A JOINT TEST STATISTIC CONSIDERING COMPLEX WISHART DISTRIBUTION: CHARACTERIZATION OF TEMPORAL POLARIMETRIC DATA

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ABSTRACT:

Polarimetric data of distributed scatterers can be fully characterized by the (3×3) Hermitian positive definite matrix which follows a complex Wishart distribution under Gaussian assumption. A second observation in time will also follow Wishart distribution. Then, these observations are correlated or uncorrelated process over time related to the monitored objects. To not to make any assumption concerning their independence, the (6×6) matrix which is also modeled as a complex Wishart distribution is used in this study to characterize the behavior of the temporal polarimetric data. According to the complex density function of (6×6) matrix, the joint statistics of two polarimetric observation is extracted. The results obtained in terms of the joint and the marginal distributions of Wishart process are based on the explicit closed-form expressions that can be used in pdf (probability density function) based statistical analysis. Especially, these statistical analysis can be a key parameter in target detection, change detection and SAR sequence tracking problem. As demonstrated the bias of the joint distribution can decrease with noise free signal and with increasing the canonical correlation parameter, number of looks and number of acquired SAR images. The results of this work are analyzed by means of simulated data.

1 INTRODUCTION

In this paper, the joint and the marginal statistics of a temporal polarimetric data (correlated complex Wishart process over time) is studied. There appears to be very little published work in the context of polarimetric data, although (Martinez et al., 2005) contains similar statistical analysis, focusing on only one polarimetric data rather than multi-temporal data set.

The polarimetric SAR measures the amplitude and phase of scattered signals in combination of the linear receive and transmit polarizations. This signals from the complex scattering mechanism are related to the incident and scattered Jones vectors. Using a straightforward lexicographic ordering of the scattering matrix elements, a complex target vector $k = [S_{hh}\ S_{hv}\ S_{vv}]^T$ is obtained for backscattering case[1], and it can be modeled as a multivariate complex Gaussian pdf $\mathcal{N}^C(0, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = \mathcal{E}\{kk^\dagger\}$. The inherent speckle in SAR data can be decreased by independent (uncorrelated pixels) averaging techniques with the cost of decreasing resolution. In this so called multilook case, $\langle \mathbf{kk}^\dagger \rangle$ follows a complex Wishart pdf $\mathcal{W}^C(n, \mathbf{\Sigma})$ (Laurent et al., 2001) with the degrees of freedom n and covariance matrix $\mathbf{\Sigma}$ where $\dagger$ indicates the conjugate transpose operator. The components of covariance matrix contains all scattering matrix elements as

$$\langle \mathbf{\Sigma} \rangle = \begin{bmatrix} \langle S_{hh}S_{hh}^\dagger \rangle & \langle S_{hh}S_{hv}^\dagger \rangle & \langle S_{hh}S_{vv}^\dagger \rangle \\ \langle S_{hv}S_{hh}^\dagger \rangle & \langle S_{hv}S_{hv}^\dagger \rangle & \langle S_{hv}S_{vv}^\dagger \rangle \\ \langle S_{vv}S_{hh}^\dagger \rangle & \langle S_{vv}S_{hv}^\dagger \rangle & \langle S_{vv}S_{vv}^\dagger \rangle \end{bmatrix}, \quad (1)$$

and the decomposition theorem of covariance matrix allows to create a set of orthonormal (independent) scattering mechanisms, whereas the corresponding eigenvalue express the individual contribution of decomposed scattering mechanism as follows

$$\Sigma = U \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} U^\dagger \rightarrow \begin{cases} U = [e_1, e_2, e_3], \text{unit eigenvectors} \\ A = \sum_{i=1}^{3} \lambda_i (e_i.e_i^\dagger). \end{cases} \quad (2)$$

Considering the potential of target decomposition (TD) in polarimetric parameter estimation, the joint and marginal distribution of eigenvalues of target vectors are discussed in coming sections.

2 CHARACTERIZATION OF TEMPORAL DATA SET

2.1 Derivation of the joint density of two matrices

To be precise, with the same notations in (Laurent et al., 2001), let $\mathbf{w} = \begin{bmatrix} \mathbf{k}_1 & \mathbf{k}_2 \end{bmatrix}^T$ be a complex target vector distributed as multivariate complex Gaussian that consist of two target vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ obtained from temporal images at time t_1 and t_2. The joint statistics $\mathbf{A} = \frac{1}{n} \sum_{j=1}^{n} \mathbf{w}_j \mathbf{w}_j^\dagger$ has a complex Wishart distribution with n degrees of freedom. Here, q represents the number of elements in one of the target vector $\mathbf{k}$, and the vector $\mathbf{w}$ has the dimension of $p = 2q$. The n look covariance matrix $\mathbf{A}$ summaries whole (joint and marginal) information from both images. If $\mathbf{A}$ is partioned as $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, the conditioned on $\mathbf{A}_{11}$, the joint density of element $\mathbf{A}_{22}$ follows the complex Wishart distribution $\mathbf{A}_{11.2} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \sim p(\mathbf{A}_{11}|\mathbf{A}_{22}) = \mathcal{W}_q^C(n-q, \mathbf{\Sigma}_{11.2})$ (Laurent et al., 2001), and it is independent from $\mathbf{A}_{12}$ and $\mathbf{A}_{22}$. Then, using the well known rule that the conditional distribution of correlation matrix $\mathbf{A}_{12}$ given $\mathbf{A}_{22}$ is a complex normal distribution $p(\mathbf{A}_{12}|\mathbf{A}_{22}) \sim \mathcal{N}_{q\times q}^C(\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{12}^{-1}\mathbf{A}_{22}, \mathbf{\Sigma}_{11.2} \otimes \mathbf{A}_{22})$ where $\otimes$ indicates Kroneker products and the *theorem 10.3.2* in (Muirhead, 1982), the conditional distribution of $\mathbf{R}^2 = \mathbf{A}_{12}\mathbf{A}_{11}^{-1}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$ on $\mathbf{A}_{22}$ $\left(p(\mathbf{R}^2|\mathbf{A}_{22})\right)$ is a noncentral Wishart distribution. Since $p\left(\mathbf{A}_{11.2}, \mathbf{A}_{22}, \mathbf{R}^2\right) = p\left(\mathbf{A}_{11.2}\right) p\left(\mathbf{R}^2|\mathbf{A}_{22}\right) p\left(\mathbf{A}_{22}\right)$,

[1] without considering the constants for the power conservation when changing from the 4 dimensional to the 3 dimension $\mathbf{k}$ polarimetric acquisition vector

after transforming $\mathbf{A}_{11.2}$ into $\mathbf{A}_{11}(\mathbf{I}-\mathbf{R}^2)$, the joint density may be given in its final form[2]

$$\begin{aligned} p\left(\mathbf{A}_{11}, \mathbf{A}_{22}, \mathbf{R}^2\right) \;\; = \;\; & {}_0\tilde{F}_1\left(q, \frac{\mathbf{P}^2 n^2 \mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{11}^{-1}}{\mathbf{I}-\mathbf{P}^2}\frac{\mathbf{A}_{11}\mathbf{A}_{22}\mathbf{R}^2}{\mathbf{I}-\mathbf{P}^2}\right) \\ & etr\left(-n\frac{\mathbf{\Sigma}_{22}^{-1}\mathbf{A}_{22}+\mathbf{\Sigma}_{11}^{-1}\mathbf{A}_{11}}{\mathbf{I}-\mathbf{P}^2}\right) \\ & \frac{n^{pn}|\mathbf{I}-\mathbf{R}^2|^{n-p}|\mathbf{A}_{11}\mathbf{A}_{22}|^{n-q}}{|\mathbf{\Sigma}|^n\tilde{\Gamma}_q(n-q)\tilde{\Gamma}_q(n)\tilde{\Gamma}_q(q)}. \end{aligned} \tag{3}$$

Here, $\mathbf{P}^2 = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}$, ${}_0\tilde{F}_1$ is the complex hypergeometric function of matrix argument, and $\tilde{\Gamma}_q(n)$ is a complex gamma function

$$\tilde{\Gamma}_q(n) = \pi^{q(q-1)/2}\prod_{i=1}^{q}\Gamma(n-i+1). \tag{4}$$

It is clear that (3) is valid for $\mathbf{0} < \mathbf{P}^2 < \mathbf{I}$, which means that both $|\mathbf{P}|$ and $|\mathbf{I}-\mathbf{P}^2|$ are positive definitive. When $\mathbf{\Sigma}_{12} = 0$, then $\mathbf{A}_{11}$, $\mathbf{A}_{22}$ and $\mathbf{R}$ are independent. As shown in (3) and well-known from SAR literature, unbiased characterization of temporal data is related to unbiased estimate of coherence ($\mathbf{P} \leftrightarrow \mathbf{R}$) and speckle free data ($\mathbf{\Sigma}_{11} \leftrightarrow \mathbf{A}_{11}$).

2.2 Derivation of the joint density of temporal eigenvalues

While statistical aspects concerning Wishart matrices have been well developed, there seem to be little work on the eigenvalue statistics of correlated Wishart process over time. Although in (Smith and Grath, 2007) and (Kuo et al., 2007) the joint density of the eigenvalues of correlated Wishart has been derived, both analysis has been performed based on the assumption that covariance matrices ($\mathbf{\Sigma}_{11}$) are unitary and the correlation between random complex variants are the same ($\mathbf{P} = p\mathbf{I}$). However, this is a too restrictive assumption for the polarimetric case, since in a general scattering scenario the covariance matrix of polarimetric data is no more unitary, and each polarimetric channel has arbitrary correlations $\left(\mathbf{P} = \mathbf{\Sigma}_{11}^{-1/2}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1/2}\right)$. Due to this missing analysis, the joint distribution of eigenvalues of temporal polarimetric images is derived in this section. The main aim of the analysis of the joint eigenvalue distribution is to study the temporal behavior of different scattering mechanism. In addition to characterization of different scattering mechanism, investigation of the dimension reduction is the second objective of this analysis.

To obtain the joint distribution of eigenvalues of correlated Wishart matrices, firstly the correlation parameter $\mathbf{R}$ must be integrated from (3). Accordingly, applying the *theorem 7.2.10* in (Muirhead, 1982) to (3) follows that

$$\begin{aligned} p\left(\mathbf{A}_{11}, \mathbf{A}_{22}\right) \;\; = \;\; & {}_0\tilde{F}_1\left(n, \frac{\mathbf{n}^2 P^2\mathbf{\Sigma}_{11}^{-1}\mathbf{A}_{11}}{\mathbf{I}-\mathbf{P}^2}\frac{\mathbf{\Sigma}_{22}^{-1}\mathbf{A}_{22}}{\mathbf{I}-\mathbf{P}^2}\right) \\ & etr\left(-n\mathbf{\Sigma}_{22.1}^{-1}\mathbf{A}_{22}\right) etr\left(-n\mathbf{\Sigma}_{11.2}^{-1}\mathbf{A}_{11}\right) \\ & \frac{n^{pn}|\mathbf{I}-\mathbf{P}^2|^{nq}|\mathbf{A}_{11}|^{n-q}|\mathbf{A}_{22}|^{n-q}}{|\mathbf{\Sigma}_{11.2}|^n|\mathbf{\Sigma}_{22.1}|^n\tilde{\Gamma}_q(n)\tilde{\Gamma}_q(n)}. \end{aligned} \tag{5}$$

Then, making the transformations $\mathbf{A}_{11} = \mathbf{U}_1\mathbf{W}_1\mathbf{U}_1^{\dagger}$, $\mathbf{A}_{22} = \mathbf{U}_2\mathbf{W}_2\mathbf{U}_2^{\dagger}$ and integrating (5) with respect to $\mathrm{d}\mathbf{U}_1$ and $\mathrm{d}\mathbf{U}_2$ over the orthogonal group $O(q)$, the joint distributions can be obtained via following theorem,

Theorem 1 (James, A.T., 1964): If $p(A)(\mathrm{d}A)$ is the pdf of a Hermitian complex matrix variate A, then the distribution of the diagonal matrix W of the latent roots of A, $A = UWU^{\dagger}$, is

$$\int_{U(m)} p(UWU^{\dagger})(\mathrm{d}U)\frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(m)}\prod_{i<j}^{m}(w_i-w_j)^2\mathrm{d}w_1....\mathrm{d}w_m. \tag{6}$$

Here, it is important to note that after applying the *theorem 1* into (5), the matrix $\mathbf{P}$ still remain in the joint eigenvalue distribution. However, it is difficult to foresee the behavior of the density function or to understand how the eigenvalues interact with each other in the presence of the matrix $\mathbf{P}$. It makes sense to make the change of variables $\mathbf{L}_1 = \mathbf{H}\mathbf{\Sigma}_{11}^{-1/2}$ and $\mathbf{L}_2 = \mathbf{Q}\mathbf{\Sigma}_{22}^{-1/2}$ with Jacobians $|\mathbf{\Sigma}_{11}|^q$ and $|\mathbf{\Sigma}_{22}|^q$ to make the matrix $\mathbf{P}^2$ diagonal. It turns out that

$$\tilde{P} = \begin{bmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_q \end{bmatrix} \qquad \begin{matrix} P = \mathbf{\Sigma}_{11}^{-1/2}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1/2} = \mathbf{H}^{\dagger}\tilde{\mathbf{P}}\mathbf{Q} \\ \mathbf{H} \in O(q), \mathbf{Q} \in O(q) \\ O : \text{orthonormal group} \end{matrix} \tag{7}$$

where $\tilde{\mathbf{P}}^2$ is a diagonal matrix consisting of square of canonical correlation coefficients ($1 > \rho_1^2 > \ ...\ > \rho_q^2 > 0$). For the detailed analysis about canonical correlation coefficients, we refer to (Muirhead, 1982)[3].

Considering (6) and (7) into (5), $p(\lambda_1', \lambda_2', \lambda_3', \lambda_1'', \lambda_2'', \lambda_3'') = p(W_1, W_2)$ results

$$\begin{aligned} & \frac{\prod_{i<j}^{q}\{(\lambda_i'-\lambda_j')^2(\lambda_i''-\lambda_j'')^2\}}{\pi^{q(1-q)}\tilde{\Gamma}_q(n)^2\prod_i^q(1-\rho_i^2)^n} \times \exp\left(-\sum_{i=1}^{q}\frac{n\lambda_i'}{l_i'(1-\rho_i^2)} + \frac{n\lambda_i''}{l_i''(1-\rho_i^2)}\right) \\ & \frac{\prod_i^q\left(\frac{\lambda_i'\lambda_i''}{l_i'l_i''}\right)^{n-q}}{\prod_{i<j}^{q}\{({l'}_i^{-1}-{l'}_j^{-1})^2({l''}_i^{-1}-{l''}_j^{-1})^2\}}\int_{U(q)} {}_0\tilde{F}_1\left(n, \Xi\tilde{U}\Upsilon\tilde{U}^{\dagger}\right)\mathrm{d}\tilde{U} \end{aligned} \tag{8}$$

where Ξ and Υ indicate the matrix parameters of hypergeometric function in (5) and λ_i', λ_i'', l_i' and l_i'' for $i = 1, ..., q$ denote the eigenvalues of $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, $\mathbf{\Sigma}_{11}$ and $\mathbf{\Sigma}_{22}$ respectively. Here, it can be noted that $\tilde{U} = U_1^{\dagger}U_2$ and $(\mathrm{d}\tilde{\mathbf{U}}) = (\mathrm{d}\mathbf{U}_1)$ over $\tilde{U} \in O(q)$.

Using the relation (James, A.T., 1964)

$$\int_{U(n)} {}_p\tilde{F}_q\left(\Xi\tilde{U}\Upsilon\tilde{U}^{\dagger}\right)\mathrm{d}\tilde{U} = {}_p\tilde{F}_q(\Xi, \Upsilon), \tag{9}$$

the integration part in (8) has been solved. Despite the joint distribution has been derived, it is expressed in terms of an infinite series (hypergeometric functions) that makes the analysis of the distribution hard. However, hypergeometric functions of matrix arguments can be expressed in terms of the matrix eigenvalues using *Zonal Polynomials* (Muirhead, 1982). For the specific case of (10), ${}_0\tilde{F}_1(.,.)$, the closed form of the hypergeometric fuction exits, and it is given by (Gross and Richards, 1989)

$${}_0\tilde{F}_1(n, s, t) = \frac{\tilde{\Gamma}_q(n)\tilde{\Gamma}_q(q)}{\pi^{q(q-1)}}\prod_{k=1}^{q} t_k^{(n-q)/2}\frac{|s_i^{(q-n)/2}I_{n-q}(2\sqrt{s_i t_j})|}{\prod_{i<j}^{q}(s_i-s_j)\prod_{i<j}^{q}(t_i-t_j)}. \tag{10}$$

Related to this expression, (8) can be solved without the need of zonal polynomials.

[2]The proof of this distribution for the real case can be found in (Muirhead, 1982) and (Lliopoulos, 2006).

[3]In (Muirhead, 1982), only real differential forms have been considered. However, the theorems have been extended to the complex case after some algebra, *e.g.* $(dz) = \mathrm{d}\mathcal{R}\{z\}\Lambda\mathrm{d}\mathcal{I}\{z\}$ where Λ, $\mathcal{R}$ and $\mathcal{I}$ indicate exterior products, real and imaginary parts of the variates respectively.

Consequently, denoting the eigenvalues of $\mathbf{A}_{11} = \langle \mathbf{k}_1\mathbf{k}_1^\dagger \rangle_n$ and $\mathbf{A}_{22} = \langle \mathbf{k}_2\mathbf{k}_2^\dagger \rangle_n$ by $\lambda_1', \lambda_2', \lambda_3'$ and $\lambda_1'', \lambda_2'', \lambda_3''$, respectively, using the *equation 92* in (James, A.T., 1964), the joint density of eigenvalues can be obtained as in the following

$$p(\lambda_1', \lambda_2', \lambda_3', \lambda_1'', \lambda_2'', \lambda_3'') = {}_0\tilde{F}_1\left(n, \frac{n\lambda_i' p_i}{l_i'(1-p_i^2)}, \frac{n\lambda_j'' p_j}{l_j''(1-p_j^2)}\right) \times \frac{\Delta(\mathbf{A}_{11})\Delta(\mathbf{A}_{22})}{\Delta(\mathbf{\Sigma}_{11})\Delta(\mathbf{\Sigma}_{22})\tilde{\Gamma}_q(n)^2\prod_i^q(1-p_i^2)^n} \times \prod_i^q\left(\frac{\lambda_i'\lambda_i''}{l_i' l_i''}\right)^{n-q} \times \exp\left(-\sum_{i=1}^q \frac{n\lambda_i'}{l_i'(1-p_i^2)} + \frac{n\lambda_i''}{l_i''(1-p_i^2)}\right) \quad (11)$$

where $\Delta_m(.)$ indicate Vandermonde determinants of matrices

$$\left.\Delta_q(\mathbf{\Sigma}) = \prod_{i<j}^{q}(\sigma_j - \sigma_i)\right\} \rightarrow \Delta_q(\mathbf{\Sigma}) = \Delta_q(diag(\mathbf{\Sigma})) \quad (12)$$

and $I_n(x)$ is the modified Bessel function of the first kind of order n.

2.3 Derivation of the joint density of the maximum eigenvalues from temporal images

In general, as in (Conradsen2003 et al., 2003), the temporal analysis based on the likelihood ratio test is performed by testing the null hypothesis that all the latent roots of $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are equal. If this hypothesis is accepted it can be concluded that all the scattering mechanism have same variance over time and hence contribute equally to the total change. It means also that there is no need to perform TD with the aim of dimension reduction. However, in practice, it is reasonable to consider the null hypothesis that deals with the comparison between individual eigenvalues (related to some specific scattering mechanism from different images) rather than all eigenvalues at once.

Therefore, to analyze the variation of maximum eigenvalue which is related to the dominant scattering mechanism after some time, the joint pdf of $p(\lambda_1', \lambda_1'')$ is required. To compute $p(\lambda_1', \lambda_1'')$, (λ_2', λ_3') and $(\lambda_2'', \lambda_3'')$ must be integrated out from (11)[4]

$$p(\lambda_1', \lambda_1'') = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty p(\lambda_1', \lambda_2', \lambda_3', \lambda_1'', \lambda_2'', \lambda_3'')\, \mathrm{d}\lambda_2'\mathrm{d}\lambda_2''\mathrm{d}\lambda_3'\mathrm{d}\lambda_3''. \quad (13)$$

In addition, the probability density function of the ratio of the joint density to the marginal density

$$\frac{p(\lambda_1', \lambda_2', \lambda_3', \lambda_1'', \lambda_2'', \lambda_3'')}{p(\lambda_1', \lambda_1'')} \quad (14)$$

can be used to analyze the contribution of specific scattering mechanism in respect to the whole temporal scattering mechanism.

The whole procedure explained above can be performed even for systems with larger dimensions. However, for large multidimensional systems, the large number of integration process related to the number of eigenvalues may become complicated.

2.4 Derivation of the marginal density of the maximum eigenvalues

The last statistical analysis is performed with the aim of characterizing the density of the maximum eigenvalue from a single polarimetric acquisition (complex Wishart distribution). In (Martinez et al., 2005), the same analyze has been performed by numerical integration. Here, the closed form expression of the pdf is given using the *teorem 2* in (McKay and et. al, 2007).

Theorem 2 (McKay and et. al, 2007): Let $\mathbf{X} \sim \mathcal{N}^C(0_{n\times q}, \Sigma \otimes \Omega)$, where $q \leq n$, and $\Omega \in C^{q\times q}$ and $\Sigma \in C^{n\times n}$ are Hermitian positive-definite matrices with eigenvalues $w_1 < ...... < w_q$ and $\sigma_1 < ...... < \sigma_n$, respectively. Then the pdf of the maximum eigenvalue λ_{max} of the complex Wishart matrix $\mathbf{X}^\dagger\mathbf{X}$ is given by

$$f_{\lambda_{max}}(x) = \frac{(-1)^{q+1}\tilde{\Gamma}_q(q)|\Omega|^{(q-1)}|\Sigma|^{n-1}}{\pi^{q(q-1)/2}\Delta_q(\Omega)\Delta_q(\Sigma)(-x)^{q(q-1)/2}} \times \left(\frac{q(q-1)|\Psi(x)|}{2x} + \sum_{l=t+1}^{n}|\Psi_l(x)|\right) \quad (15)$$

where $\Psi(x)$ is a $n \times n$ matrix with (i,j)th element

$$(\Psi(x))_{i,j} = \begin{cases} \left(\tilde{\Psi}(x)_{i,j}\right)^{(n-i)}, & i \neq l, \quad t = n-q \\ \frac{exp(-\frac{x}{w_{i-t}\sigma_j})}{w_{i-t}\sigma_j}\mathcal{P}(n, \frac{-x}{w_{i-t}\sigma_j}), & i = l \end{cases} \quad (16)$$

and where

$$\left(\tilde{\Psi}(x)\right)_{i,j} = \begin{cases} \left(\frac{1}{\sigma_j}\right)^{(n-i)}, & i \leq t, \quad t = n-q \\ exp(-\frac{x}{w_{i-t}\sigma_j})\mathcal{P}(n, \frac{-x}{w_{i-t}\sigma_j}), & i > t \end{cases} \quad (17)$$

and $\mathcal{P}(l, y) = 1 - exp(-y)\sum_{k=0}^{(l-1)}\frac{y^k}{k!}$. An alternative derivation of the maximum eigenvalue pdf has been performed for the polarimetric case and will be published in a future work. Figure 1 shows the derived pdf as a function of n and λ_{max}. As seen from Figure 1, the number of look and the true value of the eigenvalues are key parameters in the marginal distribution of the maximum eigenvalue.

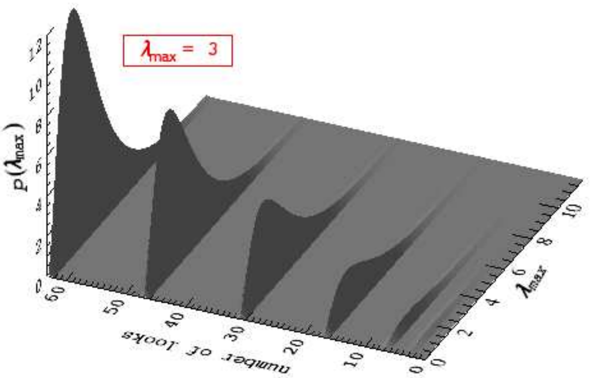

Figure 1: Distribution of the maximum eigenvalue as a function of the number of looks for $l_1 = 3$, $l_2 = 2$, $l_1 = 1$. When $n \rightarrow \infty$, $\lambda_{max} = l_1 = 3$.

3 A STATISTICAL TEST WITH APPLICATIONS

In previous sections, to investigate the temporal behavior of polarimetric data, the joint and the marginal density functions of eigenvalues from temporal images were derived in the context of target decomposition theorem. In this section, previous results are discussed considering potential applications.

[4] In (Smith and Grath, 2007), a similar analysis has been performed to test the MIMO (Multiple Input Multiple Output) channel transitions probability.

Application I: The determinant of the covariance matrix is the generalized variance of polarimetric data, and the ratio of two determinants is an important parameter in applications as in edge detection (Skriver et al., 2001), change detection (Conradsen2003 et al., 2003) and SAR image tracking (Erten et al., 2008). An unbiased estimator of the ratio of two covariance matrices determinants $\frac{|\mathbf{\Sigma}_{11}|}{|\mathbf{\Sigma}_{22}|}$ is given by $\frac{|\mathbf{A}_{11}|}{|\mathbf{A}_{22}|}$. The expectation of $\frac{|\mathbf{A}_{11}|}{|\mathbf{A}_{22}|}$ is derived from (3) as a function of the canonical correlation coefficients $r = [\rho_1\ \rho_2\ \rho_3]^T$. For that, a same procedure as in (Lliopoulos, 2006) is applied into (3), and the unbiased estimate of $\frac{|\mathbf{\Sigma}_{11}|}{|\mathbf{\Sigma}_{22}|}$ for $n \geq 2q$ results

$$\frac{|\mathbf{A}_{11}|(n-p)(n-q)^{-1}}{|\mathbf{A}_{22}|(n-q-1)} \left(2(n-p-1) + \sum_i r_i^2 + \frac{\sum_{i<j} r_i^2 r_j^2}{n-2q} \right) \tag{18}$$

To obtain (18) from (3), integrations over $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are necessary. These integrations are only valid if the variance of observations are finite, or in other words, the condition of $|\mathbf{A}_{11}| \leq \infty$ and $|\mathbf{A}_{22}| \leq \infty$ which are always satisfied in the polarimetric case. Figure 2 presents the evaluation of the unbiased estimate of $\frac{|\mathbf{\Sigma}_{11}|}{|\mathbf{\Sigma}_{22}|}$ related to the number of looks n and the canonical correlation coefficients r. It is clear that the generalized variance ratio is asymptotically unbiased for a large number of looks and the estimation tends to the true determinants ratio if the data set are highly correlated.

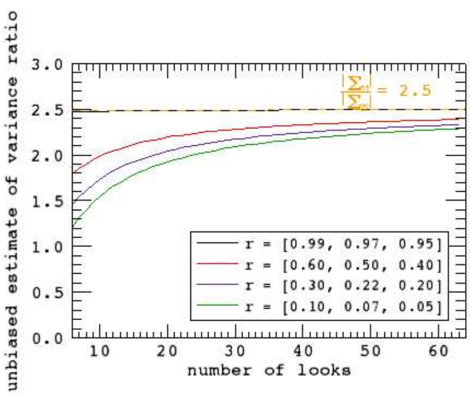

Figure 2: The expected value of the polarimetric variance ratio estimator with $r = [\rho_1\ \rho_2\ \rho_3]^T$

Application II: As indicated in (Martinez et al., 2005), the probability of the maximum eigenvalue related to the dominant scattering is a important parameter for target detection and its analysis. The probability of the detection that there is just one dominant scattering mechanism is a function of the detection threshold $\mathcal{T}$, that can be obtained from (15) as

$$F_{\lambda_{max}}(\mathcal{T}) = p(\lambda_{max} \leq \mathcal{T}) = \int_0^{\mathcal{T}} p_{\lambda_{max}}(x)\mathrm{d}x \tag{19}$$

where $F_X(x)$ indicates the cumulative density function. In addition, having the closed form of the eigenvalue pdf (15), the probability of false alarm can be also computed, as well as receiver operating characteristic (ROC) curves, allowing a complete detection problem analysis.

Application III: Another application of (3) has been detailed explained in (Erten et al., 2008) with the aim of multidimensional SAR tracking. The joint distribution of polarimetric covariance matrices over time has been used to perform maximum likelihood tracking into amplitude data without making any assumption of their independence.

4 CONCLUSIONS AND FUTURE WORK

The statistical description of two (possible) correlated Wishart distributions has been presented. Closed forms for the general distribution have been derived, as well as for the joint distributions of the eigenvalues of the two Wishart matrices and for the joint distribution of their maximum eigenvalue. This analysis can be applied to a wide field of aplications, whenever the application in question follows the statistical assumptions. Examples of applications have been given in the paper, as the assessment of different aspects in polrimetric statistical analysis over time. It has been showed that the performance of analysis may increase with high canonical correlation coefficients, the number of look and number of polarimetric channel or may decrease due to the presence of speckle that effects the calculation of true values of the parameters.

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