Adaptive Discontinuous Galerkin methods in aerodynamic flow simulations

Ralf Hartmann
Adaptive DG methods in aerodynamic flow simulations

The DG group at DLR:

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▶ Joachim Held
▶ Tobias Leicht
▶ Florian Prill

Acknowledgments:

▶ Partially joint work with Paul Houston, University of Nottingham
▶ Numerics are based on the DG flow solver PADGE which is based on deal.II.


Problem and Discretization

Laminar aerodynamic flow as governed by the compr. Navier-Stokes equations

- Symmetric interior penalty discontinuous Galerkin discretization
- with consistent and adjoint consistent discretization of boundary conditions
- with consistent and adjoint consistent discretization of the aerodynamic force coefficients
- with an optimal order penalty term for the compressible NS equations

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Multi target error estimation and adaptivity: Motivation

Given $N$ target quantities, e.g. following aerodynamic force coefficients

- the pressure induced force coefficients: $cdp$, $clp$, $cmp$
- the viscous force coefficients: $cdf$, $clf$, $cmf$

i.e. 6 force coefficients in 2d or 10 force coefficients in 3d.

The standard approach: Error estimation and adjoint-based refinement requires the solution of $N$ adjoint problems.

Goal: Replace the $N$ adjoint problems by two auxiliary problems (1 adjoint problem and 1 adjoint adjoint problem) irrespective of the number of target quantities.
Error estimation for single target quantities

Given a discretization: find $u_h \in V_h$ such that

$$\mathcal{N}(u_h, v_h) = 0 \quad \forall v_h \in V_h. \quad (1)$$

and a target quantity $J$.

Computed: $J(u_h)$, exact (but unknown): $J(u)$, what is $J(u) - J(u_h)$?!

Using a duality argument we obtain an error representation wrt. $J(\cdot)$:

$$J(u) - J(u_h) = R(u_h, z) := -\mathcal{N}(u_h, z)$$

$$\approx R(u_h, \tilde{z}_h) = \sum_{\kappa} \eta_{\kappa},$$

where $\tilde{z}_h$ is the solution to the discrete adjoint problem: find $\tilde{z}_h \in \tilde{V}_h$ such that

$$\mathcal{N}'[u_h](w_h, \tilde{z}_h) = J'[u_h](w_h) \quad \forall w_h \in \tilde{V}_h,$$

and $\eta_{\kappa}$ are adjoint-based indicators which are particularly suited for the accurate and efficient approximation of the target quantity $J(u)$. 

Single target error estimation and adaptivity applied to Discontinuous Galerkin discretizations of

- the linear advection equation
- scalar nonlinear conservation laws: inviscid Burgers equation, Buckley-Leverett equation
- 1d compressible Euler equations
- 2d compressible Euler equations: subsonic, transonic and supersonic flows
- 2d compressible Navier-Stokes equations: subsonic, transonic, supersonic flows
Single target error estimation and adaptivity applied to Discontinuous Galerkin discretizations of

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  - inviscid Burgers equation, Buckley-Leverett equation
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- 2d compressible Euler equations: subsonic, transonic and supersonic flows
- 2d compressible Navier-Stokes equations: subsonic, transonic, supersonic flows

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Example: ADIGMA MTC3 test case

Laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$ around NACA0012 airfoil:

We are interested in the

1. pressure induced drag: $J(u) = c_{dp}$
2. viscous drag: $J(u) = c_{df}$
3. total lift: $J(u) = c_l$
4. total momentum: $J(u) = c_m$
Error estimation for single target quantity: $J(u) = c_{dp}$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(u) = c_{dp}$ (pressure induced drag), Ref. value: $J_{cdp}^{ref}(u) = 0.02380$

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Error estimation for single target quantity: $J(u) = c_{df}$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(u) = c_{df}$ (viscous drag), Ref. value: $J_{c_{df}}^{ref}(u) = 0.0322835$

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Error estimation for single target quantity: $J(u) = c_l$

Example: MTC-3, laminar flow, $M = 0.5$, $\alpha = 2^\circ$, $Re = 5000$

Target quantity: $J(u) = c_l$ (total lift), Ref.value: $J_{cl}^{ref}(u) = 0.037286$

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Error estimation for single target quantity: $J(u) = c_m$

Example: MTC-3, laminar flow, $M = 0.5$, $\alpha = 2^\circ$, $Re = 5000$

Target quantity: $J(u) = c_m$ (total moment), Ref.value: $J_{cm}^{ref}(u) = -0.01661$

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Error estimation for single target quantities

$z_1$ components of adjoint solutions.  
Top: cdp, cdf;  
Bottom: cl, cm.
Error estimation for multiple target quantities

For \( N \) target quantities \( J_i(u), i = 1, \ldots, N \): Instead of computing \( N \) adjoint solutions

\[
N'[u_h](w_h, \tilde{z}_{i,h}) = J'_i[u_h](w_h) \quad \forall w_h \in \tilde{V}_h, \quad i = 1, \ldots, N,
\]

to obtain error estimates for the \( N \) target quantities

\[
J_i(u) - J_i(u_h) = R(u_h, z_i) \approx R(u_h, z_{i,h}), \quad i = 1, \ldots, N,
\]

we now solve one discrete error equation: find \( \tilde{e}_h \in \tilde{V}_h \) such that

\[
N'[u_h](\tilde{e}_h, w_h) = R(u_h, w_h) \quad \forall w_h \in \tilde{V}_h,
\]

to obtain error estimates for the \( N \) target quantities

\[
J_i(u) - J_i(u_h) \approx J'_i[u_h](e) \approx J'_i[u_h](\tilde{e}_h), \quad i = 1, \ldots, N.
\]

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Error estimation for multiple target quantities

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

On each mesh compute primal solution $u_h$ and adjoint-adjoint solution $\tilde{e}_h$.

Evaluate exact error: \[ J_{i}^{\text{ref}}(u) - J_{i}(u_h), \quad i = 1, \ldots, N, \]

Evaluate error estimate: \[ J'_{i}[u_h](\tilde{e}_h), \quad i = 1, \ldots, N, \]

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Adaptive refinement for multiple target functionals (1)

Goal-oriented mesh refinement tailored to reducing e.g.

a) the sum of relative errors or b) the (weighted) sum of absolute errors:

$$ a) \quad \sum_{i=1}^{N} \frac{|J_i(u) - J_i(u_h)|}{|J_i(u)|}, \quad b) \quad \sum_{i=1}^{N} \alpha_i |J_i(u) - J_i(u_h)|. $$

Define the combined target functional:

$$ a) \quad J_c(v) = \sum_{i=1}^{N} s_i J_i(v) / |J_i(u_h)|, \quad b) \quad J_c(v) = \sum_{i=1}^{N} \alpha_i s_i J_i(v), $$

with $s_i = \text{sign}(J_i(u) - J_i(u_h))$.

We now solve the adjoint problem: find $\tilde{z}_{c,h} \in \tilde{V}_h$ such that

$$ \mathcal{N}'[u_h](w_h, \tilde{z}_{c,h}) = J_c'[u_h](w_h) \quad \forall w_h \in \tilde{V}_h, \quad i = 1, \ldots, N, $$

and obtain the error estimate

$$ J_c(u) - J_c(u_h) = \mathcal{R}(u_h, z_c) \approx \mathcal{R}(u_h, \tilde{z}_{c,h}) = \sum_{\kappa \in T_h} \eta_{\kappa}. $$
Adaptive refinement for multiple target functionals (2)

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2°, Re = 5000$

Target quantity: sum of relative errors

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On finest mesh sum of relative errors is 2.4%. Error estimation tells us: 2.5%
Goal-oriented refinement for multiple target quantities

Example: ADIGMA MTC-3, laminar, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Goal: Accurate and efficient approximation of cdp, cdf, cl, cm

accuracy requirements (ADIGMA): cdp, cdf, cm: $|\text{error}| < 5e-4$, cl: $|\text{error}| < 5e-3$

![residual-based refinement](image1)

8896 cells, 149.4s

![adjoint-based refinement](image2)

1894 cells, 80.8s (incl. error est.)

stronger accuracy requirements: cdp, cdf, cm: $|\text{error}| < 1e-4$, cl: $|\text{error}| < 1e-3$

67660 cells, 2691.1s

8539 cells, 664.6s (incl. error est.)
The residual-based indicators

Using the error representation

\[ J(u) - J(u_h) = R(u_h, z) = -N(u_h, z) = -N(u_h, z - z_h) \]

and assuming \( z \in [H^1(\kappa)]^5 \) with \( \|z\|_{[H^1(\kappa)]^5} \leq C_{stab} \) we obtain

\[ |J(u) - J(u_h)| \leq \left( \sum_{\kappa \in T_h} \left( \eta^{(\text{res})}_\kappa \right)^2 \right)^{1/2}, \]

where the residual-based indicators \( \eta^{(\text{res})}_\kappa, \kappa \in T_h, \) are given by

\[ \eta^{(\text{res})}_\kappa = h_\kappa \| R(u_h) \|_\kappa + h_\kappa^{1/2} \| r_\kappa (u_h) \|_\kappa + h_\kappa^{-1/2} \| \rho_\kappa (u_h) \|_\kappa, \]

Note: \( \eta^{(\text{res})}_\kappa \) is independent of target quantity \( J(u) \). Mesh refinement based on the residual-based indicators \( \eta^{(\text{res})}_\kappa \) targets at resolving all flow features.
Interface to CAD data

We use an interface to CAD data (via OpenCascade). It provides additional points on curved boundaries represented by CAD data. They are required

- to allow a higher order approximation of curved boundaries
- to make sure that local mesh refinement fits the CAD boundary
Residual-based refinement for a streamlined body (BTC0)

Freestream conditions: $M = 0.5$, $\alpha = 1^\circ$, $Re = 5000$.

Coarse mesh: 768 elements, 30 720 DoFs
Residual-based refinement for a streamlined body (BTC0)

Freestream conditions: $M = 0.5$, $\alpha = 1^\circ$, $Re = 5000$.

1. refinement step: 1909 elements, 76360 DoFs
Residual-based refinement for a streamlined body (BTC0)

**Freestream conditions:** $M = 0.5, \alpha = 1^\circ, Re = 5000.$

2. refinement step: 4,912 elements, 196,480 DoFs
Residual-based refinement for a streamlined body (BTC0)

Freestream conditions: $M = 0.5, \alpha = 1^\circ, Re = 5000.$

3. refinement step: 12 437 elements, 497 480 DoFs
Residual-based refinement for a streamlined body (BTC0)

Freestream conditions: $M = 0.5$, $\alpha = 1^\circ$, $Re = 5000$.

4. refinement step: 31,582 elements, 1,263,280 DoFs
Example: Laminar delta wing (BTC3)

\[ M = 0.3, \quad \alpha = 12.5^\circ, \]
\[ Re = 4000 \]

3264 elements
for the half model

left: DG(1), 2nd order
right: DG(4), 5th order
Example: Laminar delta wing (BTC3)

\[ M = 0.3, \alpha = 12.5^\circ, \]
\[ Re = 4000 \]

3264 elements
for the half model

left: DG(1), 2nd order
right: DG(4), 5th order

DG(1), 40 DoFs/element:
130 560 DoFs

DG(4), 625 DoFs/element:
2 040 000 DoFs
Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: \( M = 0.3, \alpha = 12.5^\circ, Re = 4000. \)

Residual-based refinement

Coarse mesh: 3264 elements, 130,560 DoFs
Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$.

Residual-based refinement

1. refinement step: 8,192 elements, 327,680 DoFs
Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$.

Residual-based refinement

2. refinement step: 21352 elements, 854080 DoFs
Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: \( M = 0.3, \alpha = 12.5^\circ, Re = 4000. \)

Residual-based refinement

3. refinement step: 55 673 elements, 2 226 920 DoFs
Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: \( M = 0.3, \alpha = 12.5^\circ, Re = 4000 \).

Residual-based refinement

4. refinement step: 144 279 elements, 5 771 160 DoFs
Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3, \alpha = 12.5^\circ, Re = 4000$. 
Error estimation and goal-oriented (adjoint-based) refinement

ADIGMA BTC3 test case: laminar flow around a delta wing.

Freestream conditions: \( M = 0.3, \alpha = 12.5^\circ, \text{Re} = 4000. \)

Reference values: \( c_l^{\text{ref}} = 0.3494, c_d^{\text{ref}} = 0.1664, c_m^{\text{ref}} = -0.0311 \)

\[
J(u) - J(u_h) = \mathcal{R}(u_h, z) \approx \mathcal{R}(u_h, \tilde{z}_h) = \sum_{\kappa \in T_h} \eta_{\kappa},
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<td>-3.612e-03</td>
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</tr>
</tbody>
</table>

Similar for \( c_d \) and \( c_m \).
Local refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$. 

residual-based refinement

5 771 160 DoFs, $c_l$: $|\text{error}| = 3.2\times 10^{-3}$

adjoint-based refinement

2 462 680 DoFs, $c_l$: $|\text{error}| = 3.6\times 10^{-3}$
Summary

2d and 3d laminar compressible flows

- Error estimation and goal-oriented (adjoint-based) refinement for single and for multiple aerodynamic force coefficients
- Residual-based mesh refinement for 3d laminar flows
- Error estimation and goal-oriented mesh refinement for 3d laminar flows
Outlook

Extension to turbulent flows

Example: L1T2 three element airfoil (high lift configuration)

\[ M = 0.197, \alpha = 20.18^\circ, Re = 3.52 \cdot 10^6, \text{4th order DG discretization} \]
Outlook

Extension to complex test cases

Example: DLR-F6 wing/body configuration without fairing
Geometry used in DPW II (the second drag prediction workshop)
Institute of Aerodynamics and Flow Technology, DLR, Braunschweig

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Thank you.