DG discretizations for compressible flows: Adjoint consistency analysis, error estimation and adaptivity

Ralf Hartmann

1. Adjoint consistency - in addition to consistency - is the key requirement for discontinuous Galerkin (DG) discretizations to be of optimal order in L^2 as well as measured in terms of target functionals $J(\cdot)$. If the primal and adjoint solutions are sufficiently smooth, the order of convergence in J for an adjoint consistent discretization is twice the order of an adjoint inconsistent discretization (order doubling). In this talk we provide a general framework, see [3, 4], for analyzing the adjoint consistency of DG discretizations. We collect several conclusions which can be drawn from analyzing the adjoint consistency property of DG discretizations of the linear advection equation, Poisson's equation and the compressible Euler and Navier-Stokes equations. Consider the linear problem and linear target functional

$$Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma, \tag{1}$$

$$J(u) = \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma\, Cu \,\mathrm{d}s, \tag{2}$$

where L denotes a linear differential operator on Ω, and B and C denote linear differential (boundary) operators on Γ. The target functional $J(\cdot)$ in (2) is said to be *compatible* with (1), provided following compatibility condition holds

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma, \tag{3}$$

where L^*, B^* and C^* denote the adjoint operators to L, B and C. Then the (continuous) adjoint problem associated to (1), (2) is given by

$$L^* z = j_\Omega \quad \text{in } \Omega, \qquad\qquad B^* z = j_\Gamma \quad \text{on } \Gamma. \tag{4}$$

Let (1) be discretized as follows: find $u_h \in V_h$ such that

$$\mathcal{B}(u_h, v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h, \tag{5}$$

Then, the discretization (5) is *consistent* if the exact solution $u \in V$ to the primal problem (1) satisfies: $\mathcal{B}(u, v) = \mathcal{F}(v)$ for all $v \in V$. Similarly, the discretization (5) is *adjoint consistent* if the exact solution $z \in V$ to the continuous adjoint problem (4) satisfies: $\mathcal{B}(w, z) = J(w)$ for all $w \in V$.

Analoguously, for a nonlinear problem and nonlinear target functional

$$Nu = 0 \quad \text{in } \Omega, \qquad\qquad Bu = 0 \quad \text{on } \Gamma, \tag{6}$$

$$J(u) = \int_\Omega j_\Omega(u) \,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma(Cu) \,\mathrm{d}s, \tag{7}$$

where N is a nonlinear differential operator and B is a (possibly nonlinear) boundary operator, the continuous adjoint problem is given by

$$(N'[u])^* z = j'_\Omega[u] \quad \text{in } \Omega, \qquad\qquad (B'[u])^* z = j'_\Gamma[Cu] \quad \text{on } \Gamma. \tag{8}$$

Let $\mathcal{N} : V \times V \to \mathbb{R}$ be a semi-linear form, such that the nonlinear problem (6) is discretized as follows: find $u_h \in V_h$ such that

$$\mathcal{N}(u_h, v_h) = 0 \quad \forall v_h \in V_h. \tag{9}$$

The corresponding discrete adjoint problem is given by: find $z_h \in V_h$ such that

$$\mathcal{N}'[u_h](w_h, z_h) = J'[u_h](w_h) \quad \forall w_h \in V_h, \tag{10}$$

where $\mathcal{N}'[u]$ denotes the Fréchet derivatives of $\mathcal{N}(u,v)$ with respect to u. The discretization (9) is *consistent* if the exact solution $u \in V$ to the primal problem (6) satisfies following equation:

$$\mathcal{N}(u, v) = 0 \quad \forall v \in V. \tag{11}$$

Furthermore, the discretization (9) is *adjoint consistent* if the exact solution $z \in V$ to the adjoint problem (8) satisfies following equation:

$$\mathcal{N}'[u](w, z) = J'[u](w) \quad \forall w \in V, \tag{12}$$

In other words, a discretization is adjoint consistent if the discrete adjoint problem is a consistent discretization of the continuous adjoint problem.

For analysing adjoint consistency we rewrite the discrete adjoint problem (10) in element-based adjoint residual form: find $z_h \in V_h$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \int_\kappa w_h \, R^*[u_h](z_h) \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} w_h \, r^*[u_h](z_h) \, \mathrm{d}s + \int_\Gamma w_h \, r^*_\Gamma[u_h](z_h) \, \mathrm{d}s = 0, \tag{13}$$

for all $w_h \in V_h$, where $R^*[u_h](z_h)$, $r^*[u_h](z_h)$ and $r^*_\Gamma[u_h](z_h)$ denote the adjoint element, interior face and boundary residuals, respectively. Then, the discretization (9) is adjoint consistent if (13) holds also for the exact solutions u and z which is true provided u and z satisfy

$$R^*[u](z) = 0 \ \text{ in } \kappa, \quad r^*[u](z) = 0 \ \text{ on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \quad r^*_\Gamma[u](z) = 0 \ \text{ on } \Gamma. \tag{14}$$

The adjoint problem and consequently the adjoint consistency of a discretization depends on the specific target functional $J(\cdot)$ under consideration. Given a target functional of the form (7), we see that $R^*[u](z)$ depends on $j_\Omega(\cdot)$, and $r^*_\Gamma[u](z)$ depends on $j_\Gamma(\cdot)$. For obtaining an adjoint consistent discretization, it is in some cases necessary to replace $J(\cdot)$ by a modified target functional $\tilde{J}(\cdot)$ which is consistent if $J(u) = \tilde{J}(u)$ holds for the exact solution u. Furthermore, requiring adjoint consistency may have consequences on the discretization of boundary conditions. In the following we give several examples, see [3].

Linear advection-reaction equation: We consider the linear advection-reaction equation and a compatible target functional of the form:

$$\begin{gathered} \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma_-, \\ J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma \, u \, \mathrm{d}s \end{gathered} \tag{15}$$

The discontinuous Galerkin discretization of this problem based on upwind is adjoint consistent and the error $J(u) - J(u_h)$ in the target functional is $\mathcal{O}(h^{2p+1})$ for sufficiently smooth primal and adjoint solutions.

Poisson's equations: We consider the following elliptic model problem and a compatible target functional of the form:

$$\begin{aligned} &-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla u = g_N \quad \text{on } \Gamma_N, \\ &J(u) = \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D\, \mathbf{n}\cdot\nabla u \,\mathrm{d}s + \int_{\Gamma_N} j_N\, u\,\mathrm{d}s. \end{aligned} \tag{16}$$

The non-symmetric interior penalty DG (NIPG) discretization of this problem is adjoint inconsistent and hence the error $J(u) - J(u_h)$ in the target functional is $\mathcal{O}(h^p)$. In contrast to that the error $J(u) - \tilde{J}(u_h)$ of the symmetric version (SIPG) together with following modified target functional, see also [1],

$$\tilde{J}(u_h) = J(u_h) - \int_{\Gamma_D} \delta(u_h - g_D) j_D \,\mathrm{d}s. \tag{17}$$

is $\mathcal{O}(h^{2p})$. We note, that without this so-called *IP modification* of $J(\cdot)$ the discretization is adjoint inconsistent and $\mathcal{O}(h^p)$, only. Here, again, all orders of convergence hold provided the primal and adjoint solutions are sufficiently smooth.

The compressible Euler equations: We consider the compressible Euler equations with slip wall boundary conditions and a compatible target functional:

$$\begin{aligned} &\nabla\cdot\mathcal{F}^c(\mathbf{u}) = 0 \quad \text{in } \Omega, \qquad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_W, \\ &J(\mathbf{u}) = \int_{\Gamma_W} p(\mathbf{u})\, \mathbf{n}\cdot\boldsymbol{\psi}_{\Gamma_W} \,\mathrm{d}s. \end{aligned} \tag{18}$$

Examples of $J(\cdot)$ are the drag and lift coefficient with $\psi_{\Gamma_W} = \frac{1}{C_\infty}\psi$ and $\psi = \psi_d = (\cos(\alpha), \sin(\alpha))^\top$ for the drag and $\psi = \psi_l = (-\sin(\alpha), \cos(\alpha))^\top$ where α is the angle of attack and C_∞ is a constant depending on freestream quantities. DG discretizations of the compressible Euler equations include numerical flux functions $\mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n})$ approximating a Riemann problem connecting the states $\mathbf{u}_h^+$ and $\mathbf{u}_h^-$ between neighboring elements. The standard approach of taking the same numerical flux $\mathcal{H}_\Gamma = \mathcal{H}$ on the boundary Γ as in the interior domain and replacing the exterior value $\mathbf{u}_h^-$ by a boundary value function $\mathbf{u}_\Gamma(\mathbf{u}_h^+)$ like in $\int_\Gamma \mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n})\mathbf{v}\,\mathrm{d}s$ and computing the target functional like given in (18) is adjoint inconsistent. The adjoint consistency analysis reveals that instead using

$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)), \qquad \text{and} \quad \tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)), \tag{19}$$

see also [8], leads to an adjoint consistent discretization.

The compressible Navier-Stokes equations: We consider the compr. Navier-Stokes equations with noslip wall isothermal ($T = T_{\text{wall}}$ on Γ_{iso}) or adiabatic ($\mathbf{n}\cdot\nabla T = 0$ on Γ_{adia}) boundary conditions and a compatible target functional:

$$\begin{aligned} &\nabla\cdot(\mathcal{F}^c(\mathbf{u}) - \mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u})) = 0 \quad \text{in } \Omega, \qquad \mathbf{v} = (v_1, v_2)^\top = 0 \quad \text{on } \Gamma_W = \Gamma_{\text{iso}} \cup \Gamma_{\text{adia}}, \\ &J(\mathbf{u}) = \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n})\cdot\boldsymbol{\psi}_{\Gamma_W} \,\mathrm{d}s. \end{aligned} \tag{20}$$

Examples of $J(\cdot)$ are the total (i.e. pressure induced plus viscous) drag or lift coefficient with ψ_{Γ_W} as defined (18). For obtaining an adjoint consistent discretization, here the boundary terms must be modified analoguous to the boundary terms of the compressible Euler equations. Additionally, if the target functional $J(\cdot)$ in (20) is modified as follows, see [3],

$$\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) + \int_{\Gamma_W} \delta\left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right) \cdot \mathbf{z}_\Gamma \,\mathrm{d}s \tag{21}$$

where $\mathbf{z}_\Gamma$ is the boundary value of the continous adjoint problem then the discretization is adjoint consistent. This modification corresponds to the IP modification of J for Poisson's equation in (17). If, however, the target functional is evaluated as in (20) then the discretization is adjoint inconsistent.

To summarize: Only in combination with target functionals which are compatible with the primal equations we can expect a DG discretization to be adjoint consistent. It can be shown that only the target functionals given in (15), (16), (18) and (20) are compatible with the respective primal equations and may lead to an adjoint consistent discretization. Additionally, as shown for the compressible Euler and Navier-Stokes equations, special care is required in the discretization of boundary conditions as otherwise adjoint consistency and order doubling is lost.

2. Error estimation and adaptivity: Given a discretization (9) the error $J(u) - J(u_h)$ can be represented by $J(u) - J(u_h) = -\mathcal{N}(u_h, z)$ where z is the exact (and in general unkown) solution to the continuous adjoint problem (8). Replacing z by the solution $\tilde{z}_h \in \tilde{V}_h$ to the discrete adjoint problem (9) we obtain an approximate error representation $J(u) - J(u_h) \approx -\mathcal{N}(u_h, \tilde{z}_h) = \sum_\kappa \eta_\kappa$ which can be decomposed as a sum of local dual-weighted-residual or adjoint-based indicators η_κ. The approximate error representation and the adjoint-based indicators have been successfully applied in the *a posteriori* error estimation and goal-oriented mesh refinement for discontinuous Galerkin discretizations of inviscid and viscous laminar sub-, trans- and supersonic compressible flows, [2, 4, 5, 6], also in combination with anisotropic mesh refinement [7].

References

[1] K. Harriman, D. Gavaghan and E. Süli, *The Importance of adjoint consistency in the approximation of linear functionals using the discontinuous Galerkin finite element method*, Oxford University Computing Laboratory, NA-04/18, (2004).

[2] R. Hartmann, *Adaptive discontinuous Galerkin methods with shock-capturing for the compressible Navier-Stokes equations*, Int. J. Numer. Meth. Fluids **51**(9–10) (2006), 1131–1156.

[3] R. Hartmann, *Adjoint consistency analysis of Discontinuous Galerkin discretizations*, SIAM J. Numer. Anal. (2006), submitted.

[4] R. Hartmann, *Error estimation and adjoint based refinement for an adjoint consistent DG discretization of the compressible Euler equations*, Int. J. Comp. Sci. Math. (2007), to app.

[5] R. Hartmann and P. Houston, *Adaptive Discontinuous Galerkin Finite Element Methods for the Compressible Euler Equations*, J. Comp. Phys. **183** (2002), 508–532.

[6] R. Hartmann and P. Houston, *Symmetric Interior Penalty DG Methods for the Compressible Navier–Stokes Equations II: Goal–Oriented A Posteriori Error Estimation*, Int. J. Num. Anal. Model. **3**(2006), 141–162.

[7] T. Leicht and R. Hartmann, *Anisotropic mesh refinement for discontinuous Galerkin methods in two-dim. aerodynamic flow simulations*, Int. J. Numer. Meth. Fluids (2007), to app.

[8] J. Lu and D. L. Darmofal, *Dual-consistency analysis and error estimation for discontinuous Galerkin discretization: application to first-order conservation laws*, IMA Journal of Numerical Analysis (2006), submitted.