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Homogenized Equations of Motion for Rod Bundles in Fluid with Periodic Structure

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Summary: "Homogenized" or averaged equations of motion are deduced for linear dynamic fluid-structure interactions of rod bundles immersed in an acoustical fluid. The equations define an effective density tensor which couples the fluid and rod accelerations. In the pressure wave equation a sound speed tensor arises. The theory assumes that the bundle consists of a periodic lattice of cells with diameters which are very small in comparison to the bundle diameter and that cell averages are smooth functions in space and time. The derivation is based on Hamilton's principle. For the specific case of circular cylindrical rods in a square pattern the tensors are given numerically and the fluid-structure interaction effects are discussed.

Übersicht: Es werden „homogenisierte" oder gemittelte Bewegungs-Gleichungen für lineare, dynamische Fluid-Struktur-Wechselwirkungen von Stabbündeln in einem akustischen Fluid abgeleitet. Die Gleichungen definieren einen Tensor der effektiven Dichten, der die Fluid- und Stabbeschleunigungen koppelt. In der Wellengleichung für das Druckfeld tritt ein Schallgeschwindigkeits-Tensor auf. Die Theorie unterstellt, daß das Bündel aus einem periodischen Gitter von Zellen besteht, deren Abmaße sehr klein sind im Vergleich zum Bündeldurchmesser und daß Zellen-Mittelwerte glatte Funktionen in Raum und Zeit sind. Die Ableitungen gehen aus vom Hamiltonschen Prinzip. Für den Sonderfall von kreisförmigen, zylindrischen Stäben in quadratischer Anordnung werden die Tensoren zahlenmäßig angegeben und die Effekte der Fluid-Struktur-Wechselwirkungen diskutiert.

1 Introduction

The problem of dynamic fluid-structure interactions in large rod bundles arises, e.g., in nuclear reactor safety analysis [1]. Here one has to establish computer models to analyse the forces on the internal structures due to pressure waves. As yet, a model for the motions in the reactor core itself is missing which is defined in terms of not too many parameters so that it can be incorporated in a more general code like those described in [1, 2]. There exist analytical models [3] which describe the motion of some rods in incompressible potential flow. Such models as well as similar numerical approaches [4] are applicable for a limited number of rods but not for some 50000 rods, typical for present pressurized water reactors.

Therefore, we are looking for a set of "homogenized" partial differential equations which describe appropriate smooth mean values of the fluid and structural motion such that these equations can be solved numerically with modest amount of discretization parameters. Thus we consider the rod bundle in fluid like a porous medium or like a two-phase flow.

Reviews on homogenization are given in [5—9]. Asymptotic analysis tools for periodic structures in continuous domains are described by Bensoussan et al. [8] and have been applied by Ohayon [9] to harmonic vibrations in heterogeneous elastic bodies. Cioranescu & Paulin [10, 11] and Berdichevskii [7] have treated elastic bodies with periodic holes or incompressible flow (both inviscid and viscid) around periodically distributed fixed cavities. The present approach follows the arguments of Berdichevskii [7] but extends to the case of interacting rods and fluid and compressible flow.

The main assumptions, which are employed in the subsequent theory, are

a) the configuration of fluid and structure (rods) forms a periodic lattice of quadrilateral cells with length of periodicity of order ε which is very small in comparison to the "diameter" D of the whole domain taken by the bundle.

b) the applied forces within the domain or at the boundary and initial values, averaged over individual cells, are smooth functions of space and time.

c) the fluid behaves locally like an "acoustic fluid", which means, the flow velocities are very small in comparison to the speed of sound and so that convective accelerations are negligible; the flow is inviscid and the speed of sound is a constant.

d) the sturcture behaves locally like "rods", which means the local deflections are approximately constant across the cross section.

Formal statements of these assumptions are given below.

In the present paper, a general theory is given to construct the homogenized equations for a rod bundle with arbitrary periodic cell geometry. As a result effective densities and sound speeds arise. These quantities are then evaluated and discussed for the special case of a bundle composed of circular cylindrical rods in a square arrangement.

The main idea in the construction of the homogenized equations is the following. We write down (in Section 2) the local equations and show (in Section 3) that an equivalent variational formulation exists which is basically Hamilton's principle [12]. We then specify (in Section 4) the solutions in terms of trial functions which are the product (Berdichevskii [7] used sums) of periodic local functions and smooth global functions. Equations for the local functions are obtained by extremizing the variational functional for fixed global functions. Thereafter (in Section 6) the global functions follow from the variational principle for fixed local functions. An important aspect is the introduction of the averaged pressure as a Lagrangian multiplier [12] in order to account for the cell averaged mass conservation law (deduced in Section 5).

2 Statement of the Problem

The configuration consists of a closed domain V in R^3 with boundary ∂V and diameter D. Let V be composed of rectangular cells $V_{\mathbf{m}}$, see Fig. 1, with side lengths ε^1, ε^2, ε^3 along each of the Cartesian coordinates x_i $(i = 1, 2, 3)$, $\varepsilon = \max(\varepsilon^1, \varepsilon^2, \varepsilon^3) \ll D$, $\min(\varepsilon^1, \varepsilon^2, \varepsilon^3) > 0$. The cells are numbered with the integer vector $\mathbf{m} = (m_1, m_2, m_3)$. The volume V is filled out by a fluid and a bundle of rods with the rod axis parallel to the x_3 coordinate. Each cell contains a section $S_{\mathbf{m}}$ of one or some rods such that the cells form a periodic lattice. The rod section $S_{\mathbf{m}}$ is surrounded by the fluid domain $F_{\mathbf{m}} = V_{\mathbf{m}} - S_{\mathbf{m}}$ in the cell. The whole fluid domain $F = \sum_{\mathbf{m}} F_{\mathbf{m}}$ is multiply connected. The volume porosity is

$$\alpha := |F_{\mathbf{m}}|/|V_{\mathbf{m}}|, \qquad 0 < \alpha < 1 . \tag{1}$$

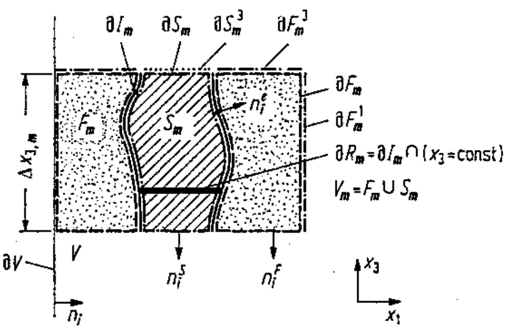

Fig. 1. Domain notations

Let $\partial V_{\mathbf{m}}^i$ be the cell face with outward unit normal in the direction of the coordinate x_i and $\partial F_{\mathbf{m}}^i$ the corresponding fluid part $(i = 1, 2, 3)$. Then we define the surface permeability

$$\gamma^i := |\partial F_{\mathbf{m}}^i| / |\partial V_{\mathbf{m}}^i| , \qquad 0 < \gamma^i \le 1 , \tag{2}$$

and the surface permeability tensor

$$\gamma_{ij} := \gamma^i \delta_{ij} , \qquad (i = 1, 2, 3, j = 1, 2, 3) . \tag{3}$$

These quantities are independet of $\mathbf{m}$. The surface of the rod section $S_{\mathbf{m}}$ is $\partial S_{\mathbf{m}}$, that of the fluid part $F_{\mathbf{m}}$ is $\partial F_{\mathbf{m}}$, both have the common interface $\partial I_{\mathbf{m}} = \partial F_{\mathbf{m}} \cap \partial S_{\mathbf{m}}$. Examples of such a configuration are sketched in Fig. 2.

For the fluid we assume the acoustic model which means

- inviscid motion
- constant speed of sound a, $0 \le 1/a < \infty$
- equation of state relating local density ϱ^ε and pressure p^ε in the form

$$\varrho^\varepsilon - \varrho_0 = (p^\varepsilon - p_0)/a^2 , \qquad p_0 = \text{const}, \varrho_0 = \text{const} \tag{4}$$

- small local fluid flow velocity components $u_i^\varepsilon (|u_i^\varepsilon| \ll a)$ which satisfy the momentum equation

$$\varrho_0 \dot{u}_i^\varepsilon = -\partial p^\varepsilon / \partial x_i + \varrho_0 g_i^\varepsilon \tag{5}$$

and mass conservation law or continuity equation

$$\dot{\varrho}^\varepsilon = -\partial(\varrho_0 u_i^\varepsilon)/\partial x_i \tag{6}$$

with given accelerations g_i^ε. All these fields are assumed to be smooth functions in F. We adpot Einstein's summation convention for repeated lower indices and $\dot{y} \equiv \partial y/\partial t$. It is implicitely assumed that the gradient in the pressure head $\varrho_0 (u_i^\varepsilon)^2/2$ is small in comparison to the static pressure gradient so that the equations are linear.

Rods are characterized by the fact that the local deflections $w_i^\varepsilon = w_i^\varepsilon(\mathbf{m}, x_1, x_2, x_3, t)$ are approximately constant across the rod section so that $w_i^\varepsilon = w_i^\varepsilon(\mathbf{m}, x_3, t)$, and satisfy

$$M_{ij}^\varepsilon \ddot{w}_j^\varepsilon + \oint_{\partial R_{\mathbf{m}}} p^\varepsilon n_i^\varepsilon \, ds = f_i^\varepsilon(\mathbf{m}, x_3, t) , \tag{7}$$

where M_{ij}^ε is a periodic function of x_3 with periodicity ε^3, positive definite and symmetric, and M_{ij}^ε measures the effective mass per unit length of a rod. The line integral is taken over the "wetted" circumference in the $x_1 - x_2$-plane $\partial R_{\mathbf{m}} = \partial I_{\mathbf{m}} \cap \{x_3 = \text{const}\}$, and n_i^ε is the unit normal of $\partial I_{\mathbf{m}}$ pointing into the fluid as sketched in Fig. 1. At this stage, the forces f_i^ε per unit length are taken as prescribed. The local rod density ϱ_s^ε is assumed to be related to the surrounding pressure as

$$a_s^2 \iiint_{S_{\mathbf{m}}} (\varrho_s^\varepsilon - \varrho_s^0) \, dV / |\partial S_{\mathbf{m}}| = \iint_{\partial I_{\mathbf{m}}} (p^\varepsilon - p_0) \, dO / |\partial I_{\mathbf{m}}| , \tag{8}$$

$$a_s = \text{const} , \qquad 0 \le 1/a_s < \infty ,$$

where dO is a surface element.

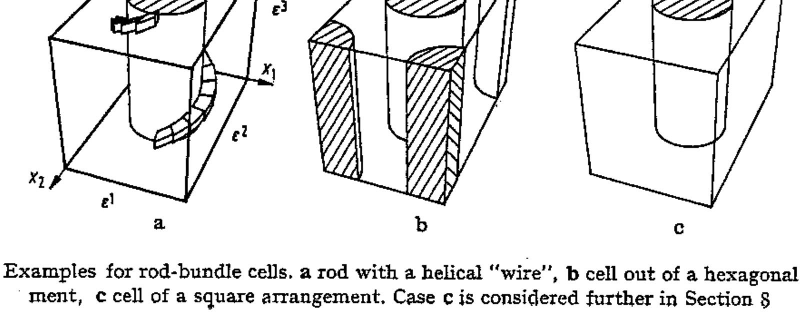

Fig. 2 a—c. Examples for rod-bundle cells. **a** rod with a helical "wire", **b** cell out of a hexagonal rod arrangement, **c** cell of a square arrangement. Case **c** is considered further in Section 8

This relationship is particular useful if the rods are in fact thin walled pipes in order to account for quasi-static compression under pressure load. Some generalizations are discussed in Section 7.

At the fluid-rod interface $\partial I_{\mathbf{m}}$ the normal velocity is continuous,

$$n_i^\varepsilon u_i^\varepsilon = n_i^\varepsilon \dot{w}_i^\varepsilon \,. \tag{9}$$

The outer surface ∂V of the domain with inward unit normal n_i is composed of cuts through the rods, $\partial V_{w^\varepsilon}$, and the fluid, $\partial V_{p^\varepsilon} \cup \partial V_{u^\varepsilon} = \partial V - \partial V_{w^\varepsilon}$. On these surfaces, boundary conditions prescribe either pressure or the normal motions, e.g.,

$$\left.\begin{aligned} p^\varepsilon &= p_0 \quad \text{on} \quad \partial V_{p^\varepsilon}\,, \\ n_i u_i^\varepsilon &= u_n^\varepsilon \quad \text{on} \quad \partial V_{n^\varepsilon}\,, \\ n_i w_i^\varepsilon &= w_n^\varepsilon \quad \text{on} \quad \partial V_{w^\varepsilon}\,. \end{aligned}\right\} \tag{10}$$

We assume that the "forces" g_i^ε and f_i^ε as well as the boundary values u_n^ε, w_n^ε are continuous functions of time and of space in the sense that the differences of their average values taken over their definition domains within two neighbouring cells tend to zero for $\varepsilon/D \to 0$. In the same sense the initial values must be continuous in space. Further, these cell averages must be slowly varying functions of the cell index so that resultant pressure waves have wave-lengths which are very large in comparison to ε and so that the differences between the deflections of two adjacent rods are small in comparison to ε.

3 Variational Formulation, Hamilton's Principle

The problem of finding solutions u_i^ε, w_i^ε which satisfy (5, 6, 7, 9) is equivalent to finding such functions which extremize the functional

$$J := \int_{t_1}^{t_2} \left\{ \iiint_F \left[\frac{\varrho_0}{2} (\dot{v}_i^\varepsilon)^2 + \varrho_0 g_i^\varepsilon v_i^\varepsilon\right] dV + \sum_{\mathbf{m}} \int_{\Delta x_{3,\mathbf{m}}} \left[\frac{1}{2}\, \dot{w}_i^\varepsilon M_{ij}^\varepsilon \dot{w}_j^\varepsilon + f_i^\varepsilon w_i^\varepsilon\right] dx_3 \right\} dt \tag{11}$$

($\dot{v}_j^\varepsilon \equiv u_i^\varepsilon$, $\Delta x_{3,\mathbf{m}}$ = interval which the cell $\mathbf{m}$ covers on the coordinate x_3) with respect to virtual displacements δv_i^ε, δw_i^ε which vanish at $t = t_1, t_2$, which are consistent with the constraints (6, 9) and boundary conditions (10), and which are taken for fixed density ϱ^ε. The integral contains the specific kinetic energy of motion and work done by external forces. The above statement corresponds, therefore, to Hamilton's principle [12].

The proof of this statement is illustrative because its basic arguments will be used later again. The extremum requires $\delta J = 0$:

$$\delta J = \int_{t_1}^{t_2} \left\{ \iiint_F [\varrho_0 \dot{v}_i^\varepsilon \delta\dot{v}_i^\varepsilon + \varrho_0 g_i^\varepsilon\, \delta v_i^\varepsilon]\, dV + \sum_{\mathbf{m}} \int_{\Delta x_{3,\mathbf{m}}} [M_{ij}^\varepsilon \dot{w}_j^\varepsilon\, \delta\dot{w}_i^\varepsilon + f_i^\varepsilon\, \delta w_i^\varepsilon]\, dx_3 \right\} dt\,.$$

Partial integration in time yields

$$\delta J = \int_{t_1}^{t_2} \left\{ \iiint_F [-\varrho_0 \ddot{v}_i^\varepsilon + \varrho_0 g_i^\varepsilon]\, \delta v_i^\varepsilon\, dV + \sum_{\mathbf{m}} \int_{\Delta x_{3,\mathbf{m}}} [-M_{ij}^\varepsilon \ddot{w}_j^\varepsilon + f_i^\varepsilon]\, \delta w_i^\varepsilon\, dx_3 \right\} dt\,.$$

The virtual displacements are not independent but have to satisfy the constraints (6, 9) which require, for fixed density,

$$\partial(\delta v_i^\varepsilon)/\partial x_i = 0\,, \qquad n_i^\varepsilon\, \delta v_i^\varepsilon = n_i^\varepsilon\, \delta w_i^\varepsilon\,.$$

These constraints are taken into account by means of the Lagrangian multiplier [12] $(p^\varepsilon - p_0)$:

$$\delta J' := \int_{t_1}^{t_2} \left\{ \iiint_F (p^\varepsilon - p_0)\, \partial(\delta v_i^\varepsilon)/\partial x_i\, dV + \sum_{\mathbf{m}} \int_{\Delta x_{3,\mathbf{m}}} \oint_{\partial R_{\mathbf{m}}} (p^\varepsilon - p_0)\, n_i^\varepsilon(\delta v_i^\varepsilon - \delta w_i^\varepsilon)\, ds\, dx_3 \right\} dt = 0\,.$$

Partial integration in space gives

$$\delta J' = \int_{t_1}^{t_2} \left\{ \iiint_F -(\partial p^\varepsilon/\partial x_i)\, \delta v_i^\varepsilon\, dV - \sum_{\mathbf{m}} \int_{\Delta x_{3,\mathbf{m}}} \oint_{\partial R_{\mathbf{m}}} p^\varepsilon n_i^\varepsilon\, \delta w_i^\varepsilon\, ds\, dx_3 - \right.$$

$$\left. - \oiint_{\partial V_{p^\varepsilon} \cup \partial V_{u_\varepsilon}} (p^\varepsilon - p_0)\, \delta v_i^\varepsilon n_i\, dO \right\} dt = 0\,.$$

Because of the boundary conditions (10) the integral over the surface ∂V vanishes. The sum of $\delta J'$ and δJ results

$$\delta J = \int_{t_1}^{t_2} \left\{ \iiint_F [-\varrho_0 \dot{u}_i^\varepsilon + \varrho_0 g_i^\varepsilon - \partial p^\varepsilon/\partial x_i^\varepsilon]\, \delta v_i^\varepsilon\, dV + \right.$$

$$\left. + \sum_{\mathbf{m}} \int_{\Delta x_{3,\mathbf{m}}} \left[-M_{ij}^\varepsilon \ddot{w}_j^\varepsilon + f_i^\varepsilon - \oint_{\partial R_{\mathbf{m}}} p^\varepsilon n_i^\varepsilon\, ds \right] \delta w_i^\varepsilon\, dx_3 \right\} dt = 0\,.$$

Obviously we have $\delta J = 0$ if the integrands are zero which is equivalent to (5, 7), q.e.d.

It might be noted, that we have taken virtual displacements for fixed density. This side-condition can be abandoned by including the specific potential energy $\varrho_0 a^2 (\partial v_i^\varepsilon \partial x_i)^2$ in the fluid integral in (11) and using (4). However, the contributions from this term are irrelevant under the condition of long pressure waves which we assumed and would unduly complicate the further analysis.

4 Averages and Local Solutions

Let $\mathbf{x} = (x_1, x_2, x_3) \in F_{\mathbf{m}}$, $x_3 \in \Delta x_{3,\mathbf{m}}$, then we introduce the approximative ansatz

$$u_i^\varepsilon(\mathbf{x}, t) \approx u_i'(\mathbf{x}, t)\,, \qquad w_i^\varepsilon(\mathbf{m}, x_3, t) \approx w_i'(\mathbf{m}, x_3, t)\,, \tag{12}$$

with

$$u_i'(\mathbf{x}, t) = \varphi_{ij}(\mathbf{x})\, \bar{u}_j(\mathbf{m}, t) + \psi_{ij}(\mathbf{x})\, \dot{\bar{w}}_j(\mathbf{m}, t)\,, \qquad w_i'(\mathbf{m}, x_3, t) = \bar{w}_i(\mathbf{m}, t)\,, \tag{13}$$

where we will define φ, $\bar{u}$, ψ, $\bar{w}$ such that suitable averages of $u_i^\varepsilon - u_i$, $w_i^\varepsilon - w_i'$ taken over the cell vanish for $\varepsilon/D \to 0$ and such that $\bar{u}$, $\bar{w}$ are "smear" velocities and deflections. For later reference in the mean continuity equation (Section 5) it is convenient to define cell-face averages: For any quantity y valid in F we define surface averages

$$^i\bar{y} = \iint_{\partial F_{\mathbf{m}}^i} y\, dO / |\partial F_{\mathbf{m}}^i| \tag{14}$$

and require

$$^i\overline{u_i'} = {}^i\overline{\psi_{ij}}\, \bar{u}_j + \overline{\varphi_{ij}}\, \dot{\bar{w}}_j \overset{!}{=} \bar{u}_i\,. \tag{15a}$$

At the fluid-structure interface, (9) implies

$$n_i^\varepsilon u_i' = n_i^\varepsilon \varphi_{ij} \bar{u}_j + n_i^\varepsilon \psi_{ij} \dot{\bar{w}}_j \overset{!}{=} n_i^\varepsilon \dot{w}_i' = n_i^\varepsilon \dot{\bar{w}}_i\,. \tag{15b}$$

From these requirements we obtain the conditions

$$^i\overline{\varphi_{ij}} = \delta_{ij}\,, \qquad ^i\overline{\psi_{ij}} = 0\,, \tag{16a}$$

$$n_i^\varepsilon \varphi_{ij}|_{\partial I_{\mathbf{m}}} = 0\,, \qquad n_i^\varepsilon \psi_{ij}|_{\partial I_{\mathbf{m}}} = n_i^\varepsilon\,. \tag{16b}$$

These conditions serve as boundary conditions to determine φ_{ij} and ψ_{ij}. In addition we require

$$\varphi_{ij} \text{ and } \psi_{ij} \text{ be periodic in } x_k \tag{16c}$$

with length of periodicty equal to ε^k.

Due to this condition, u_i' becomes a continuous function in the fluid domain for $\varepsilon/D \to 0$.

According to the prerequisite of large pressure wave lengths we may assume that the velocity field behaves locally incompressible so that

$$\partial\varphi_{ij}/\partial x_i = 0\,, \qquad \partial\psi_{ij}/\partial x_i = 0\,, \qquad j = 1, 2, 3\,, \qquad \mathbf{x} \in F_{\mathbf{m}}\,. \tag{17}$$

For short waves one would have to introduce extra terms $\varphi_{ij}^{(k)}\bar{u}_j^{(k)}$ in (13) where the $\varphi_{ij}^{(k)}$ are dependent on the wave length or frequency of the pressure waves, see [7].

In terms of the ansatz-functions (13), the functional (11) becomes

$$\bar{J} = \int_{t_1}^{t_2} \sum_{\mathbf{m}} \left\{ \iiint_{F_{\mathbf{m}}} \left[\frac{\varrho_0}{2} (\varphi_{ij}\dot{\bar{v}}_j + \psi_{ij}\dot{\bar{w}}_j)^2 + \varrho_0 g_i^e(\varphi_{ij}\bar{v}_j + \psi_{ij}\bar{w}_j) \right] dV + \right.$$

$$\left. + \int_{\Delta x_{3,\mathbf{m}}} \left[\frac{1}{2}\, \dot{\bar{w}}_i M_{ij} \dot{\bar{w}}_j + f_i^e \bar{w}_i \right] dx_3 \right\} dt\,, \qquad \dot{\bar{v}}_j \equiv \bar{u}_j\,. \tag{18}$$

Now we determine φ_{ij}, ψ_{ij} under the constraints (15, 16, 17) such that they extremize (18) for fixed $\bar{v}_i$, $\bar{w}_i$ and zero forces. Because of the periodicity and the time independence of φ_{ij}, ψ_{ij}, it suffices to consider one fluid cell $F_{\mathbf{m}}$ for any time instant where we have the condition

$$\delta\dot{\bar{J}}_{\mathbf{m}} := \iiint_{F_{\mathbf{m}}} \varrho_0(\varphi_{ij}\dot{\bar{v}}_j + \psi_{ij}\dot{\bar{w}}_j)\,(\delta\varphi_{ik}\dot{\bar{v}}_k + \delta\psi_{ik}\dot{\bar{w}}_k)\,dV = 0\,.$$

The constraints (17) are taken into account by Lagrangian multipliers Φ_j, Ψ_j:

$$\delta\dot{\bar{J}}'_{\mathbf{m}} := \iiint_{F_{\mathbf{m}}} \varrho_0(\Phi_j\dot{\bar{v}}_j + \Psi_j\dot{\bar{w}}_j)\, \frac{\partial}{\partial x_i}\, (\delta\varphi_{ik}\dot{\bar{v}}_k + \delta\psi_{ik}\dot{\bar{w}})\,dV = 0\,.$$

Partial integration in space yields

$$\delta\dot{\bar{J}}'_{\mathbf{m}} = \iiint_{F_{\mathbf{m}}} -\, \varrho_0 \frac{\partial}{\partial x_i} (\Phi_j\dot{\bar{v}}_j + \Psi_j\dot{\bar{w}}_j)\,(\delta\varphi_{ik}\dot{\bar{v}}_k + \delta\psi_{ik}\dot{\bar{w}}_k)\,dV\, -$$

$$- \iint_{\partial I_{\mathbf{m}}} \varrho_0(\Phi_j\dot{\bar{v}}_j + \Psi_j\dot{\bar{w}}_j)\, n_i^s(\delta\varphi_{ik}\dot{\bar{v}}_k + \delta\psi_{ik}\dot{\bar{w}}_k)\,dO\,.$$

Here, the surface integral is zero because φ_{ik} and ψ_{ik} have to satisfy (16b). The sum $\delta\dot{\bar{J}}_{\mathbf{m}} + \delta\dot{\bar{J}}'_{\mathbf{m}}$ vanishes if

$$\varphi_{ij} = \partial\Phi_j/\partial x_i\,, \qquad \psi_{ij} = \partial\Psi_j/\partial x_i\,, \tag{19}$$

and, again because of (17),

$$\partial^2\Phi_j/\partial x_i^2 = 0\,, \qquad \partial^2\Psi_j/\partial x_i^2 = 0\,, \qquad j = 1, 2, 3\,, \qquad \mathbf{x} \in F_{\mathbf{m}}\,. \tag{20}$$

We note that (19) is consistent with irrotational motion but we did not use this assumption. We see that φ_{ij}, ψ_{ij} are determined by solutions of Laplace equations (20) with periodic boundary conditions and prescribed normal gradients at the fluid-rod interface (16). After suitable transformations, see appendix, these solutions can be computed with standard numerical methods.

The local solutions φ_{ij}, ψ_{ij} extremize the functional for zero forces. In this sense they are local eigensolutions of the differential equations with zero eigenvalues and (16a, b) appear as normalization conditions on these eigenfunctions.

5 Averaged Continuity Equations and Equations of State

A prerequisite for the next step, namely the determination of equations for $\bar{u}_i, \bar{w}_j$, is the establishment of a pendant to the local continuity equation (6) in terms of averaged quantities. This averaged continuity equation defines the constraint which will be satisfied by means of the cell averaged pressure. Also we have to establish relations between the averaged pressure and averaged densities from the equation of state (4) and (8).

For smooth fluid fields the local continuity equation (6) is a consequence of the integral mass conservation principle

$$\frac{\partial}{\partial t} \iiint_{F_{\mathbf{m}}} \varrho^\varepsilon \, dV = - \oiint_{\partial F_{\mathbf{m}} - \partial I_{\mathbf{m}}} n_i^F u_i^\varepsilon \varrho^\varepsilon \, dO \tag{21a}$$

(n_i^F = outward normal on the surface $\partial F_{\mathbf{m}}$ of the fluid cell $F_{\mathbf{m}}$). Here, $F_{\mathbf{m}}$ and $S_{\mathbf{m}}$ are time-dependent according to the moving interface $\partial I_{\mathbf{m}}$. For the same reason the flux across $\partial I_{\mathbf{m}}$ is zero and this part is excluded from the surface integral, therefore. Likewise we have for the structural (rod-) density ϱ_s^ε

$$\frac{\partial}{\partial t} \iiint_{S_{\mathbf{m}}} \varrho_s^\varepsilon \, dV = - \oiint_{\partial S_{\mathbf{m}} - \partial I_{\mathbf{m}}} n_i^S \dot{w}_i^\varepsilon \varrho_i^\varepsilon \, dO \, . \tag{21b}$$

Because of the assumed long pressure waves and for $\varepsilon/D \to 0$, $\varrho^\varepsilon \approx \bar{\varrho}_s = \text{const}$, $\varrho_s^\varepsilon \approx \bar{\varrho} = \text{const}$ within each cell so that

$$\iiint_{F_{\mathbf{m}}} \varrho^\varepsilon \, dV = \bar{\varrho}\alpha \, |V_{\mathbf{m}}| \, , \qquad \iiint_{S_{\mathbf{m}}} \varrho_s^\varepsilon \, dV = \bar{\varrho}_s(1 - \alpha) \, |V_{\mathbf{m}}| \, , \tag{22a}$$

where α is the porosity defined in (1). The surface integrals on the right of (21) can be taken as the sum of integrals over the different faces $\partial F_{\mathbf{m}}^i$, $\partial S_{\mathbf{m}}^i$ of the considered and adjacent cells. E.g.,

$$\oiint_{\partial F_{\mathbf{m}} - \partial I_{\mathbf{m}}} n_i^F u_i^\varepsilon \varrho^\varepsilon \, dO = \bar{\varrho} \sum_{i=1}^{3} \left[\iint_{\partial F_{\mathbf{m}}^i} u_i^\varepsilon \, dO - \iint_{\partial F_{\mathbf{m}-\delta_{ij}}} u_i^\varepsilon \, dO \right] . \tag{22b}$$

The integrals over $\partial F_{\mathbf{m}}^i$ are related to the velocities $\bar{u}_i$ defined in (15a)

$$\iint_{\partial F_{\mathbf{m}}^i} u_i^\varepsilon \, dO \approx \bar{u}_i \, |\partial F_{\mathbf{m}}^i| = \gamma_{ij} \bar{u}_j \, |V_{\mathbf{m}}|/\varepsilon^i \, . \tag{22c}$$

Similarily,

$$\begin{aligned} \iint_{\partial S_{\mathbf{m}}^i} \dot{w}_i^\varepsilon \, dO &\approx \dot{\bar{w}}_i \, |\partial S_{\mathbf{m}}^i| \\ &= (\delta_{ij} - \gamma_{ij}) \, \dot{\bar{w}}_j \, |V_{\mathbf{m}}|/\varepsilon^i \, . \end{aligned} \tag{22d}$$

From these relations together with the definitions of finite differences

$$\Delta_i y := [y(x_i + \varepsilon^i/2) - y(x_i - \varepsilon^i/2)]/\varepsilon^i \, , \tag{23}$$

and $n_i^s = -n_i^F$, $n_i^s u_i^s = n_i^s \dot{w}^s$, one obtains

$$\frac{\partial}{\partial t} [\alpha \bar{\varrho}] = -\bar{\varrho} \Delta_i \gamma_{ij} \bar{u}_j \, , \tag{24}$$

$$\frac{\partial}{\partial t} [(1 - \alpha) \, \bar{\varrho}_s] = -\bar{\varrho}_s \Delta_i [(\delta_{ij} - \gamma_{ij}) \, \dot{\bar{w}}_j] \, . \tag{25}$$

For abbreviation, we define the "connectivity" tensor

$$\pi_{ij} := \delta_{ij} - \gamma_{ij} \, . \tag{26}$$

From (24, 25) one can eliminate $\partial\alpha/\partial t$ and obtains

$$\frac{\alpha}{\bar{\varrho}} \frac{\partial \bar{\varrho}}{\partial t} + \frac{(1 - \alpha)}{\bar{\varrho}_s} \frac{\partial \bar{\varrho}_s}{\partial t} + \Delta_i [\gamma_{ij} \bar{u}_j + \pi_{ij} \dot{\bar{w}}_j] = 0 \, . \tag{27}$$

In the limit $\varepsilon/D \to 0$ one can replace the discrete functions $\bar{u}_i$, $\bar{w}_i$, $\bar{\varrho}$, $\bar{\varrho}_s$ by fields u_i, w_i, ϱ, ϱ_s which have the following properties: they are defined everywhere in V, are "maximally smooth" interpolations of the discrete fields as defined in [7], and, for a certain $\mathbf{x}' \in V_\mathbf{m}$ they satisfy

$$(\bar{u}_j, \bar{w}_j, \bar{\varrho}, \bar{\varrho}_s)(\mathbf{m}, t) = (u_j, w_j, \varrho, \varrho_s)(\mathbf{x}', t) . \tag{28}$$

In the same limit, finite differences become differentials so that the averaged continuity equation becomes

$$\frac{\alpha}{\varrho_0}\frac{\partial\varrho}{\partial t} + \frac{(1-\alpha)}{\varrho_s^0}\frac{\partial\varrho_s}{\partial t} + \frac{\partial}{\partial x_i}[\gamma_{ij}u_j + \pi_{ij}\dot{w}_j] = 0 . \tag{29}$$

Finally we need averaged equations of state. From (22a) and (4) follows

$$\bar{\varrho} - \varrho_0 = \frac{1}{a^2}\iiint_{F_\mathbf{m}} (p^\varepsilon - p_0)\, dV/|F_\mathbf{m}| , \tag{30a}$$

and from (22a) and (8)

$$\bar{\varrho}_s - \varrho_s^0 = \frac{1}{a^2}\iint_{\partial I_\mathbf{m}} (p^\varepsilon - p_0)\, dO/|\partial I_\mathbf{m}| . \tag{30b}$$

As p^ε is a smooth function in the fluid,

$$\bar{p} := \iiint_{F_\mathbf{m}} p^\varepsilon\, dV/|F_\mathbf{m}| \approx \iint_{\partial I_\mathbf{m}} p^\varepsilon\, dO/|\partial I_\mathbf{m}| \tag{31}$$

and the discrete function $\bar{p}(\mathbf{m}, t)$ approaches the smooth function $p(\mathbf{x}, t)$ for $\varepsilon/D \to 0$ in the sense of (28). Therefore, for $\varepsilon/D \to 0$, the averaged equations of state are

$$\varrho - \varrho_0 = (p - p_0)/a^2 , \tag{32a}$$

$$\varrho_s - \varrho_s^0 = (p - p_0)/a_s^2 . \tag{32b}$$

6 Homogenized Momentum Equations

We now have the tools to establish the homogenized momentum equations. We require that the discrete functions $\bar{u}_j$, respectively $\bar{v}_j$ with $\bar{u}_j \equiv \dot{\bar{v}}_j$, and $\bar{w}_j$ extremize the functional $\bar{J}$ given in (18) with respect to variations $\delta\bar{v}_j$, $\delta\bar{w}_j$ which vanish at $t = t_1$ and $t = t_2$, and are consistent with the averaged continuity equation (29). By the same way, we also will obtain homogenized boundary conditions. The variation of $\bar{J}$, after partial time integration, yields

$$\delta\bar{J} = \int_{t_1}^{t_2} \sum_\mathbf{m} \left\{ \iiint_{F_\mathbf{m}} [-\varrho_0(\varphi_{ij}\ddot{\bar{v}}_j + \psi_{ij}\ddot{\bar{w}}_j) + \varrho_0 g_i^\varepsilon] (\varphi_{ik}\delta\bar{v}_k + \psi_{ik}\delta\bar{w}_k)\, dV + \int_{\Delta x_{3,\mathbf{m}}} [-M_{ij}^\varepsilon \ddot{\bar{w}}_j + f_i^\varepsilon]\, dx_3\, \delta\bar{w}_i \right\} dt . \tag{33}$$

As φ_{ij}, ψ_{ij} are now known functions, we introduce abbreviations which have the meaning of "effective" densities and forces per unit volume.

$$\varrho_{ij}^{ll} := \frac{1}{|V_\mathbf{m}|}\iiint_{F_\mathbf{m}} \varrho_0\varphi_{ki}\varphi_{kj}\, dV , \tag{34a}$$

$$\varrho_{ij}^{ss} := \frac{1}{|V_\mathbf{m}|}\left[\iiint_{F_\mathbf{m}} \varrho_0\, \psi_{ki}\psi_{kj}\, dV + \int_{\Delta x_{3,\mathbf{m}}} M_{ij}^\varepsilon\, dx_3\right] \tag{34b}$$

$$\varrho_{ij}^{ls} := \frac{1}{|V_\mathbf{m}|}\iiint_{F_\mathbf{m}} \varrho_0\varphi_{ki}\psi_{kj}\, dV , \qquad \varrho_{ji}^{sl} \equiv \varrho_{ij}^{ls} , \tag{34c}$$

$$\bar{g}_i(\mathbf{m}, t) := \frac{1}{|V_\mathbf{m}|}\iiint_{F_\mathbf{m}} \varrho_0 g_j^\varepsilon \varphi_{ji}\, dV , \tag{34d}$$

$$\bar{f}_i(\mathbf{m}, t) := \frac{1}{|V_\mathbf{m}|}\left[\iiint_{F_\mathbf{m}} \varrho_0 g_j^\varepsilon \psi_{ji}\, dV + \int_{\Delta x_{3,\mathbf{m}}} f_i^\varepsilon\, dx_3\right] . \tag{34e}$$

The densities in (34a—c) are independent of ε/D. Because of (17, 19), the volume integrals in (34a—c), by Green's theorem, convert into surface integrals, e.g.,

$$\iiint_{F_{\mathbf{m}}} \varphi_{ki}\psi_{kj}\, dV = \oint\!\!\oint_{\partial F_{\mathbf{m}}} \Phi_i n_k^F (\partial\Psi_j/\partial x_k)\, dO\,. \tag{35}$$

With the definitions (34), Eq. (33) reads

$$\left.\begin{aligned} &\delta\bar{J} = \int_{t_1}^{t_2} \delta\dot{\bar{J}}\, dt = 0\,,\\ &\delta\dot{\bar{J}} := \sum_{\mathbf{m}} |V_{\mathbf{m}}| \{[-\varrho_{ij}^{ff}\ddot{\bar{v}}_j - \varrho_{ij}^{fs}\ddot{\bar{w}}_j + \bar{g}_i]\, \delta\bar{v}_i + [-\varrho_{ij}^{sf}\ddot{\bar{v}}_j - \varrho_{ij}^{ss}\ddot{\bar{w}}_j + \bar{f}_i]\, \delta\bar{w}_i\,.\end{aligned}\right\} \tag{36}$$

Now, we take the limit $\varepsilon/D \to 0$ so that the sum over all cells becomes the integral over V and the discrete fields are replaced by continuous ones according to (28) and similarily $\bar{g}_i(\mathbf{m}, t)$, $\bar{f}_i(\mathbf{m}, t)$ by $g_i(\mathbf{x}, t)$, $f_i(\mathbf{x}, t)$

$$\delta\dot{\bar{J}} = \iiint_V [-\varrho_{ij}^{ff}\ddot{v}_j - \varrho_{ij}^{fs}\ddot{w} + g_i]\, \delta v_i + [-\varrho_{ij}^{sf}\ddot{v}_j - \varrho_{ij}^{ss}\ddot{w}_j + f_i]\, \delta w_i\}\, dV\,. \tag{37}$$

The virtual displacements δv_i, δw_i have to satisfy the constraint

$$\partial[\gamma_{ij}\delta v_j + \pi_{ij}\delta w_j]/\partial x_i = 0\,, \tag{38}$$

resulting from (29). This constraint is taken into account by a Lagrangian multiplier $\lambda(\mathbf{x}, t)$. It appears to be consistent to identify λ with the averaged pressure, $\lambda(\mathbf{x}, t) \equiv p(\mathbf{x}, t) - p_0$. Partial integration in space as in Section 3 yields

$$\begin{aligned} \delta\dot{\bar{J}}' &:= \iiint_V (p - p_0)\, \partial[\gamma_{ij}\delta v_j + \pi_{ij}\delta w_j]/\partial x_i\, dV\\ &= -\iiint_V (\partial p/\partial x_i)\, [\gamma_{ij}\delta v_j + \pi_{ij}\delta w_j]\, dV - \oint\!\!\oint_{\partial V} (p - p_0)\, n_i[\gamma_{ij}\delta v_j + \pi_{ij}\delta w_j]\, dO\,. \end{aligned} \tag{39}$$

The surface integral vanishes if the homogenized boundary conditions have the form

$$p - p_0 = 0 \quad \text{on} \quad \partial V_p \subset \partial V\,, \tag{40a}$$

$$n_i u_i = u_n \quad \text{on} \quad \partial V - \partial V_p\,, \tag{40b}$$

$$n_i w_i = w_n \quad \text{on} \quad \partial V - \partial V_p\,, \tag{40c}$$

where

$$u_n := \lim_{\varepsilon/D \to 0} \left\{ \iint_{\partial F_{\mathbf{m}} \cap \partial V} u_n^s\, dO / |\partial F_{\mathbf{m}} \cap \partial V| \right\}, \tag{40d}$$

$$w_n := \lim_{\varepsilon/D \to 0} \left\{ \iint_{\partial S_{\mathbf{m}} \cap \partial V} w_n^s\, dO / |\partial S_{\mathbf{m}} \cap \partial V| \right\}. \tag{40e}$$

From the sum $\delta\bar{J} + \delta\bar{J}' \overset{!}{=} 0$ one finally obtains the homogenized momentum equations

$$\varrho_{ij}^{ff}\ddot{u}_j + \varrho_{ij}^{fs}\ddot{w}_j = g_i - \gamma_{ij}\, \partial p/\partial x_j\,, \tag{41a}$$

$$\varrho_{ij}^{sf}\ddot{u} + \varrho_{ij}^{ss}\ddot{w}_j = f_i - \pi_{ij}\, \partial p/\partial x_j\,. \tag{41b}$$

7 Discussion

Equations (29, 32, 40, 41) specify the homogenized problem in terms of smooth functions which are defined everywhere in V. These equations can be solved with much less numerical effort than the original equations. Moreover, the equations offer some interesting physical insight: Equations (41) show that the effective density is essentially different from the local scalar densities or their volumetric mean values, the "smear density"

$$\bar{\varrho} := \alpha\varrho_0 + (1 - \alpha)\, \varrho_s^0\,. \tag{42}$$

"Virtual" or effective tensors arise. This means that the acceleration vectors of the fluid and rods may have directions which differ from each other and the direction of the pressure gradient. Moreover, there will generally be a slip in the accelerations between fluid and rods (or structure).

From (29, 32, 41) one can eliminate the velocities and density changes so that one obtains a wave equation for the pressure in the form

$$\ddot{p} = \partial[(a^2)_{ij} (\partial p/\partial x_j) + k_i]/\partial x_i , \tag{43}$$

where effective speeds of sound a appear. We do not write down explicitely the lengthy expressions which define these quantities and the "sources" k_i but note that the effective speed of sound is a tensor as one might have expected for a nonisotropic body.

The "connectivity" π_{ij}, defined in (26) in terms of the permeability γ_{ij} can be zero, which is the case for rods which have a geometry like "beads" on a string and where the cells are defined such that the beads are completely internal to the cell ($\partial S_m \cap \partial V_m = 0$). In such cases the pressure gradient has formally no direct impact on the structural motion rather than only by means of the induced fluid accelerations. This example also points out the fact that the homogenized densities and connectivities change if we change the definition of our cells by all shifting them by fractions of ε in any direction. For example, one can define the cells such that the rod centerline passes through the cell mid point or through the cell corners. On the other hand, if the cell contains rods which have the form of many small particles of more or less random distribution and characteristic diameters which are small in comparison to ε then the permeabilities γ^i become all equal to the porosity α as has been shown by Bear [13, p. 20]. For such porous bodies the sensitivity to the geometric definitions disappears.

This dependence on shifts in the cell definition has its reason in the fact that $\bar{u}^i$ is defined by (15a) such that it represents the surface-mean fluid-velocity. This definition was technically necessary for the deduction of the averaged continuity equation (27). Also, boundary conditions at ∂V are more naturally expressed in terms of surface mean values. The surface mean values differ from the volume mean values unless $\gamma^i = \alpha$, $i = 1, 2, 3$ (in which case $\pi_{ij} = \delta_{ij}(1 - \alpha)$). The difference is due to a nonzero-volume-mean fluid displacement velocity implied by a unit rod motion within the cell. The transformation to volume mean velocities with equations which are independent on cell shifts is possible and will be given in a subsequent paper.

In the present theory we have assumed that the local fluid and structure motions in each cell in each coordinate direction are resolvable by one degree of freedom ($\bar{u}_j$ or $\bar{w}_j$) only. This is sufficient if only long pressure waves have to be taken into account and if relative motions and rotations within the rod cross-sections contribute only negligible amounts of specific kinetic and potential energy, see the note at the end of Section 3. Higher order approximations can be constructed by introducing extra terms $\varphi_j^{(k)}\bar{u}_j^{(k)} + \psi_{ij}^{(k)}\bar{w}_j^{(k)}$ and $\chi_{ij}^{(k)}\bar{w}_j^{(k)}$ in (13) where $\varphi_{ij}^{(k)}$ etc. are the higher eigensolutions of the local field equations. Such higher order approximations become necessary if ε/D is not very small. For example this might be the case if the rod in the cell is effectively a rod bundle itself, like the fuel element in a nuclear reactor. It is planned to investigate these questions further by means of direct numerical integration of the local field equations for some relevant examples and comparison with the results of homogenized equations, as has been done for elasticity problems by Ohayon [9].

A further question concerns the treatment of flows with large convective accelerations. We propose to treat such accelerations like the prescribed acceleration g_i^s.

It should be noted that the present theory is readily extendable to cases where the rods experience conservative forces which are gradients of a potential $U = U(w_i^s(\mathbf{m}, t), \partial w_i^s(\mathbf{m}, t)/\partial x_3, \partial^2 w_i^s(\mathbf{m}, t)/\partial x_3^2)$, e.g. due to rod bending, because such a potential is easily included in Hamilton's principle. Also friction forces, provided that they can be expressed as the gradient of Rayleigh's dissipation function [12], can be included in this manner.

In comparison to theories in which the homogenization corresponds simply to averaging the local equations (see e.g. Sha et al. [14]), the present approach has the advantage that it explicitely constructs the effective material properties which account for the local phase interactions; it has the disadvantage of being restricted to problems for which a variational principle can be defined. Ultimately, perhaps both approaches should be merged.

8 An Example: Circular Cylindrical Rods in a Square Pattern

For the case of circular cylindrical rods of radius R in a square pattern with center-to-center distance ("pitch") d (see Fig. 3), the effective densities (34a–c) are given quantitatively and in terms of asymptotic formulae below.

Let the cells be defined as indicated in Fig. 3 with side-lengths $\varepsilon = \varepsilon^1 = \varepsilon^2 = \varepsilon^3 = d$. Then we have

$$\left.\begin{aligned} &\alpha = 1 - \pi R^2/d^2\,, \qquad \gamma^1 = \gamma^2 = 1\,, \qquad \gamma^3 = \alpha\,,\\ &\Phi_1(x_1, x_2) = \Phi_2(x_2, -x_1) = \chi^{(f)}(x_1, x_2)\,, \quad \Psi_1(x_1, x_2) = \Psi_2(x_2, -x_1) = \chi^{(s)}(x_1, x_2)\,,\\ &\Phi_3 = x_3\,, \qquad \Psi_3 = 0\,,\\ &M^s_{ij} = \varrho^0_s \pi R^2 \delta_{ij} \end{aligned}\right\} \tag{44}$$

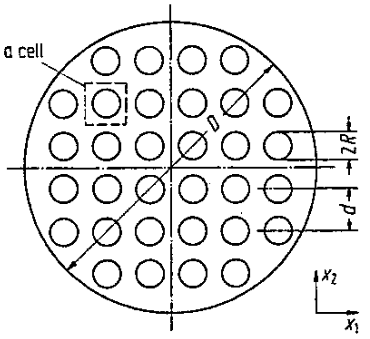

Fig. 3. The considered rod bundle (with only a few rods)

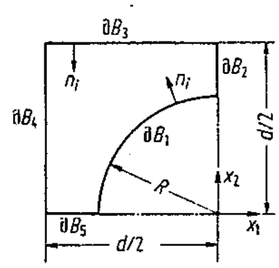

Fig. 4. Section of cross-section with definition of boundary parts

The functions $\chi^{(k)}$, $k = f, s$, are in principle solutions of Laplace equations as defined in (16–20). However, because of symmetry and continuity it suffices to consider a quarter of a two-dimensional $x_1 - x_2$-cross-section of a cell as shown in Fig. 4 with boundary conditions as given in Table 1.

Table 1. Boundary Conditions of $\chi^{(k)}$, $k = f, s$

boundary, see Fig. 4	quantity	value for χ: $\chi^{(f)}$	value for χ: $\chi^{(s)}$
∂B_1	$n_i(\partial\chi/\partial x_i) =$	0	n_1
∂B_2	$\chi =$	0	0
$\partial B_3 \cup \partial B_5$	$n_i(\partial\chi/\partial x_i) =$	0	0
∂B_4	$\chi = \text{const}, \ \frac{2}{d}\int_{\partial B_4} \frac{\partial\chi}{\partial x_1}\,dx_2 =$	1	0

The solutions $\chi^{(k)}$ have been computed using the integral equation method of Papamichael & Symm [15]. From the results the nondimensional integrals

$$I^{ij} = -\frac{4}{d^2}\int_{\partial B_1 \cup \partial B_4} \chi^{(i)} n_k \partial\chi^{(j)}/\partial x_k\, ds \tag{45}$$

are evaluated so that the density tensors, see (34, 35), are

$$\left.\begin{aligned} &\varrho^{ff}_{ij} = \operatorname{diag}(\varrho_0 I^{ff}, \varrho_0 I^{ff}, \alpha\varrho_0)\,,\\ &\varrho^{fs}_{ij} = \operatorname{diag}(\varrho_0 I^{fs}, \varrho_0 I^{fs}, 0)\,,\\ &\varrho^{ss}_{ij} = \operatorname{diag}(\varrho_0 I^{ss} + (1-\alpha)\,\varrho^0_s, \varrho_0 I^{ss} + (1-\alpha)\,\varrho^0_s, (1-\alpha)\,\varrho^0_s)\,. \end{aligned}\right\} \tag{46}$$

We could now report the numerical values for I^{ij} to complete the equations. We prefer, however, to give these integrals in the form

$$I^{ij} = k_{ij}\tilde{I}^{ij}, \tag{47}$$

where $\tilde{I}^{ij}$ are approximative values for which we give analytical expressions below, and k_{ij} are "correction factors" which are plotted in Fig. 5. Approximate analytical formulae are desirable for further insight. Such approximations are obtained if we replace the quadrilateral cell by a circular cylindrical one with the same volume, i.e. outer radius $R_a = d/\sqrt{\pi}$, as shown

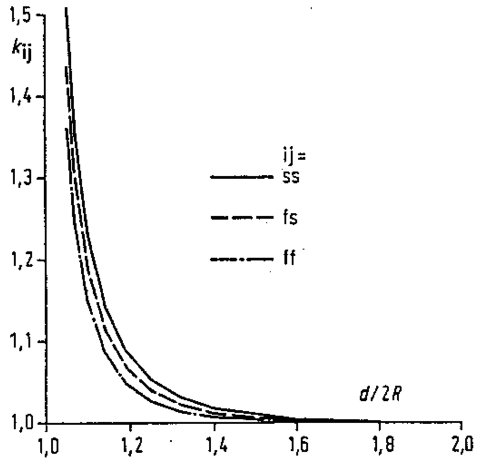

Fig. 5. Numerically evaluated correction factors, see Eq. (48)

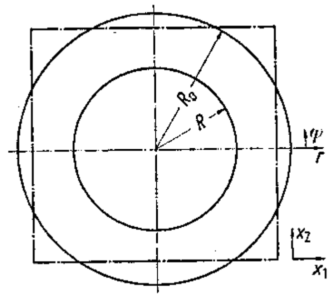

Fig. 6. The circularly approximated cell

in Fig. 6. This particular configuration has been considered also by Fritz [16]. Let us define functions $\tilde{\chi}^{(k)}$, $k = f, s$, which are solutions of Laplace equations on the circular cell, analogues to $\chi^{(k)}$, and satisfy the boundary conditions

$$\left.\begin{aligned} &\frac{\partial\tilde{\chi}^{(f)}}{\partial r} = \cos\varphi, \qquad \frac{\partial\tilde{\chi}^{(s)}}{\partial r} = 0 \quad \text{at} \quad r = R_a, \\ &\frac{\partial\tilde{\chi}^{(f)}}{\partial r} = 0, \qquad \frac{\partial\chi^{(s)}}{\partial r} = \cos\varphi \quad \text{at} \quad r = R. \end{aligned}\right\} \tag{48}$$

The solutions are

$$\tilde{\chi}^{(f)} = \frac{R_a^2}{R_a^2 - R^2}\,\frac{r^2 + R^2}{r}\cos\varphi, \qquad \tilde{\chi} = -\frac{R^2}{R_a^2 - R^2}\,\frac{r^2 + R_a^2}{r}\cos\varphi. \tag{49}$$

With these potentials, the approximate integrals are defined by

$$\tilde{I}^{ij} := \frac{1}{\pi R_a^2}\int_0^{2\pi} \tilde{\chi}^{(i)}(\partial\tilde{\chi}^{(j)}/\partial r)\, r\, d\varphi\Big|_{r=R}^{r=R_a}, \tag{50}$$

with the results

$$\left.\begin{aligned} &\tilde{I}^{ff} = 1 + \tilde{I}_V, \qquad \tilde{I}^{ss} = (1 + \tilde{I}_V)\, R^2/R_a^2, \\ &\tilde{I}^{fs} = \tilde{I}^{sf} = -\tilde{I}_V, \qquad \tilde{I}_V := 2R^2/(R_a^2 - R^2). \end{aligned}\right\} \tag{51}$$

Fig. 5 shows that for large values of $d/(2R)$ the correction factors tend to unity so that the integrals $\tilde{I}^{ij}$ are the asymptotic values of I^{ij} for $d/(2R) \to \infty$. The sketches in Fig. 7 illustrate the flow fields which are represented by the potentials $\chi^{(k)}$ or $\tilde{\chi}^{(k)}$: $\chi^{(f)}$ is the potential for unit flow through the rod bundle in the x_1-direction for fixed rods. The quantity $\chi^{(s)}$ is the potential induced by a unit motion of the rods in x_1-direction with zero fluid flow across the cell boundaries, as required in (16). In the x_2-direction, the flow field has the same picture.

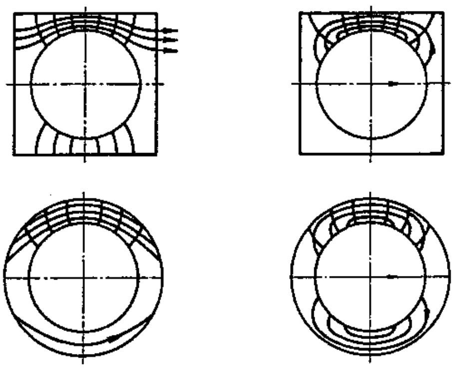

Fig. 7. Sketch of flow fields corresponding to potentials $\chi^{(f)}$ (left) and $\chi^{(s)}$ (right) for the quadratic cell and its circular approximation

In case of zero external forces, one can eliminate the rod accelerations from (41 b) and obtains for the present example, e.g. for the x_1-components,

$$\bar{\varrho}_{\mathrm{eff}}\dot{u}_1 = -\partial p/\partial x_1 \,, \tag{52a}$$

$$\bar{\varrho}_{\mathrm{eff}} := \varrho_{11}^{ff} - (\varrho_{11}^{fs})^2/\varrho_{11}^{ss} \,, \tag{52b}$$

$$\ddot{w}_1/\dot{u}_1 = -\varrho_{11}^{fs}/\varrho_{11}^{ss} \,. \tag{52c}$$

Using the asymptotic results, one finds for $d/(2R) \gg 1$ with $\varkappa := \varrho_s^0/\varrho_0$

$$\frac{\bar{\varrho}_{\mathrm{eff}}}{\varrho_0} = \frac{R_a^2 + R^2}{R_a^2 - R^2} - \frac{2R^2}{R_a^2 - R^2}\,\frac{2R_a^2}{\varkappa(R_a^2 - R^2) + R_a^2 + R^2} \,, \tag{53a}$$

$$\frac{\bar{\varrho}_{\mathrm{eff}}}{\bar{\varrho}} = \frac{R_a^2[\varkappa(R_a^2 + R^2) + R_a^2 - R^2]}{[\varkappa(R_a^2 - R^2) + R_a^2 + R^2]\,[\varkappa R^2 + R_a^2 - R^2]} \,, \tag{53b}$$

$$\frac{\ddot{w}_1}{\dot{u}_1} = \frac{2R_a^2}{\varkappa(R_a^2 - R^2) + R_a^2 + R^2} \,. \tag{53c}$$

Thus, the "dynamically effective" density $\bar{\varrho}_{\mathrm{eff}}$, (52b), is essentially different from the "statically effective" smear density, (42). For $\varrho_s^0 < \varrho_0$ the effective density $\bar{\varrho}_{\mathrm{eff}}$ is smaller than min $(\varrho_0, \bar{\varrho})$; the rod represents a hole which enlarges the "fluidity" of the fluid. For $\varrho_s^0 > \varrho_0$ the effective density is larger than the fluid density; the rod narrows the flow path and thus enlarges the effective fluid inertia. For $\varrho_s^0 < \varrho_0$ the local rod acceleration is larger than the mean fluid acceleration and vice versa.

The pressure wave Eq. (43), for zero applied forces, takes the form

$$\ddot{p} = a_r^2 \left[\frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2}\right] + a_z^2 \frac{\partial^2 p}{\partial x_3^2} \,, \tag{54a}$$

where

$$a_r^2 = \frac{1}{\bar{\varrho}_{\mathrm{eff}}}\left[\frac{\alpha}{\varrho_0 a^2} + \frac{(1-\alpha)}{\varrho_s^0 a_s^2}\right]^{-1} , \tag{54b}$$

$$a_z^2 = \left[\frac{\alpha}{\varrho_0} + \frac{(1-\alpha)}{\varrho_s^0}\right]\left[\frac{\alpha}{\varrho_0 a^2} + \frac{(1-\alpha)}{\varrho_s^0 a_s^2}\right]^{-1} \tag{54c}$$

are squared components of the tensor of effective speed of sound. Obviously, these quantities are strongly influenced by the fluid-structure interactions and anisotropic.

Appendix: Computation of Φ_j, Ψ_j

The computation of the potentials $\Phi_j, \Psi_j, j = 1, 2, 3$, as defined by (16, 19, 20), is not amenable to standard numerical procedures, because (16c) requires periodicity of the gradients rather than of the potentials themselves. However, Φ_j and Ψ_j can be expressed in terms of a single potential X^j, which is periodic

$$X^j(i_1\varepsilon^1 + x_1, i_2\varepsilon^2 + x_2, i_3\varepsilon^3 + x_3) = X^j(x_1, x_2, x_3) \tag{55}$$

for integers i_k, satisfies the Laplace equation, and the boundary conditions

$$n_i^\varepsilon(\partial X^j/\partial x_i) = n_j^\varepsilon \quad \text{at} \quad \partial I_{\mathbf{m}}\,. \tag{56}$$

Let

$$b_{kj} := {}^{i}\overline{\partial X^k/\partial x_j} - \delta_{kj}\,. \tag{57}$$

(Often, e.g. for $\gamma^k = 0$ or for planes ∂V^k through which the rod passes such that the normal n_i^ε lies within this plane, one can show that $b_{kj} = 0$ for $j \neq k$. This means that in such cases b_{kj} is a diagonal tensor. In general this is not the case but we assume, without proof, det $(b_{kj}) \neq 0$.) Then the solution of

$$b_{kj}\Phi_j = X^k - x_k\,, \qquad \psi_j = (x_j - \Phi_j) \tag{58}$$

are the requested solutions satisfying (16, 17, 19, 20).

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