Computation of minimal order dynamic covers for periodic systems

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***Abstract*— Minimal dimension dynamic covers play an important role in solving the structural synthesis problems of minimum order functional observers or fault detectors, or in computing minimal order inverses or minimal degree solutions of model matching problems. We propose numerically reliable algorithms to compute two basic types of minimal dimension dynamic covers for a linear periodic system. The proposed approach is based on a special reachability staircase condensed form, which can be computed using exclusively periodic orthogonal similarity transformations. Using such a condensed form minimal dimension periodic covers and corresponding periodic feedback/feedforward matrices can be easily computed. The overall algorithm has a satisfactory computational complexity and is provably numerically reliable.**

I. INTRODUCTION

We consider linear periodic time-varying systems of the form

$$\begin{array}{rcl} x(k+1) & = & A_k x(k) + B_k u(k) \\ y(k) & = & C_k x(k) + D_k u(k) \end{array} \qquad (1)$$

where the matrices $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m}$, $C_k \in \mathbb{R}^{p\times n_k}$, $D_k \in \mathbb{R}^{p\times m_k}$ are periodic with period $N \geq 1$. The periodic system (1) will be alternatively denoted by the periodic quadruple $\mathcal{S} := (A_k, B_k, C_k, D_k)$. The vector of state-dimensions $\mathbf{n} = [\, n_1,\ n_2,\ \cdots, n_N\,]$ characterizes the state-space order of the periodic system. The series and parallel coupling of two periodic systems $\mathcal{S}_1$ of order $\mathbf{n}_1$ and $\mathcal{S}_2$ of order $\mathbf{n}_2$ we denote simply with $\mathcal{S}_1 \star \mathcal{S}_2$ and $\mathcal{S}_1 \oplus \mathcal{S}_2$, respectively, and both have orders $\mathbf{n}_1 + \mathbf{n}_2$.

The main motivation to address the computational aspects of determining minimal dimension periodic dynamic covers is the solution of the least order detector design problem formulated in [1]. The approach suggested in [1] extends to periodic case the design method of least order fault detectors for standard systems proposed in [2]. The main computation consists of determining a *Type II* minimal dynamic cover by using reliable numerical algorithms proposed in [3]. Recently, an alternative solution to the least order design problem for standard systems has been proposed [4] using *Type I* dynamic covers. It is aimed to extend this approach to periodic systems by developing analogous algorithms to compute periodic minimal dynamic covers.

The basic computation in the above problems consists of determining, for given periodic systems $\mathcal{S}_1$ and $\mathcal{S}_2$ with the same number of outputs, an appropriate periodic system $\widetilde{\mathcal{S}}$ such that $\mathcal{S}_1 \oplus (\mathcal{S}_2 \star \widetilde{\mathcal{S}})$ has the least possible state-space order. Assume that $\mathcal{S}_1 = (A_k, B_{k,1}, C_k, D_{k,1})$ and $\mathcal{S}_2 = (A_k, B_{k,2}, C_k, D_{k,2})$, such that the compound system $(A_k, [\, B_{k,1}\ \ B_{k,2}\,], C_k, [\, D_{k,1}\ \ D_{k,2}\,])$ is a minimal state-space realization. In analogy with the results of [5] for standard systems, we recast the above problem to compute a periodic state feedback matrix F_k and a periodic feedforward matrix G_k to achieve that the system $\mathcal{S}^{F,G} := (A_k + B_{k,1}F_k, B_{k,1}G_k + B_{k,2}, C_k + D_{k,1}F_k, D_{k,1}G_k + D_{k,2})$ becomes maximally unreachable. Different instances of this problem for standard systems appear in solving various structural synthesis problems, as for example, the design of minimum order functional observers [6], determining minimal order inverses [7] or computation of minimal degree solutions of rational equations [5], [8], with important applications in fault detection [2], [9], [10]. In all these cases, the proposed solution procedures reformulate these problems as minimum dynamic cover problems. A prerequisite for extending these results to the periodic case is the availability of similar computational tools.

The computational problem of determining minimal order dynamic covers for standard state space systems has been recently addressed in [3]. The proposed computational algorithm is essentially a modified staircase reachability form computation as that proposed in [11]. A similar algorithm for periodic systems has been proposed recently [12], and this algorithm serves as basis to develop a similar cover design algorithm for periodic systems.

In this paper we propose a numerically reliable and computationally efficient approach to compute a periodic feedback matrix F_k and a possibly nonzero periodic feedforward matrix G_k to achieve the largest reduction of state-order of $\mathcal{S}^{F,G}$. We solve the problems of determining both F_k and G_k or only F_k which lead to largest reduction of state-order. We solve these problems by computing periodic orthogonal bases for periodic subspaces representing minimal dimension dynamic covers of *Type II* and *Type I*, respectively (according to the terminology of [6] for standard systems). The main computational ingredient in these computations is bringing the system matrices into special condensed forms which exhibit the structural information necessary to solve the problem. For the matrices in these condensed forms the computation of appropriate F_k and G_k is a simple, almost trivial task, and a minimal realization for $\mathcal{S}^{F,G}$ can be simply determined without additional computations.

The algorithm to compute the condensed form has two stages: (1) an orthogonal reduction of the structured periodic pair $(A_k, [\, B_{k,1}\, B_{k,2}\,])$ to a special reachability staircase form; and (2) a non-orthogonal transformation to zero additionally a minimum number of elements followed by special

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row/column block permutations. The orthogonal reduction part is based on employing techniques similar to those used in the reachability staircase form algorithms for periodic systems [12]. This part involves many rank decisions which can be computed by using reliable techniques (e.g., singular values based rank evaluations). The non-orthogonal part of the reduction does not involve any rank computations and is performed to allow an easy computation of appropriate feedback/feedforward matrices. The overall algorithm has a satisfactory computational complexity and is provably numerically reliable.

In the last part we also address shortly the solution of minimum cover problems with stability constraints. In the case the minimum cover problem with stabilization is solvable, we propose a reliable computational solution to this problem by exploiting the existing parametric freedom in the cover determination problem.

II. Computation of Type II periodic minimal dynamic covers

The computational problem which we solve is the following: given the periodic pair (A_k, B_k) with $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m}$, and B_k partitioned as $B_k = [\, B_{k,1}\ B_{k,2}\,]$ with $B_{k,1} \in \mathbb{R}^{n_{k+1}\times m_1}$, $B_{k,2} \in \mathbb{R}^{n_{k+1}\times m_2}$, determine the periodic matrices F_k and G_k such that the periodic pair $(A_k + B_{k,1}F_k, B_{k,1}G_k + B_{k,2})$ is maximally unreachable. This problem can be recast (see [5] for the standard case) to compute a periodic subspace $\mathcal{V}_k$ having least possible dimension satisfying

$$\begin{array}{rcl} (A_k + B_{k,1}F_k)\mathcal{V}_k & \subset & \mathcal{V}_{k+1} \\ \operatorname{span}(B_{k,1}G_k + B_{k,2}) & \subset & \mathcal{V}_{k+1} \end{array}$$

If we denote $\mathcal{B}_{k,1} = \operatorname{span} B_{k,1}$ and $\mathcal{B}_{k,2} = \operatorname{span} B_{k,2}$, then the above conditions can be rewritten also as conditions defining a *Type II* dynamic cover (see [13], [6])

$$\begin{array}{rcl} A_k\mathcal{V}_k & \subset & \mathcal{V}_{k+1} + \mathcal{B}_{k,1} \\ \mathcal{B}_{k,2} & \subset & \mathcal{V}_{k+1} + \mathcal{B}_{k,1} \end{array} \tag{2}$$

The computation of the minimal dynamic covers relies on the reduction of the periodic pair $(A_k, [\, B_{k,1}\ B_{k,2}\,])$ to a particular condensed form, for which the solution of the problem is simple. This reduction is performed in two stages. The first stage is an orthogonal reduction which represents a particular instance of the reachability staircase procedure of [11] applied to the periodic pair $(A_k, [\, B_{k,1}\ B_{k,2}\,])$. This procedure can be seen as an extension to the periodic case of the basis selection approach of [6] by employing only orthogonal transformation and therefore will be useful to construct both *Type II* and *Type I* minimal covers. In the second stage, additional zero blocks are generated in the reduced matrices using non-orthogonal transformations and by applying appropriate feedback and feedforward matrices. From the resulting overall periodic transformation matrix, a periodic basis for the minimum dynamic cover can be easily obtained. In what follows we present in detail these two stages.

Stage I: Special Reachability Staircase Algorithm

1. Set $j=1$, $r_k=0$, $t=2$, $\nu_{k,1}^{(0)} = m_1$, $\nu_{k,2}^{(0)} = m_2$, $A_k^{(0)} = A_k$, $B_{k,1}^{(0)} = B_{k,1}$, $B_{k,2}^{(0)} = B_{k,2}$, $Z_k = I_{n_k}$ for $k = 1, \ldots, N$.
2. For $k = 1, \ldots, N$, compute the orthogonal matrix $U_{k+1,1}$ to compress the matrix $B_{k,1}^{(j-1)} \in \mathbb{R}^{(n_{k+1}-r_{k+1})\times \nu_{k,1}^{(j-1)}}$ to a full row rank matrix
$$U_{k+1,1}^T B_{k,1}^{(j-1)} := \begin{array}{c} \left[\begin{array}{c} A_{k;t-1,t-3} \\ 0 \end{array}\right] \begin{array}{l} \nu_{k+1,1}^{(j)} \\ \rho_{k+1,1}^{(j)} \end{array} \\ \nu_{k,1}^{(j-1)} \end{array}$$
3. For $k = 1, \ldots, N$, compute $U_{k+1,1}^T B_{k,2}^{(j-1)}$ and partition it in the form
$$U_{k+1,1}^T B_{k,2}^{(j-1)} := \begin{array}{c} \left[\begin{array}{c} A_{k;t-1,t-2} \\ X_k \end{array}\right] \begin{array}{l} \nu_{k+1,1}^{(j)} \\ \rho_{k+1,1}^{(j)} \end{array} \\ \nu_{k,2}^{(j-1)} \end{array}$$
4. For $k = 1, \ldots, N$, compute the orthogonal matrix $U_{k+1,2}$ to compress the matrix $X_k \in \mathbb{R}^{(n_{k+1}-r_{k+1}-\nu_{k+1,1}^{(j)})\times \nu_{k,2}^{(j-1)}}$ to a full row rank matrix
$$U_{k+1,2}^T X_k := \begin{array}{c} \left[\begin{array}{c} A_{k;t,t-2} \\ 0 \end{array}\right] \begin{array}{l} \nu_{k+1,2}^{(j)} \\ \rho_{k+1,2}^{(j)} \end{array} \\ \nu_{k,2}^{(j-1)} \end{array}$$
5. For $k = 1, \ldots, N$, compute the transformed matrix $\operatorname{diag}(I, U_{k+1,2}^T)U_{k+1,1}^T A_k^{(j-1)} U_{k,1}\operatorname{diag}(I, U_{k,2})$ and partition it in the form
$$\begin{array}{c} \left[\begin{array}{ccc} A_{k;t-1,t-1} & A_{k;t-1,t} & A_{k;t-1,t+1} \\ A_{k;t,t-1} & A_{k;t,t} & A_{k;t,t+1} \\ B_{k,1}^{(j)} & B_{k,2}^{(j)} & A_k^{(j)} \end{array}\right] \begin{array}{l} \nu_{k+1,1}^{(j)} \\ \nu_{k+1,2}^{(j)} \\ \rho_{k+1,2}^{(j)} \end{array} \\ \begin{array}{ccc} \nu_{k,1}^{(j)} & \nu_{k,2}^{(j)} & \rho_{k,2}^{(j)} \end{array} \end{array}$$
6. For $k = 1, \ldots, N$, compute for $i = 1, \ldots, t-2$
$$A_{k;i,t-1}U_{k,1}\operatorname{diag}(I, U_{k,2}) := \begin{array}{c} [\, A_{k;i,t-1}\ \ A_{k;i,t}\ \ A_{k;i,t+1}\,] \\ \nu_{k,1}^{(j)}\quad \nu_{k,2}^{(j)}\quad \rho_{k,2}^{(j)} \end{array}$$
7. For $k = 1, \ldots, N$
$$Z_k \leftarrow Z_k \operatorname{diag}(I_{r_k}, U_{k,1})\operatorname{diag}(I_{r_k+\nu_{k,1}^{(j)}}, U_{k,2})$$
8. For $k = 1, \ldots, N$, $r_k \leftarrow r_k + \nu_{k,1}^{(j)} + \nu_{k,2}^{(j)}$; if $\rho_{k,2}^{(j)} = 0$ for $k = 1, \ldots, N$, then $\ell = j$ and **Exit 1**.
9. If $\nu_{k,1}^{(j)} + \nu_{k,2}^{(j)} = 0$ for $k = 1, \ldots, N$, then $t \leftarrow t-2$, $\ell = j-1$, **Exit 2**; else, $j \leftarrow j+1$, $t \leftarrow t+2$, and go to Step 2.

At the end of this algorithm $\widehat{A}_k := Z_{k+1}^T A_k Z_k$ and $\widehat{B}_k := Z_{k+1}^T B_k$ have the following forms

$$\widehat{A}_k = \begin{array}{c} \left[\begin{array}{cc} A_k^r & * \\ O & A_k^{\bar{r}} \end{array}\right] \begin{array}{l} r_{k+1} \\ n_{k+1} - r_{k+1} \end{array} \\ \begin{array}{cc} r_k & n_k - r_k \end{array} \end{array},$$

$$\widehat{B}_k = \left[\begin{array}{c} B_k^r \\ O \end{array}\right] \begin{array}{l} r_{k+1} \\ n_{k+1} - r_{k+1} \end{array}$$

where the periodic pair (A_k^r, B_k^r) is reachable, and $A_k^{\bar{r}}$ is the unreachable part of A_k. The pair (A_k^r, B_k^r) is in the special staircase form with

$$[\, B_k^r \,|\, A_k^r \,] = \left[\begin{array}{cc|cccccc} A_{k;1,-1} & A_{k;1,0} & A_{k;11} & A_{k;12} & \cdots & A_{k;1,2\ell-2} & * & * \\ O & A_{k;2,0} & A_{k;21} & A_{k;22} & \cdots & A_{k;2,2\ell-2} & * & * \\ O & O & A_{k;31} & A_{k;32} & \cdots & A_{k;3,2\ell-2} & * & * \\ O & O & O & A_{k;42} & \cdots & A_{k;4,2\ell-2} & * & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \cdots & A_{k;2\ell,2\ell-2} & * & * \end{array}\right]$$

where $A_{k;2j-1,2j-3} \in \mathbb{R}^{\nu_{k+1,1}^{(j)} \times \nu_{k,1}^{(j-1)}}$ and $A_{k;2j,2j-2} \in \mathbb{R}^{\nu_{k+1,2}^{(j)} \times \nu_{k,2}^{(j-1)}}$ are full row rank matrices for $j = 1, \ldots, \ell$.

To compute a *Type II* minimal cover, in the second reduction stage we use non-orthogonal upper triangular periodic transformation matrices $U_k = \mathrm{diag}\,(U_k^r, I_{n_k - r_k})$ to annihilate a minimum set of blocks in A_k^r. Assume U_k^r has an upper block diagonal structure, with the diagonal blocks identity matrices of the same size as the column dimensions of the blocks $A_{k;ii}$ of A_k^r. Moreover the supra-diagonal block structure U_k^r corresponds to that of A_k^r and the corresponding blocks are denoted similarly. The following procedure performs the second reduction stage by exploiting the full row rank of submatrices $A_{k;2t-1,2t-3}$ to zero the blocks $A_{k;2t-1,2j}$, for $j = t-1, t, \ldots, \ell$ in block row $2t-1$ of A_k^r.

Stage II: Special reduction for Type II Covers

For $k = 1, \ldots, N$, set $U_k^r = I_{r_k}$.

for $t = \ell, \ell - 1, \ldots, 2$

  for $j = t-1, t, \ldots, \ell$

   For $k = 1, \ldots, N$ compute $U_{k;2t-3,2j}$ such that

$$A_{k;2t-1,2t-3} U_{k;2t-3,2j} + A_{k;2t-1,2j} = 0$$

   for $k = 1, \ldots, N$

    For $i = 1, 2, \ldots, 2t-1$ compute

$$A_{k;i,2j} \leftarrow A_{k;i,2j} + A_{k;i,2t-3} U_{k;2t-3,2j}$$

    For $i = 2j-2, \ldots, 2\ell$ compute

$$A_{k;2t-3,i} \leftarrow A_{k;2t-3,i} - U_{k+1;2t-3,2j} A_{k;2j,i}$$

   end

  end

end

At the end of Stage II, the upper triangular periodic matrix U_k contains the accumulated non-orthogonal transformations performed in the reduction. Let $\widetilde{A}_k := U_{k+1}^{-1} \widehat{A}_k U_k$, and $\widetilde{B}_k = [\, \widetilde{B}_{k,1} \;\; \widetilde{B}_{k,2} \,] := U_{k+1}^{-1} \widehat{B}_k$ be the system matrices resulted at the end of Stage II. Define also the periodic feedback matrix $\widetilde{F}_k^r \in \mathbb{R}^{m_1 \times r_k}$ partitioned column-wise compatibly with $\widetilde{A}_k$

$$\widetilde{F}_k^r = [\, O \; F_{k,2} \; \cdots \; F_{k,2\ell-2} \; O \; F_{k,2\ell} \,]$$

where $F_{k,2j}$ are chosen such that $A_{k;1,-1} F_{k,2j} + A_{k;1,2j} = 0$ for $j = 1, \ldots, \ell$. Choose also G_k such that $A_{k;1,-1} G_k + A_{k;1,0} = 0$. These choices are always possible since $A_{k;1,-1}$ has full row rank.

With the computed $\widetilde{F}_k$ and G_k we achieved that

$$\left[\, \widetilde{B}_{k,1}^r \;\; \widetilde{B}_{k,1}^r G_k + \widetilde{B}_{k,2}^r \,\right] = \left[\begin{array}{cc} A_{k;1,-1} & O \\ O & A_{k;2,0} \\ O & O \\ O & O \\ \vdots & \vdots \\ O & O \end{array}\right]$$

$$\widetilde{A}_k^r + \widetilde{B}_{k,1}^r \widetilde{F}_k^r = \left[\begin{array}{cccccc} \overline{A}_{k;11} & O & \cdots & O & * & O \\ \overline{A}_{k;21} & \overline{A}_{k;22} & \cdots & \overline{A}_{k;2,2\ell-2} & * & * \\ A_{k;31} & O & \cdots & O & * & O \\ O & A_{k;42} & \cdots & \overline{A}_{k;4,2\ell-2} & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & \cdots & A_{k;2\ell,2\ell-2} & * & * \end{array}\right]$$

where the elements with bars have been modified in Stage II.

Consider now the permutation matrix defined by

$$P_k^T = \left[\begin{array}{cc|c|cc|c} O & I_{\nu_{k,2}^{(1)}} & \cdots & O & O & O \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline O & O & \cdots & O & I_{\nu_{k,2}^{(\ell)}} & O \\ \hline I_{\nu_{k,1}^{(1)}} & O & \cdots & O & O & O \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline O & O & \cdots & I_{\nu_{k,1}^{(\ell)}} & O & O \\ \hline O & O & \cdots & O & O & I_{n_k - r_k} \end{array}\right] \qquad (3)$$

If we define $V_k = Z_k U_k P_k$ and $F_k = [\, \widetilde{F}_k \;\; 0 \,] V_k^{-1}$, then overall we achieved that

$$V_{k+1}^{-1}(B_{k,1} G_k + B_{k,2}) = \left[\begin{array}{c} \breve{B}_{k,1} \\ O \\ O \end{array}\right]$$

$$V_{k+1}^{-1}(A_k + B_{k,1} F_k) V_k = \left[\begin{array}{ccc} \breve{A}_{k,1} & * & * \\ O & \breve{A}_{k,2} & * \\ O & O & A_k^{\bar{r}} \end{array}\right],$$

where

$$[\,\breve{B}_{k,1}\,|\,\breve{A}_{k,1}\,] = \left[\begin{array}{c|ccccc} A_{k;2,0} & \overline{A}_{k;2,2} & \overline{A}_{k;2,4} & \cdots & & \overline{A}_{k;2,2\ell} \\ O & A_{k;4,2} & \overline{A}_{k;4,4} & \cdots & & \overline{A}_{k;4,2\ell} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ O & O & O & A_{k;2\ell,2\ell-2} & & A_{k;2\ell,2\ell} \end{array}\right]$$

$$\breve{A}_{k,2} = \left[\begin{array}{cccc} \overline{A}_{k;1,1} & \overline{A}_{k;1,3} & \cdots & \overline{A}_{k;1,2\ell-1} \\ A_{k;3,1} & \overline{A}_{k;3,3} & \cdots & \overline{A}_{k;3,2\ell-1} \\ \vdots & \ddots & \ddots & \vdots \\ O & O & A_{k;2\ell-1,2\ell-3} & \overline{A}_{k;2\ell-1,2\ell-1} \end{array}\right]$$

It follows by inspection that the periodic pair $(\breve{A}_{k,1}, \breve{B}_{k,1})$ is reachable. Thus, by the above choice of F_k and G_k, the dimension of k-th reachability subspace has been reduced by $\sum_{i=1}^{\ell} \nu_{k,1}^{(i)}$. The first $n_k^r := \sum_{i=1}^{\ell} \nu_{k,2}^{(i)}$ columns $V_{k,1}$ of V_k satisfy

$$A_k V_{k,1} = V_{k+1,1}\breve{A}_{k,1} - B_{k,1}F_k V_{k,1},$$

$$B_{k,2} = V_{k+1,1}\breve{B}_{k,1} - B_{k,1}G_k$$

and thus, according to (2), span a *Type II* periodic dynamic cover of dimension n_k^r for the pair $(A_k, [\,B_{k,1}\ B_{k,2}\,])$. Using arguments similar to the standard case (see [6]), the following result can be shown:

Theorem 1: The *Type II* periodic dynamic cover $\mathcal{V} = \operatorname{span} V_{k,1}$ has minimum dimension $\mathbf{n}^r$.

III. Computation of Type I periodic minimal dynamic covers

The computational problem which we solve in this section is the following: given the periodic pair (A_k, B_k) with $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m}$, and B_k partitioned as $B_k = [\,B_{k,1}\ B_{k,2}\,]$ with $B_{k,1} \in \mathbb{R}^{n_{k+1}\times m_1}$, $B_{k,2} \in \mathbb{R}^{n_{k+1}\times m_2}$, determine the periodic state feedback matrix F_k such that the pair $(A_k + B_{k,2}F_k, B_{k,1})$ is maximally unreachable. This problem is equivalent to compute a periodic subspace $\mathcal{V}_k$ having least possible dimension satisfying

$$\begin{array}{rcl} (A_k + B_{k,2}F_k)\mathcal{V}_k & \subset & \mathcal{V}_{k+1} \\ \operatorname{span} B_{k,1} & \subset & \mathcal{V}_{k+1} \end{array}$$

These conditions can be rewritten also as conditions defining a *Type I* periodic minimum dynamic cover which are similar to the standard case [13], [6]

$$\begin{array}{rcl} A_k\mathcal{V}_k & \subset & \mathcal{V}_{k+1} + \mathcal{B}_{k,2} \\ \mathcal{B}_{k,1} & \subset & \mathcal{V}_{k+1} \end{array} \qquad (4)$$

To compute Type I covers, we perform first the Stage I orthogonal reduction on the periodic pair $(A_k, [B_{k,1}, B_{k,2}])$, as done in the previous section. However, at Stage II the non-orthogonal reduction annihilates a different set of blocks in A_k^r. The following procedure performs the second reduction stage by exploiting the full row rank of submatrices $A_{2t,2t-2}$ to zero the blocks $A_{2t,2j-1}$, for $j = t, t+1, \ldots, \ell$ in row $2t$ of A_k^r.

Stage II: Special reduction for Type I Covers

For $k = 1, \ldots, N$, set $U_k^r = I_{r_k}$.

for $t = \ell, \ell-1, \ldots, 2$

  for $j = t, t+1, \ldots, \ell$

    For $k = 1, \ldots, N$ compute $U_{k;2t-2,2j-1}$ such that

$$A_{k;2t,2t-2}U_{k;2t-2,2j-1} + A_{k;2t,2j-1} = 0$$

    for $k = 1, \ldots, N$,

      For $i = 1, 2, \ldots, 2t$ compute

$$A_{k;i,2j-1} \leftarrow A_{k;i,2j-1} + A_{k;i,2t-2}U_{k;2t-2,2j-1}$$

      For $i = 2j-3, \ldots, 2\ell$ compute

$$A_{k;2t-2,i} \leftarrow A_{k;2t-2,i} - U_{k+1;2t-2,2j-1}A_{k;2j-1,i}$$

    end

  end

end

Let $\widetilde{A}_k := U_{k+1}^{-1}\widehat{A}_k U_k$ and $\widetilde{B}_k = [\,\widetilde{B}_{k,1}\ \widetilde{B}_{k,2}\,] := U_{k+1}^{-1}\widehat{B}_k$ be the system matrices resulted at the end of Stage II. Define also the feedback matrix $\widetilde{F}_k \in \mathbb{R}^{m_2\times n_k}$ partitioned column-wise compatibly with $\widehat{A}_k$

$$\widetilde{F}_k = [\,F_{k,1}\ O\ F_{k,3}\ \cdots\ O\ F_{k,2\ell-1}\ O\ \ O\,]$$

where $F_{k,2j-1}$ are such that $A_{k;2,0}F_{k,2j-1} + A_{k;2,2j-1} = 0$ for $j = 1, \ldots, \ell$.

Consider now the permutation matrix defined by

$$P_k^T = \left[\begin{array}{cc|c|cc|c} I_{\nu_{k,1}^{(1)}} & O & \cdots & O & O & O \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline O & O & \cdots & I_{\nu_{k,1}^{(\ell)}} & O & O \\ \hline O & I_{\nu_{k,2}^{(1)}} & \cdots & O & O & O \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline O & O & \cdots & O & I_{\nu_{k,2}^{(\ell)}} & O \\ \hline O & O & \cdots & O & O & I_{n_k - r_k} \end{array}\right]$$

If we define $V_k = Z_k U_k P_k$ and $F_k = \widetilde{F}_k V_k^{-1}$, then overall we achieved that

$$V_{k+1}^{-1}B_{k,1} = \left[\begin{array}{c} \breve{B}_{k,1} \\ O \\ O \end{array}\right]$$

$$V_{k+1}^{-1}(A_k + B_{k,2}F_k)V_k = \left[\begin{array}{ccc} \breve{A}_{k,1} & * & * \\ O & \breve{A}_{k,2} & * \\ O & O & A_k^{\bar{r}} \end{array}\right],$$

where

$$[\,\breve{B}_{k,1}\,|\,\breve{A}_{k,1}\,] = \left[\begin{array}{c|cccc} A_{k;1,-1} & \overline{A}_{k;1,1} & \overline{A}_{k;1,3} & \cdots & * \\ O & A_{k;3,1} & \overline{A}_{k;3,3} & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & A_{k;2\ell-1,2\ell-3} & * \end{array}\right]$$

$$\breve{A}_{k,2} = \left[\begin{array}{cccc} \overline{A}_{k;2,2} & \overline{A}_{k;2,4} & \cdots & \overline{A}_{k;2,2\ell} \\ A_{k;4,2} & \overline{A}_{k;4,4} & \cdots & \overline{A}_{k;4,2\ell} \\ \vdots & \ddots & \ddots & \vdots \\ O & O & A_{k;2\ell,2\ell-2} & A_{k;2\ell,2\ell} \end{array}\right]$$

It follows by inspection that the pair $(\breve{A}_{k,1}, \breve{B}_{k,1})$ is reachable. Thus, by the above choice of F_k, we reduced by $\sum_{i=1}^{\ell} \nu_{k,2}^{(i)}$ the dimension of the k-th reachable subspace. The first $n_k^r := \sum_{i=1}^{\ell} \nu_{k,1}^{(i)}$ columns $V_{k,1}$ of V_k satisfy

$$A_k V_{k,1} = V_{k+1,1}\breve{A}_{k,1} - B_{k,2}F_k V_{k,1}, \quad B_{k,1} = V_{k+1,1}\breve{B}_{k,1}$$

and thus span a *Type I* periodic dynamic cover of dimension n_k^r for the pair $(A_k, [\,B_{k,2}\,B_{k,1}\,])$. The following result can be shown extending the results of [6]:

Theorem 2: The *Type I* periodic dynamic cover $\mathcal{V}_k = \operatorname{span} V_{k,1}$ has minimum dimension $\mathbf{n}^r$.

IV. NUMERICAL ASPECTS

The key reduction of system matrices to the special reachability form can be performed by using exclusively periodic orthogonal Lyapunov transformations. It can be shown that the computed condensed matrices $\widehat{A}_k$ and $\widehat{B}_k$ are exact for matrices which are nearby to the original matrices A_k and B_k, respectively. Thus this part of the reduction is *numerically backward stable*. In implementing the algorithm, the row compressions are usually performed using rank revealing QR-factorizations with column pivoting [14]. To make rank determinations even more reliable, QR-decompositions and singular value decompositions can be combined (see [11]).

The rank revealing QR-decomposition is performed by employing Householder transformations, and these transformations are immediately applied to B_k, A_k and Z_k, without accumulating them in $U_{k,1}$ and $U_{k,2}$. Thus, the reduction is essentially the same as that required to compute the periodic Hessenberg form of the periodic matrix A_k, which amounts in worst case to $7/3n^3N$ floating-point operations (flops), where n is the maximum state dimension. Note that for solving the motivating original problem, the accumulation of Z_k is not even necessary, since all right transformations can be directly applied to C_k.

The computations at Stage II to determine a basis for the minimal dynamic cover and the computation of feedback/feedforward matrices involve the solution of many, generally overdetermined, linear equations. For the computation of the basis for $\mathcal{V}_k$, we can estimate the condition numbers of the overall transformation matrix by computing $\|V_k\|_F^2 = \|U_k\|_F^2$. If this norm is relatively small (e.g., $\|V_k\|_F^2 \le 10000$) then practically there is no danger for a significant loss of accuracy due to nonorthogonal reduction. Note that it is very important to compute these condition numbers, since large values of them provide a clear hint of *possible* accuracy losses. In practice, it suffices to look at the largest magnitudes of elements of U_k used at Stage II to obtain equivalent information. For the computation of the feedback/feedforward matrices, condition numbers for solving the underlying equations can be also easily estimated. For the Stage II reduction, a simple operation count is possible by assuming all blocks 1×1 and this amounts to about $n^3/4N$ flops.

V. MINIMUM COVERS WITH STABILIZATION

In some applications it is important to achieve simultaneously that the resulting feedback is stabilizing. For a Type II cover, this amounts to determine F_k, G_k and V_k such that the resulting periodic $\breve{A}_{k,1}$ has characteristic values in an appropriate stability domain $\mathbb{C}^-$ (e.g., interior of the unit circle). This goal can not always be achieved, but it is always possible to move a maximum number of characteristic values in this domain. To show how this is possible, consider the matrix pair $(P_{k+1}^T\widetilde{A}_k P_k, P_{k+1}^T\widetilde{B}_k)$, where $\widetilde{A}_k$ and $\widetilde{B}_k$ are the resulting matrices at the end of Stage II and P_k^T is the permutation matrix (3). The matrices of this pair have the form

$$P_{k+1}^T\widetilde{B}_k = \left[\begin{array}{cc} O & \widetilde{B}_{k;12} \\ \widetilde{B}_{k;21} & \widetilde{B}_{k;22} \\ O & O \\ \hline O & O \end{array}\right]$$

$$P_{k+1}^T\widetilde{A}_k P_k = \left[\begin{array}{ccc|c} \widetilde{A}_{k;11} & \widetilde{A}_{k;12} & \widetilde{A}_{k;13} & * \\ \widetilde{A}_{k;21} & \widetilde{A}_{k;22} & \widetilde{A}_{k;23} & * \\ O & \widetilde{A}_{k;32} & \widetilde{A}_{k;33} & * \\ \hline O & O & O & A_k^r \end{array}\right]$$

where the periodic pair $(\widetilde{A}_{k;11}, \widetilde{B}_{k;12})$ is reachable, and $\widetilde{B}_{k;21}$ has full row rank. Note that the Stage II special reduction achieves basically to zero the blocks $\widetilde{A}_{k;31}$, while the periodic feedback matrix F_k and feedforward matrix G_k achieve additionally to zero $\widetilde{A}_{k;21}$ and $\widetilde{B}_{k;22}$, respectively, by exploiting the full rank property of $\widetilde{B}_{k;21}$.

Consider the transformation matrix

$$T_k = \left[\begin{array}{ccc|c} I & O & O & O \\ X_k & I & O & O \\ O & O & I & O \\ \hline O & O & O & I \end{array}\right]$$

partitioned in accordance with the structure of $P_{k+1}^T\widetilde{A}_k P_k$. It follows that

$$T_{k+1}^{-1}P_{k+1}^T\widetilde{B}_k = \left[\begin{array}{cc} O & \widetilde{B}_{k;12} \\ \widetilde{B}_{k;21} & \overline{B}_{k;22} \\ O & O \\ \hline O & O \end{array}\right]$$

$$T_{k+1}^{-1}P_{k+1}^T\widetilde{A}_k P_k T_k = \left[\begin{array}{ccc|c} \widetilde{A}_{k;11} + \widetilde{A}_{k;12}X_k & \widetilde{A}_{k;12} & \widetilde{A}_{k;13} & * \\ \overline{A}_{k;21} & \overline{A}_{k;22} & \overline{A}_{k;23} & * \\ \widetilde{A}_{k;32}X_k & \widetilde{A}_{k;32} & \widetilde{A}_{k;33} & * \\ \hline O & O & O & A_k^r \end{array}\right]$$

where we denoted with bars the changed quantities. If we choose X_k such that $\widetilde{A}_{k;32}X_k = 0$, we can preserve the structure of the original pair $(P_{k+1}^T\widetilde{A}_kP_k, P_{k+1}^T\widetilde{B}_k)$. Thus, defining V_k as $V = Z_kU_kP_kT_k$, we can compute the feedback and feedforward matrices F_k and G_k exactly as before.

With T_k chosen as above, the resulting $\breve{A}_{k,1}$ is $\widetilde{A}_{k;11} + \widetilde{A}_{k;12}X_k$ and we can try to exploit this parametric freedom to move the characteristic values of $\widetilde{A}_{k;11}$ to stable locations. The following straightforward computations are necessary for this purpose:

1) For $k = 1,\dots,N$, compute X_k^N with orthonormal columns such that span X_k^N is the right nullspace of $\widetilde{A}_{k;32}$.
2) Compute a periodic $\widetilde{F}_k$ to place a maximum number of characteristic values of $\widetilde{A}_{k;11}+\widetilde{A}_{k;12}X_k^N\widetilde{F}_k$ into the stability domain $\mathbb{C}^-$.
3) Define $X_k = X_k^N\widetilde{F}_k$.

All steps of this algorithms can be performed using numerically reliable computations. The computation of X_k^N is straightforward, since $\widetilde{A}_{k;32}$ is part of a staircase form. Thus, no further rank determination is necessary and X_k^N results from an RQ-like decomposition of $\widetilde{A}_{k;32}$ which exploits the full row rank of its leading nonzero rows. To determine $\widetilde{F}_k$, the most appropriate method is to apply a partial pole assignment technique like that of [15], provided the dimensions of $\widetilde{A}_{k;11}$ are constant. This approach can easily accommodate with non-stabilizable pairs, by moving only the reachable unstable eigenvalues of $\widetilde{A}_{k;11}$ into $\mathbb{C}^-$. If the pair $(\widetilde{A}_{k;11}, \widetilde{A}_{k;12}X_k^N)$ is stabilizable then this algorithm can assign all unstable eigenvalues to arbitrary stable locations using minimum norm local feedbacks. In this way, the norm of X_k is minimized as well and thus also the condition number of the transformation matrix T_k. A similar approach can be devised for determining *Type I* minimal covers with stabilization. If the periodic matrix $\widetilde{A}_{k;11}$ has time-varying dimensions, then an enhanced version of the algorithm of [15] must be still developed.

A specific aspect of determining minimal dynamic covers is the non-uniqueness of the resulting periodic solution triple (F_k, G_k, V_k). This non-uniqueness manifests at several points of the proposed approach and can have negative or positive influence on the stabilizability properties determined by the triple $(\widetilde{A}_{k;11}, \widetilde{A}_{k;12}, \widetilde{A}_{k;32})$. For example, selecting differently at Stage I the linearly independent columns in $B_{k,1}^{(j-1)}$ and $B_{k,2}^{(j-1)}$ or computing differently the blocks of U_k at Stage II when solving the underdetermined linear systems can lead to different minimal covers and different stabilizability properties. For numerical implementations, we recommend those solutions which ensure the best numerical properties of the proposed approach (e.g., selecting independent columns using column pivoting or determining least-norm solutions of all underdetermined linear systems).

VI. Conclusions

We proposed efficient algorithms to compute two types of periodic minimal dynamic covers, which have many potential applications in various structural synthesis problems of linear discrete-time periodic systems. The proposed algorithms rely on the extensive use of orthogonal transformations. The use of non-orthogonal transformations at the final step of the reduction process allows also to obtain a precise estimation of possible accuracy losses induced by the overall reduction. Thus the proposed algorithm, although not numerically stable, can be considered numerically reliable.

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