ON COMPUTING MINIMAL REALIZATIONS OF PERIODIC DESCRIPTOR SYSTEMS

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Abstract: We propose computationally efficient and numerically reliable algorithms to compute minimal realizations of periodic descriptor systems. The main computational tool employed for the structural analysis of periodic descriptor systems (i.e., reachability and observability) is the orthogonal reduction of periodic matrix pairs to Kronecker-like forms. Specializations of a general reduction algorithm are employed for particular type of systems. One of the proposed minimal realization methods relies exclusively on structure preserving manipulations via orthogonal transformations for which the backward numerical stability can be proved.

Keywords: Periodic systems, discrete-time systems, time-varying systems, Kalman decomposition, numerical methods.

1. INTRODUCTION

We consider periodic time-varying descriptor systems of the form

$$\begin{aligned} E_k x(k+1) &= A_k x(k) + B_k u(k) \\ y(k) &= C_k x(k) \end{aligned} \qquad (1)$$

where the matrices $E_k \in \mathrm{R}^{\mu_{k+1}\times n_{k+1}}$, $A_k \in \mathrm{R}^{\mu_{k+1}\times n_k}$, $B_k \in \mathrm{R}^{\mu_{k+1}\times m}$, $C_k \in \mathrm{R}^{p\times n_k}$, are periodic with period $N \geq 1$, and the dimensions fulfil the condition $\sum_{k=1}^{N}\mu_k = \sum_{k=1}^{N} n_k$. We denote alternatively the periodic system (1) as the quadruple $\Sigma := (E_k, A_k, B_k, C_k)$. The absence of a direct term in the second equation has no importance for our developments and therefore we assume the simpler form above for the system equations.

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Periodic descriptor systems with constant dimensions have been considered in a series of papers (Conte *et al.*, 1990; Sreedhar and Dooren, 1997; Sreedhar *et al.*, 1999; Sreedhar and Dooren, 1999). This type of system may naturally arise even in the context of standard periodic systems, when, for instance, forming an inverse or a conjugate periodic system. Moreover, intermediary computations, as for example, in designing fault detectors for periodic systems (Varga, 2004*d*) or solving periodic model-matching problems lead to periodic descriptor system representations.

From the beginning we would like to emphasize the importance of considering the more general case of time-varying dimensions. The development of general algorithms able to address the case of time-varying dimensions, was one of the requirements formulated for a *satisfactory* numerical algorithm for periodic systems (Varga and Dooren, 2001). However, one even more important reason is intrinsic to the minimal realization problem itself. It is well known that the minimal realization (i.e., reachable and observable) of standard periodic systems (i.e., $E_k = I_{n_{k+1}}$) have, in general, time-varying state dimensions (Colaneri and Longhi, 1995; Gohberg *et al.*, 1999). As it will be apparent in Sections 3 and 4 the minimal realization of periodic descriptor systems leads generally to rectangular descriptor matrices E_k even in the case of originally constant dimensions.

Although standard periodic systems with time-varying dimensions have been already considered earlier in (Grasselli and Longhi, 1991*b*; Gohberg *et al.*, 1992), only recently, numerically reliable algorithms for such systems have been developed. Relevant examples are the algorithms for the computation of periodic Kalman reachability and observability forms (Varga, 2004*a*), computation of minimal periodic realizations (Varga, 1999; Varga, 2004*c*), or the evaluation of the lifted transfer-function matrix of a periodic system (Varga, 2003). Recently, the first algorithms have been proposed for the more general periodic descriptor systems with time-varying dimensions, as for example, the computation of system zeros (Varga and Van Dooren, 2003) or the evaluation of $\mathcal{L}_\infty$-norm of a periodic system (Varga, 2006).

In this paper we propose computationally efficient and numerically reliable approaches to compute minimal realizations of periodic descriptor systems. The proposed algorithms employ similarity transformations to bring the system matrices in condensed forms which allow to eliminate redundant non-reachable and/or non-observable parts of the system. The main computational tool employed for the structural analysis of periodic descriptor systems is the orthogonal reduction of periodic matrix pairs to Kronecker-like forms (Varga, 2004*b*). Specializations of the general reduction algorithm are employed for particular type of systems (e.g., backward- or forward-time systems), to separate non-reachable and non-observable parts, or to compute backward/forward spectral separations. One of the two proposed minimal realization methods relies exclusively on structure preserving manipulations via orthogonal transformations for which the backward numerical stability can be proved. Although the structural analysis of periodic descriptor systems with constant dimensions has been addressed by several authors using geometric methods (Conte *et al.*, 1990) or the backward/forward separation technique (Coll *et al.*, 2004; Chu *et al.*, 1995), none of these theoretical methods are suited for reliable numerical computations.

2. MINIMAL REALIZATION PROBLEM

We formulate the minimal realization problem for the periodic system (1) in terms of the associated lifted *transfer-function matrix* (TFM). We also discuss shortly the relevant aspects related to the *solvability* to guarantee the existence of the associated TFM.

To define the associated lifted TFM for the system $\Sigma = (E_k, A_k, B_k, C_k)$, we employ the lifting technique introduced in (Grasselli and Longhi, 1991*a*) for constant dimensions to build an equivalent time-invariant descriptor system with the input, state and output vectors defined over time intervals of length N, rather than 1. For a given sampling time k, the corresponding mN-dimensional input vector, pN-dimensional output vector and $(\sum_{k=1}^{N} n_k)$-dimensional state vector are

$$\begin{aligned} u_k^S(h) &= [u^T(k+hN)\cdots u^T(k+hN+N-1)]^T, \\ y_k^S(h) &= [y^T(k+hN)\cdots y^T(k+hN+N-1)]^T, \\ x_k^S(h) &= [x^T(k+hN)\cdots x^T(k+hN+N-1)]^T. \end{aligned}$$

The corresponding constant descriptor system has the form

$$\begin{aligned} L_k x_k^S(h+1) &= F_k x_k^S(h) + G_k u_k^S(h) \\ y_k^S(h) &= H_k x_k^S(h) \end{aligned} \qquad (2)$$

where $G_k = \text{diag}(B_k, B_{k+1}, ..., B_{k+N-1})$, $H_k = \text{diag}(C_k, C_{k+1}, ..., C_{k+N-1})$ and

$$F_k - zL_k = \begin{bmatrix} A_k & -E_k & O & \cdots & O \\ O & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots -E_{k+N-3} & O \\ O & & & \ddots\; A_{k+K-2} & -E_{k+N-2} \\ -zE_{k+N-1} & O & \cdots & O & A_{k+N-1} \end{bmatrix} \qquad (3)$$

Assuming the square pencil $F_k - zL_k$ is regular (i.e. $\det(F_k - zL_k) \not\equiv 0$), the TFM of the lifted system at time k is

$$W_k^\Sigma(z) := H_k(zL_k - F_k)^{-1}G_k. \qquad (4)$$

Obviously $W_{k+N}^\Sigma(z) = W_k^\Sigma(z)$ and the TFMs at two successive values of k are related by the relation

$$W_{k+1}^\Sigma(z) = \begin{bmatrix} 0 & I_{p(N-1)} \\ zI_p & 0 \end{bmatrix} W_k^\Sigma(z) \begin{bmatrix} 0 & z^{-1}I_m \\ I_{m(N-1)} & 0 \end{bmatrix}.$$

It is easy to show that if Q_k and Z_k are invertible periodic matrices of appropriate orders then the two systems $\Sigma = (E_k, A_k, B_k, C_k)$ and $\widetilde{\Sigma} = (Q_k E_k Z_{k+1}, Q_k A_k Z_k, Q_k B_k, C_k Z_k)$ related by a similarity transformation have the same TFMs, i.e., $W_k^\Sigma(z) = W_k^{\widetilde{\Sigma}}(z)$, $k = 1, \ldots, N$.

The *minimal realization problem* (MRP) can be formulated as follows: Given the periodic system (1) defined by $\Sigma = (E_k, A_k, B_k, C_k)$ of state dimensions n_k, $k = 1, \ldots, N$ determine a periodic system $\overline{\Sigma} = (\overline{E}_k, \overline{A}_k, \overline{B}_k, \overline{C}_k)$ of least order state dimensions $\overline{n}_k \leq n_k$, $k = 1, \ldots, N$ such that $W_k^\Sigma(z) = W_k^{\overline{\Sigma}}(z)$, $k = 1, \ldots, N$.

To compute minimal realizations, we will only employ similarity transformations to achieve various compressions of system matrices leading to dimensional reductions. Further reduction of order achieved by eliminating non-dynamic parts is not considered in this paper.

Before we start presenting the computational approaches we discuss shortly the significance of the regularity assumption of the pencil $F_k - zL_k$. The *solvability* conditions of system (1) for arbitrary inputs have been discussed in (Conte *et al.*, 1990; Sreedhar and Dooren, 1999) and are equivalent to *regularity* condition established by Luenberger (1977). In accordance with this, we say the system (1) is *regular* if the associated pencil $F_k - zL_k$ is regular, that is, it has no left or right Kronecker structure. In what follows we will assume throughout the paper that the solvability condition is fulfilled.

To check the regularity of the pencil $F_k - zL_k$ we can use the *fast* structure exploiting method as proposed for zeros computation to separate a low-order subpencil with the same left/right Kronecker structure as $F_k - zL_k$ (Varga and Van Dooren, 2003). Alternatively, the structurally stable reduction of the periodic pair (A_k, E_k) to a periodic Kronecker-like form can be performed using the algorithm proposed in (Varga, 2004*b*). If this pair has no left or right Kronecker structure, then the periodic descriptor system (1) is solvable.

3. FORWARD/BACKWARD DECOMPOSITION BASED TECHNIQUES

The backward/forward separation technique for structural analysis of peridic descriptor systems with constant dimensions has been considered in (Coll *et al.*, 2004; Chu *et al.*, 1995). This technique can be employed to compute minimal realizations as follows. First compute an additive decomposition of the system Σ into Σ_f, representing the forward (or proper) part, and Σ_b representing the backward (or improper) part, such that $W_k^{\Sigma}(z) = W_k^{\Sigma_f}(z)+W_k^{\Sigma_b}(z)$, $k = 1, \ldots, N$. In a second step, minimal realization of each part are computed using special algorithms to compute appropriate reachability/observability staircase forms. In what follows we discuss in some details the main steps of this approach.

3.1 Forward/backward decomposition

A computational algorithm to perform this decomposition for constant dimensions has been proposed in (Sreedhar and Dooren, 1997) based on the periodic generalized Schur decomposition of the periodic pair (A_k, E_k) (Bojanczyk *et al.*, 1992). The forward/backward separation is achieved by employing eigenvalue reordering techniques, a procedure which is generally considered not being the best numerical approach because of possible eigenvalue sensitivity problems in the case of multiple eigenvalues.

As an alternative, we sketch an enhanced procedure based on the separation of finite-infinite structures which is automatically achieved by applying the basic reduction procedure of (Varga, 2004*b*) to the periodic pair (A_k, E_k). The result of this reduction are orthogonal N-periodic matrices Q_k and Z_k such that

$$\widetilde{A}_k := Q_k A_k Z_k = \begin{bmatrix} A_k^b & A_k^{bf} \\ 0 & A_k^f \end{bmatrix},$$

$$\widetilde{E}_k := Q_k E_k Z_{k+1} = \begin{bmatrix} E_k^b & E_k^{bf} \\ 0 & E_k^f \end{bmatrix}$$

where, for $k = 1, \ldots, N$, $A_k^b \in \mathrm{R}^{n_k^b \times n_k^b}$ is upper triangular and nonsingular, $E_k^b \in \mathrm{R}^{n_k^b \times n_{k+1}^b}$ has the part formed from the trailing non-zero columns of full-column rank, $A_k^f \in \mathrm{R}^{n_{k+1}^f \times n_k^f}$, and $E_k^f \in \mathrm{R}^{n_{k+1}^f \times n_{k+1}^f}$ is upper triangular and nonsingular.

In a second step, we determine periodic matrices L_k and R_k such that

$$\begin{bmatrix} A_k^b & 0 \\ 0 & A_k^f \end{bmatrix} := \begin{bmatrix} I & L_k \\ 0 & I \end{bmatrix} \begin{bmatrix} A_k^b & A_k^{bf} \\ O & A_k^f \end{bmatrix} \begin{bmatrix} I & R_k \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} E_k^b & 0 \\ 0 & E_k^f \end{bmatrix} := \begin{bmatrix} I & L_k \\ 0 & I \end{bmatrix} \begin{bmatrix} E_k^b & E_k^{bf} \\ O & E_k^f \end{bmatrix} \begin{bmatrix} I & R_{k+1} \\ 0 & I \end{bmatrix}$$

by solving the periodic generalized Sylvester systems of equations

$$\left. \begin{array}{rcl} A_k^b R_k + L_k A_k^f & = & -A_k^{bf} \\ E_k^b R_{k+1} + L_k E_k^f & = & -E_k^{bf} \end{array} \right\}, k = 1, \ldots, N$$

To solve the above equation there is presently no general algorithm available in the case of time-varying dimensions. However, since the upper triangular E_k^f is invertible, it is possible to eliminate L_k from the second set of equations

$$L_k = -(E_k^{bf} + E_k^b R_{k+1})(E_k^f)^{-1}$$

and replace it in the first one

$$A_k^b R_k - E_k^b R_{k+1}(E_k^f)^{-1} A_k^f = -A_k^{bf} + E_k^{bf}(E_k^f)^{-1}$$

which is a periodic generalized Sylvester equation of the form considered in (Byers and Rhee, 1995) for constant dimensions. We believe that it is straightforward to derive a general algorithm to solve the above equations with time-varying dimensions by extending the approach proposed in (Byers and Rhee, 1995). Note that the above periodic generalized Sylvester equation can be further reduced to a more standard form (although still involving time-varying dimensions) by employing that A_k^b is also invertible (and upper triangular).

The resulting equations can be solved by extending the methods for solving periodic Lyapunov equations in (Varga, 1997).

If we apply the overall transformation matrices to B_k and C_k we obtain after appropriate row and column partitioning

$$\begin{bmatrix} I & L_k \\ 0 & I \end{bmatrix} Q_k B_k := \begin{bmatrix} B_k^b \\ B_k^f \end{bmatrix}$$

$$C_k Z_k \begin{bmatrix} I & R_k \\ 0 & I \end{bmatrix} := [\, C_k^b \;\; C_k^f \,]$$

The original periodic system (1) has been additively decomposed in the backward part defined by the quadruple $\Sigma_b := (E_k^b, A_k^b, B_k^b, C_k^b)$ and the forward part defined by $\Sigma_f := (E_k^f, A_k^f, B_k^f, C_k^f)$. The analysis of each of these parts can be performed by extending or adapting the techniques of (Varga, 2004*a*) to the periodic descriptor representations with invertible E_k or invertible A_k.

Since the backward/forward separation involves using non-orthogonal transformation, the numerical reliability of this approach depends on the conditioning of the employed transformations. This can be easily assessed by simply evaluating the norms of matrices L_k and R_k. If this norms are large, then the separation is expected to be inaccurate and the alternative technique presented in the next section is a more appropriate.

3.2 Minimal realization of forward part Σ_f

To solve the MRP for the forward subsystem $\Sigma_f := (E_k^f, A_k^f, B_k^f, C_k^f)$ with state dimensions n_k^f, $k = 1, \ldots, N$, it is possible to reduce this system to an equivalent standard system $(I_{n_{k+1}^f}, (E_k^f)^{-1}A_k^f, (E_k^f)^{-1}B_k^f, C_k^f)$ and apply the minimal realization procedure of (Varga, 2004*a*). Alternatively, a numerically more reliable approach is to extend the algorithms for Periodic Kalman Reachability Decomposition (PKRD) and Periodic Kalman Observability Decomposition (PKOD) to the descriptor system representation Σ_f, where additionally we can assume that E_k^f is upper triangular.

The extension of the PKRD algorithm to the descriptor case is also discussed in (Varga, 2004*a*, Remark 3.). The result of applying the extended algorithm PKRD to the periodic system Σ_f is an equivalent system $\widetilde{\Sigma}_f = (\widetilde{E}_k^f, \widetilde{A}_k^f, \widetilde{B}_k^f, \widetilde{C}_k^f)$ with the matrices having the following forms (generalized periodic Kalman reachability decomposition)

$$\widetilde{E}_k^f = \begin{bmatrix} E_k^{f,r} & * \\ 0 & E_k^{f,\overline{r}} \end{bmatrix}, \quad \widetilde{A}_k^f = \begin{bmatrix} A_k^{f,r} & * \\ 0 & A_k^{f,\overline{r}} \end{bmatrix},$$

$$\widetilde{B}_k^f = \begin{bmatrix} B_k^{f,r} \\ 0 \end{bmatrix}, \quad \widetilde{C}_k^f = [\, C_k^{f,r} \;\; C_k^{f,\overline{r}} \,]$$

where $E_k^{f,r} \in \mathrm{R}^{r_{k+1}^f \times r_{k+1}^f}$ is invertible and upper triangular, $A_k^{f,r} \in \mathrm{R}^{r_{k+1}^f \times r_k^f}$, r_k^f is the dimension of the corresponding reachability subspace at time k and the periodic system $\Sigma_f^r := (E_k^{f,r}, A_k^{f,r}, B_k^{f,r}, C_k^{f,r})$ is completely reachable and $W_k^{\Sigma_f}(z) = W_k^{\Sigma_f^r}(z)$. Moreover, the resulting matrices $[\, B_k^{f,r} \;\; A_k^{f,r} \,]$ are in a staircase form, with the structure described in (Varga, 2004*a*).

Important remark: Preserving the upper triangular form of resulting $\widetilde{E}_k^f$ during the row compressions involved in the extended PKRD algorithm of (Varga, 2004*a*) is a key aspect to guarantee a satisfactory worst-case computational complexity of $O(Nn^3)$, where n is an upper bound on the state space dimensions. The basic technique is the same as described in (Varga, 1990).

We can further apply the extension of Algorithm PKOD of (Varga, 2004*a*) to the reduced periodic system $\Sigma_f^r := (E_k^{f,r}, A_k^{f,r}, B_k^{f,r}, C_k^{f,r})$ to obtain an equivalent system $\widetilde{\Sigma}_f^r := (\widetilde{E}_k^{f,r}, \widetilde{A}_k^{f,r}, \widetilde{B}_k^{f,r}, \widetilde{C}_k^{f,r})$ with the matrices having the following forms (generalized periodic Kalman observability decomposition)

$$\widetilde{E}_k^{f,r} = \begin{bmatrix} E_k^{f,ro} & * \\ 0 & E_k^{f,r\overline{o}} \end{bmatrix}, \quad \widetilde{A}_k^{f,r} = \begin{bmatrix} A_k^{f,ro} & * \\ 0 & A_k^{f,r\overline{o}} \end{bmatrix},$$

$$\widetilde{B}_k^{f,r} = \begin{bmatrix} B_k^{f,ro} \\ B_k^{f,r\overline{o}} \end{bmatrix}, \quad \widetilde{C}_k^{f,r} = [\, C_k^{f,ro} \;\; 0 \,]$$

where $E_k^{f,ro} \in \mathrm{R}^{q_{k+1}^f \times q_{k+1}^f}$ is invertible and upper triangular, $A_k^{f,ro} \in \mathrm{R}^{q_{k+1}^f \times q_k^f}$, q_k^f is the dimension of the corresponding observability subspace at time k and the periodic system

$$\Sigma_f^{ro} := (E_k^{f,ro}, A_k^{f,ro}, B_k^{f,ro}, C_k^{f,ro})$$

is completely reachable and completely observable, and satisfies $W_k^{\Sigma_f}(z) = W_k^{\Sigma_f^{ro}}(z)$.

3.3 Minimal realization of backward part Σ_b

To solve the MRP for the backward subsystem $\Sigma_b := (E_k^b, A_k^b, B_k^b, C_k^b)$ with state dimensions n_b^f, $k = 1, \ldots, N$ we apply the same algorithms to the dual forward system defined as $\Sigma_D := (A_{N-k+1}^b, E_{N-k+1}^b, B_{N-k+1}^b, C_{N-k+1}^b)$. The resulting system

$$\Sigma_D^{ro} := (A_{N-k+1}^{b,ro}, E_{N-k+1}^{b,ro}, B_{N-k+1}^{b,ro}, C_{N-k+1}^{b,ro})$$

is completely reachable and completely observable and its dual represents a minimal realization of Σ_b as

$$\Sigma_b^{ro} := (E_k^{b,ro}, A_k^{b,ro}, B_k^{b,ro}, C_k^{b,ro})$$

The minimal realization Σ_{ro} of the system Σ can be assembled as $\Sigma_{ro} := (E_k^{ro}, A_k^{ro}, B_k^{ro}, C_k^{ro})$, where

$$E_k^{ro} = \begin{bmatrix} E_k^{b,ro} & 0 \\ 0 & E_k^{f,ro} \end{bmatrix}, \quad A_k^{ro} = \begin{bmatrix} A_k^{b,ro} & 0 \\ 0 & A_k^{f,ro} \end{bmatrix},$$

$$B_k^{ro} = \begin{bmatrix} B_k^{b,ro} \\ B_k^{f,ro} \end{bmatrix}, \quad C_k^{ro} = \left[\, C_k^{b,ro} \;\; C_k^{f,ro} \,\right]$$

4. GENERALIZED PERIODIC KALMAN DECOMPOSITIONS BASED METHOD

The second approach we propose is inspired by the method described in (Van Dooren, 1981) for standard descriptor systems and essentially concerns with the structured reduction of the periodic pair (S_k, T_k)

$$S_k := \left[\, B_k \middle| A_k \,\right], \quad T_k := \left[\, 0 \middle| E_k \,\right] \tag{5}$$

to a periodic Kronecker-like form in which the completely reachable part and the part containing the input-decoupling zeros are separated. By deleting the part containing the input decoupling zeros, a reachable realization can be computed.

We present in more detail the reduction of the pair (S_k, T_k) defined by (5) which allows the elimination of the non-reachable part. Let $U_k^{(1)}$ be a periodic orthogonal state-space transformation such that $U_k^{(1)}$ compresses B_k to a full row rank matrix

$$U_k^{(1)} B_k := \left[\begin{array}{c} B_{k,1} \\ \hline 0 \end{array} \right], \tag{6}$$

where $B_{k,1}$ has full row rank ν_k. We apply the transformation to S_k and T_k and partition $U_k^{(1)} E_k$ and $U_k^{(1)} A_k$ according to the row partitioning of $U_k^{(1)} B_k$ in (6)

$$U_k^{(1)} S_k := \left[\begin{array}{c|c} B_{k,1} & A_{k,1} \\ \hline 0 & A_{k,2} \end{array} \right], \; U_k^{(1)} T_k := \left[\begin{array}{c|c} 0 & E_{k,1} \\ \hline 0 & E_{k,2} \end{array} \right]$$

We now apply to the reduced periodic pairs $(A_{k,2}, E_{k,2})$ the periodic Kronecker-like algorithm of (Varga, 2004*b*). Note that this pair has no left Kronecker structure since the original pair (S_k, T_k) does not have either (guaranteed by the regularity assumption). We obtain periodic orthogonal matrices $U_k^{(2)}$ and V_k such that

$$U_k^{(2)} A_{k,2} V_k = \begin{bmatrix} B_{k,1}^r & A_{k,1}^r & * \\ 0 & 0 & A_k^{\overline{r}} \end{bmatrix},$$

$$U_k^{(2)} E_{k,2} V_{k+1} = \begin{bmatrix} 0 & E_{k,1}^r & * \\ 0 & 0 & E_k^{\overline{r}} \end{bmatrix},$$

where the periodic pair $\left([\, B_{k,1}^r \; A_{k,1}^r \,], [\, 0 \; E_{k,1}^r \,] \right)$ has only right Kronecker structure, with $[\, B_{k,1}^r \; A_{k,1}^r \,]$ full row rank and in a staircase form, $E_{k,1}^r$ square, upper-triangular and invertible, and the pair $(A_k^{\overline{r}}, E_k^{\overline{r}})$ is regular (i.e., has no left or right Kronecker structures). The corresponding generalized eigenvalues represent the input-decoupling zeros of the system (Varga and Van Dooren, 2003).

Overall we have with $U_k := \operatorname{diag}\left(I_{\nu_k}, U_k^{(2)} \right) U_k^{(1)}$

$$U_k E_k V_{k+1} = \left[\begin{array}{cc|c} E_{k,11} & * & * \\ 0 & E_{k,1}^r & * \\ \hline 0 & 0 & E_k^{\overline{r}} \end{array} \right] := \left[\begin{array}{c|c} E_k^r & * \\ \hline 0 & E_k^{\overline{r}} \end{array} \right],$$

$$U_k A_k V_k = \left[\begin{array}{cc|c} A_{k,11} & * & * \\ B_{k,1}^r & A_{k,1}^r & * \\ \hline 0 & 0 & A_k^{\overline{r}} \end{array} \right] := \left[\begin{array}{c|c} A_k^r & * \\ \hline 0 & A_k^{\overline{r}} \end{array} \right],$$

$$U_k B_k = \left[\begin{array}{c} B_{k,1} \\ 0 \\ \hline 0 \end{array} \right] := \left[\begin{array}{c} B_k^r \\ \hline 0 \end{array} \right], \quad C_k V_k = \left[\, C_k^r \middle| C_k^{\overline{r}} \,\right]$$

It can be easily shown using similar arguments as in the constant system case (Van Dooren, 1981) that the reduced system defined as

$$\Sigma_r := (E_k^r, A_k^r, B_k^r, C_k^r)$$

is completely reachable (i.e., the corresponding periodic pair (S_k, T_k) in (5) has only right Kronecker structure). The resulting system $\widetilde{\Sigma} := (U_k E_k V_{k+1}, U_k A_k V_k, U_k B_k, C_k V_k)$ is in a *generalized periodic Kalman reachability decomposition* form, where $[\, B_k^r \; A_k^r \,]$ is full row rank and is in a staircase form, and $[\, B_k^r \; E_k^r \,]$ has full row rank.

In a dual setting, the periodic pairs (S_k^r, T_k^r) with

$$S_k^r := \begin{bmatrix} A_k^r \\ C_k^r \end{bmatrix}, \quad T_k^r := \begin{bmatrix} E_k^r \\ 0 \end{bmatrix} \tag{7}$$

can be reduced to a Kronecker-like form in which the completely observable part and the part containing the output-decoupling zeros are separated. By deleting the part corresponding to the output decoupling zeros, a minimal order realization can be computed. The resulting reduced system

$$\Sigma_{ro} := (E_k^{ro}, A_k^{ro}, B_k^{ro}, C_k^{ro})$$

is both reachable and observable, and thus minimal.

Note that in contrast to the approach based on backward/foward decomposition, this method relies exclusively on orthogonal transformations. Thus, it is possible to prove that the computed matrices of the minimal realization are exact for slightly perturbed original matrices. It follows that this algorithm is backward numerically stable.

The overall worst-case computational complexity of the algorithm is $O(N(n^3+mn^2+pn^2))$ which indicates that, in light of requirements formulated in (Varga and Dooren, 2001), the proposed approach is satisfactory.

5. CONCLUSION

We proposed computationally efficient and numerically reliable algorithms to determine minimal realizations of periodic descriptor systems. The main advantage of the backward/forward separation based method is the additional structural insight which can be gained by separately analyzing the proper and improper parts. Although not numerically stable, still the method can be considered numerically reliable because possible instabilities caused by ill-conditioned transformations can be easily detected. The second approach uses exclusively orthogonal transformations and thus its numerical stability can be proven. Both methods employ powerful analysis techniques relying on the computation of periodic Kronecker-like forms (Varga, 2004*b*). For efficient and robust software implementations a specialization of the underlying algorithms is highly recommended.

REFERENCES

Bojanczyk, A. W., G. Golub and P. Van Dooren (1992). The periodic Schur decomposition. Algorithms and applications. *Proceedings SPIE Conference* (F. T. Luk, Ed.). Vol. 1770. pp. 31–42.

Byers, R. and N. Rhee (1995). Cyclic Schur and Hessenberg-Schur numerical methods for solving periodic Lyapunov and Sylvester equations. Technical report. Dept. of Mathematics, Univ. of Missouri at Kansas City.

Chu, E.K.W., H.Y. Fan and W.W. Lin (1995). Reachability and observability of periodic descriptor systems. Preprint 2005-1-009 NCTS. National Tsing Hua University, Taiwan.

Colaneri, P. and S. Longhi (1995). The realization problem for linear periodic systems. *Automatica* **31**, 775–779.

Coll, C., M. J. Fullana and E. Sánchez (2004). Reachability and observability indices of a discrete-time periodic descriptor system. *Applied Mathematics and Computation* **153**, 485–496.

Conte, G., S. Longhi and A. M. Perdon (1990). On the geometry of singular periodic systems. *Proc. of IMACS Int. Symp. on Mathematical and Intelligent Models in System Simulation, Brussels, Belgium.* pp. 119–124.

Gohberg, I., M. A. Kaashoek and J. Kos (1999). Classification of linear periodic difference equations under periodic or kinematic similarity. *SIAM J. Matrix Anal. Appl.* **21**, 481–507.

Gohberg, I., M. A. Kaashoek and L. Lerer (1992). Minimality and realization of discrete time-varying systems. *Operator Theory: Advances and Applications* **56**, 261–296.

Grasselli, O. M. and S. Longhi (1991*a*). Finite zero structure of linear periodic discrete-time systems. *Int. J. Systems Sci.* **22**, 1785–1806.

Grasselli, O. M. and S. Longhi (1991*b*). Pole-placement for nonreachable periodic discrete-time systems. *Math. Control Signals Syst.* **4**, 439–455.

Luenberger, D. G. (1977). Time-invariant descriptor systems. *Automatica* **14**, 473–480.

Sreedhar, J. and P. Van Dooren (1997). Forward/backward decomposition of periodic descriptor systems. *Proc. ECC, Brussels, Belgium.*

Sreedhar, J. and P. Van Dooren (1999). Periodic descriptor systems: Solvability and conditionability. *IEEE Trans. Automat. Control* **44**(2), 310–313.

Sreedhar, J., P. Van Dooren and P. Misra (1999). Minimal order time invariant representation of periodic descriptor systems. *Proc. of ACC.* pp. 1309–1313.

Van Dooren, P. (1981). The generalized eigenstructure problem in linear systems theory. *IEEE Trans. Automat. Control* **26**, 111–129.

Varga, A. (1990). Computation of irreducible generalized state-space realizations. *Kybernetika* **26**, 89–106.

Varga, A. (1997). Periodic Lyapunov equations: some applications and new algorithms. *Int. J. Control* **67**, 69–87.

Varga, A. (1999). Balancing related methods for minimal realization of periodic systems. *Systems & Control Lett.* **36**, 339–349.

Varga, A. (2003). Computation of transfer functions matrices of periodic systems. *Int. J. Control* **76**, 1712–1723.

Varga, A. (2004*a*). Computation of Kalman decompositions of periodic systems. *European Journal of Control* **10**, 1–8.

Varga, A. (2004*b*). Computation of Kronecker-like forms of periodic matrix pairs. *Proc. of MTNS'04, Leuven, Belgium.*

Varga, A. (2004*c*). Computation of minimal periodic realizations of transfer-function matrices. *IEEE Trans. Automat. Control* **46**, 146–149.

Varga, A. (2004*d*). Design of fault detection filters for periodic systems. *Proc. of CDC'04, Paradise Island, Bahamas.*

Varga, A. (2006). Computation of $\mathcal{L}_\infty$-norm of linear discrete-time periodic systems. *Proc. of MTNS'06, Kyoto, Japan.*

Varga, A. and P. Van Dooren (2001). Computational methods for periodic systems - an overview. *Proc. of IFAC Workshop on Periodic Control Systems, Como, Italy*, pp. 171–176.

Varga, A. and P. Van Dooren (2003). Computing the zeros of periodic descriptor systems. *Systems & Control Lett.* **50**, 371–381.