu)\, z\,\mathrm{d}\mathbf{x} = -\int_\Omega (\mathbf{b}u)\cdot\nabla z\,\mathrm{d}\mathbf{x} + \int_\Omega cuz\,\mathrm{d}\mathbf{x} + \int_\Gamma \mathbf{b}\cdot\mathbf{n}\, uz\,\mathrm{d}s.$

After splitting the boundary $\Gamma = \Gamma_-\cup\Gamma_+$ we obtain:

$$(\nabla\cdot(\mathbf{b}u) + cu, z)_\Omega + (u, -\mathbf{b}\cdot\mathbf{n}\,z)_{\Gamma_-} = (u, -\mathbf{b}\cdot\nabla z + cz)_\Omega + (u, \mathbf{b}\cdot\mathbf{n}\,z)_{\Gamma_+}.$$

Comparing with the compatibility condition

$$(Lu, z)_\Omega + (Bu, C^*z)_\Gamma = (u, L^*z)_\Omega + (Cu, B^*z)_\Gamma,$$

we see that for $Lu = \nabla\cdot(\mathbf{b}u) + cu$ in Ω and

$$\begin{aligned} &Bu = u, && Cu = 0 && \text{on } \Gamma_-,\\ &Bu = 0, && Cu = u && \text{on } \Gamma_+, \end{aligned}$$

the adjoint operators are given by $L^*z = -\mathbf{b}\cdot\nabla z + cz$ in Ω and

$$\begin{aligned} &B^*z = 0, && C^*z = -\mathbf{b}\cdot\mathbf{n}\,z && \text{on } \Gamma_-,\\ &B^*z = \mathbf{b}\cdot\mathbf{n}\,z, && C^*z = 0 && \text{on } \Gamma_+. \end{aligned}$$

The continuous adjoint problem to the linear advection equation

Primal problem:

$$Lu := \nabla\cdot(\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma_-.$$

For the operators $Lu = \nabla\cdot(\mathbf{b}u) + cu$ in Ω and

$$\begin{aligned} Bu &= u, & Cu &= 0 & &\text{on } \Gamma_-,\\ Bu &= 0, & Cu &= u & &\text{on } \Gamma_+, \end{aligned}$$

the adjoint operators are given by $L^*z = -\mathbf{b}\cdot\nabla z + cz$ in Ω and

$$\begin{aligned} B^*z &= 0, & C^*z &= -\mathbf{b}\cdot\mathbf{n}\, z & &\text{on } \Gamma_-,\\ B^*z &= \mathbf{b}\cdot\mathbf{n}\, z, & C^*z &= 0 & &\text{on } \Gamma_+. \end{aligned}$$

In particular,

$$J(u) = \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma\, Cu \,\mathrm{d}s = \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma\, u \,\mathrm{d}s,$$

is **compatible** the continuous adjoint problem is given by

$$-\mathbf{b}\cdot\nabla z + cz = j_\Omega \quad \text{in } \Omega, \qquad \mathbf{b}\cdot\mathbf{n}\, z = j_\Gamma \quad \text{on } \Gamma_+.$$

Primal residual form of the upwind DG discretization

We rewrite the **discrete primal problem:** find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

in element-based **primal residual form:** find $u_h \in V_h$ such that

$$\int_\Omega R(u_h) v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} r(u_h) v_h \,\mathrm{d}s + \int_\Gamma r_\Gamma(u_h) v_h \,\mathrm{d}s = 0 \quad \forall v_h \in V_h,$$

where the **primal residuals** are given by $R(u_h) = f - \nabla_h \cdot (\mathbf{b} u_h) - c u_h$ on Ω, and

$$\begin{aligned} r(u_h) &= \mathbf{b} \cdot \mathbf{n} \left(u_h^+ - u_h^-\right) && \text{on } \partial\kappa_- \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_\Gamma(u_h) &= \mathbf{b} \cdot \mathbf{n}(u_h - g) && \text{on } \Gamma_-. \end{aligned}$$

Primal residual form of the upwind DG discretization

We rewrite the **discrete primal problem:** find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

in element-based **primal residual form:** find $u_h \in V_h$ such that

$$\int_\Omega R(u_h) v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} r(u_h) v_h \,\mathrm{d}s + \int_\Gamma r_\Gamma(u_h) v_h \,\mathrm{d}s = 0 \quad \forall v_h \in V_h,$$

where the **primal residuals** are given by $R(u_h) = f - \nabla_h \cdot (\mathbf{b} u_h) - c u_h$ on Ω, and

$$\begin{aligned} r(u_h) &= \mathbf{b}\cdot\mathbf{n}\,(u_h^+ - u_h^-) && \text{on } \partial\kappa_- \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_\Gamma(u_h) &= \mathbf{b}\cdot\mathbf{n}(u_h - g) && \text{on } \Gamma_-. \end{aligned}$$

The exact solution $u \in H^{1,\mathbf{b}}(\Omega)$ to the **primal problem:**

$$\nabla\cdot(\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma_-,$$

satisfies

$$\begin{aligned} R(u) &= 0 && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ r(u) &= 0 && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_\Gamma(u) &= 0 && \text{on } \Gamma. \end{aligned}$$

Thereby, the upwind DG discretization is **consistent**.

Adjoint residual form of the upwind DG discretization

We rewrite the **discrete adjoint problem**: find $z_h \in V_h$ such that

$$B_h(w_h, z_h) = J(w_h) \quad \forall w_h \in V_h,$$

in following element-based **adjoint residual form**: find $z_h \in V_h$ such that

$$\int_\Omega w_h\, R^*(z_h)\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma} w_h\, r^*(z_h)\,\mathrm{d}s + \int_\Gamma w_h\, r_\Gamma^*(z_h)\,\mathrm{d}s = 0 \quad \forall w_h \in V_h,$$

where the **adjoint residuals** are given by

$$\begin{aligned} R^*(z_h) &= j_\Omega + \mathbf{b}\cdot\nabla_h z_h - c z_h && \text{on } \Omega\\ r^*(z_h) &= -\mathbf{b}\cdot\mathbf{n}\,[z_h] && \text{on } \partial\kappa\setminus\Gamma, \kappa\in\mathcal{T}_h,\\ r_\Gamma^*(z_h) &= j_\Gamma - \mathbf{b}\cdot\mathbf{n}\,z_h && \text{on } \Gamma_+. \end{aligned}$$

Adjoint residual form of the upwind DG discretization

The **adjoint residuals** are given by

$$
\begin{aligned}
R^*(z_h) &= j_\Omega + \mathbf{b}\cdot\nabla_h z_h - c z_h && \text{on } \Omega \\
r^*(z_h) &= -\mathbf{b}\cdot\mathbf{n}\,[z_h] && \text{on } \partial\kappa\setminus\Gamma, \kappa\in\mathcal{T}_h, \\
r_\Gamma^*(z_h) &= j_\Gamma - \mathbf{b}\cdot\mathbf{n}\,z_h && \text{on } \Gamma_+.
\end{aligned}
$$

The exact solution $z \in H^{1,\mathbf{b}}(\Omega)$ to the continuous **adjoint problem:**

$$
-\mathbf{b}\cdot\nabla z + cz = j_\Omega \quad \text{in } \Omega, \qquad \mathbf{b}\cdot\mathbf{n}\,z = j_\Gamma \quad \text{on } \Gamma_+,
$$

satisfies

$$
\begin{aligned}
R^*(z) &= 0 && \text{on } \Omega \\
r^*(z) &= 0 && \text{on } \partial\kappa\setminus\Gamma, \kappa\in\mathcal{T}_h, \\
r_\Gamma^*(z) &= 0 && \text{on } \Gamma_+.
\end{aligned}
$$

Thereby, the upwind DG discretization is **adjoint consistent**.

Example: A priori error estimates for target functionals $J(\cdot)$

For the linear advection equation

$$Lu := \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad\qquad u = g \quad \text{on } \Gamma_-,$$

we have the *a priori* error estimate:

$$|||u - u_h|||_{b_0} \leq Ch^{p+1/2}|u|_{H^{p+1}(\Omega)} \qquad \forall u \in H^{p+1}(\Omega),$$

and the approximation estimate:

$$|||v - P^d_{h,p}v|||_{b_0} \leq Ch^{p+1/2}|v|_{H^{p+1}(\Omega)} \qquad \forall v \in H^{p+1}(\Omega).$$

If we now had continuity

$$|B_h(u, v)| \leq C|||u|||_{b_0}|||v|||_{b_0}$$

we could employ the error estimate: $|J(u) - J(u_h)|$ **is of order** $\mathcal{O}(h^{r+\tilde{r}})$.
Here for $r = p + 1/2$ and $\tilde{r} = p + 1/2$.

Example: A priori error estimates for target functionals $J(\cdot)$

For the linear advection equation

$$Lu := \nabla\cdot(\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad\qquad u = g \quad \text{on } \Gamma_-,$$

we have the *a priori* error estimate:

$$|||u-u_h|||_{b_0} \le Ch^{p+1/2}|u|_{H^{p+1}(\Omega)} \qquad \forall u\in H^{p+1}(\Omega),$$

and the approximation estimate:

$$|||v-P^d_{h,p}v|||_{b_0} \le Ch^{p+1/2}|v|_{H^{p+1}(\Omega)} \qquad \forall v\in H^{p+1}(\Omega).$$

If we now had continuity

$$|B_h(u,v)| \le C|||u|||_{b_0}|||v|||_{b_0}$$

we could employ the error estimate: $|J(u)-J(u_h)|$ **is of order** $\mathcal{O}(h^{r+\tilde r})$.
Here for $r=p+1/2$ and $\tilde r = p+1/2$.

The error $|J(u)-J(u_h)|$ for the upwind DG discretization is of $\mathcal{O}(h^{2p+1})$ [35,23].

Outline

A priori error estimates for target functionals $J(\cdot)$: Summary

- A discretization is adjoint consistent if the corresponding discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Adjoint consistency and thus optimal order estimates can be obtained only for target functionals which are compatible with the primal equations.

A priori error estimates for target functionals $J(\cdot)$: Summary

- A discretization is adjoint consistent if the corresponding discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Adjoint consistency and thus optimal order estimates can be obtained only for target functionals which are compatible with the primal equations.
- The upwind DG(p) discretization of the linear advection equation in combination with **compatible** target quantities is **adjoint consistent**:
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p+1})$

A priori error estimates for target functionals $J(\cdot)$: Summary

- A discretization is adjoint consistent if the corresponding discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Adjoint consistency and thus optimal order estimates can be obtained only for target functionals which are compatible with the primal equations.
- The upwind DG(p) discretization of the linear advection equation in combination with **compatible** target quantities is **adjoint consistent**:
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p+1})$
- For an **adjoint consistent** DG(p) discretization of Poisson's equation:
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p})$

A priori error estimates for target functionals $J(\cdot)$: Summary

- A discretization is adjoint consistent if the corresponding discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Adjoint consistency and thus optimal order estimates can be obtained only for target functionals which are compatible with the primal equations.
- The upwind DG(p) discretization of the linear advection equation in combination with **compatible** target quantities is **adjoint consistent**:
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p+1})$
- For an **adjoint consistent** DG(p) discretization of Poisson's equation:
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{2p})$
- For an **adjoint inconsistent** DG(p) discretization of Poisson's equation:
  - The error measured in terms of $J(\cdot)$ behaves like $\mathcal{O}(h^{p})$