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Goal-oriented mesh adaptation using a dissipation-based error indicator

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SUMMARY

The accuracy of functionals of solutions of the Euler equations, solved using a finite volume code, are examined under grid refinement. It is shown that a commonly used adaptation indicator based on local solution gradients is ineffective in reducing functional error for flows with supersonic regions. A novel indicator is introduced which attempts to quantify that part of the error in the functional due to the explicitly added dissipation present in the numerical flux. The scheme is considerably simpler and computationally cheaper than other recently proposed *a posteriori* error estimators for finite volume schemes, but does not account for all sources of error. In mind of this, emphasis is placed on numerical evaluation of the performance of the scheme, and it is shown to be extremely effective in both estimating and reducing error for a wide range of flows. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: *A posteriori* error estimation; mesh adaptation; finite volume; sensitivity analysis

1. INTRODUCTION

One of the principal advantages of unstructured mesh methods is the readiness with which the mesh can be locally refined to suit the solution. The expectation is that by concentrating points in regions of interest, important flow field features, such as shocks and vortices, can be resolved with significantly less points than required for a globally finer mesh. However it is not necessarily the case that adding points locally will reduce global measures of error.

To ensure this an extremely successful theory of *a posteriori* error estimation has been developed in the context of finite element methods [2]. Using the adjoint problem, which serves to relate local errors in the field to the error in a specified functional of interest J, it is possible to derive *error representation formulas* and corresponding local error indicators, which do not require knowledge of a more accurate solution of the original problem.

However the dominant methods in use for aerodynamic applications are currently second-order finite volume codes, in which context the application of these techniques has proven

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difficult. The leading approach uses a globally refined grid to obtain the necessary estimate of the local residual error [9], but this involves delicate second-order interpolation of possibly discontinuous solutions between grids. Further storage of the fine grid represents a memory bottleneck for the entire solution process. These problems come in addition to the unsolved issue of robust and efficient solution of the adjoint problem for viscous flows in complex geometries [4].

In light of these difficulties an alternative approach is proposed here, which also requires an adjoint solution, but is otherwise considerably simpler to realize and cheaper to evaluate. We consider the sensitivity of J to parameters controlling the level of explicitly added stabilizing dissipation in the numerical method (here of the DLR *TAU*-code [5]). Since grid converged solutions are independent of the value of these parameters, any sensitivity on finite grids is a numerical artifact, which is seen to correspond to numerical errors introduced by the dissipation. The sensitivities themselves may be accurately evaluated using the adjoint method, and it will be shown that the scheme may be regarded as an approximation to an error representation formula.

An obvious deficit of the approach is that sources of error other than those due to the dissipation are invisible to the sensor. Therefore emphasis is placed on numerical evaluation of the performance of the method.

2. DISCRETIZATION OF THE GOVERNING EQUATIONS

Consider the stationary Euler equations for a compressible fluid in conservative variables w, with analytic flux f, subject to slip boundary conditions, and discretized with the finite volume method on a control volume Ω_i with inner boundary Γ_i:

$$\int_{\Omega_i} \nabla \cdot f \, \mathrm{d}\Omega \simeq \int_{\Gamma_i} \hat{f} \cdot n \, \mathrm{d}\Gamma + \int_{\Gamma_b} \hat{f}_b \cdot n \, \mathrm{d}\Gamma = 0,$$

where $\hat{f}$ and $\hat{f}_b$ are numerical internal and boundary fluxes accounting for discontinuities at volume interfaces. It is well known that $\hat{f}$ must contain some added dissipation if the discretization is to be stable. This may be achieved by upwinding, or explicit addition of artificial viscosity. The Jameson-Schmidt-Turkel (JST) scheme [7] employing the latter approach is applied here, for which the flux across a grid face $\{ij\}$ with normal n_{ij} is

$$\hat{f}_{ij} = \frac{1}{2}\left(f(w_i) + f(w_j)\right) \cdot n_{ij} - \frac{1}{2}|\lambda_{ij}| \left[\varepsilon_{ij}^{(2)} \{w_j - w_i\} - \varepsilon_{ij}^{(4)} \{L_j(w) - L_i(w)\} \right], \qquad (1)$$

where $|\lambda_{ij}|$ is the maximum convective eigenvalue at the face, and

$$\varepsilon_{ij}^{(2)} = k^{(2)} \max(\Psi_i, \Psi_j) \Phi^{(2)}, \quad \varepsilon_{ij}^{(4)} = \max(k^{(4)} - \varepsilon_{ij}^{(2)}, 0) \Phi^{(4)},$$

$$L_i(w) = \sum_{j \in \mathcal{N}(i)} (w_j - w_i), \quad \Psi_i = \frac{\sum_{j \in \mathcal{N}(i)} (p_j - p_i)}{\sum_{j \in \mathcal{N}(i)} (p_j + p_i)},$$

where $\mathcal{N}(i)$ is the set of neighbours of i, Ψ is a shock switch, the Φ contain some mesh anisotropy corrections, and $k^{(2)}$, $k^{(4)}$ are user-defined constants, typically $\frac{1}{2}$ and $\frac{1}{64}$ respectively, through which the level of 2nd- and 4th-order dissipation, and hence the compromise between stability and accuracy may be adjusted.

3. A DISSIPATION-ERROR ESTIMATOR

An *a posteriori* error estimator and mesh adaptation indicator is proposed on the basis of two observations: (i) explicitly added artificial dissipation in unstructured finite volume schemes is responsible for a large proportion of the error in the solution under typical conditions of mesh resolution, and (ii) if non-linear effects are neglected, the sensitivity of a goal function J to variation of the level of dissipation is proportional to the error in J due to that dissipation.

The first observation is the result of extensive experience with the JST scheme for practical engineering problems. However it is not true in general: for a smooth solution the error due to dissipation is $O(h^3)$, while the method is second-order, and for a sufficiently refined mesh the error due to the central difference in (1) dominates.

The second observation is a consequence of the fact that grid converged solutions are independent of the level of dissipation. As $h \to 0$ the dissipation, as well as the sensitivity of J to the coefficients of dissipation, tend to zero. From this perspective any extant dissipation on finite grids may be regarded as entirely spurious with respect to accuracy, and its quantitative effect on J may be extrapolated using the sensitivity of J with respect to the dissipation coefficients.

On this basis we propose a dissipation-error estimator for the JST scheme:

$$\eta_J = k^{(2)} \frac{\mathrm{d}J}{\mathrm{d}k^{(2)}} + k^{(4)} \frac{\mathrm{d}J}{\mathrm{d}k^{(4)}}. \tag{2}$$

In order to build an indicator for adaptation however, local sensitivity information is required, and to this end the dissipation coefficients may be *interpreted* as being defined independently for each control volume, with the coefficient on a face being an average of neighbours, $k_{ij} = \frac{1}{2}(k_i + k_j)$, and the sensitivity of J evaluated with respect to all these parameters. In order to do this efficiently and accurately an adjoint approach, more often used for evaluating sensitivities with respect to geometry modifications in gradient based optimization [3], is applied. Consider the *Lagrangian*: $\mathcal{L}(w, k, \psi) = J(w) + \psi^T R(w, k)$, which always takes the value J provided the state equation $R(w, k) = 0$ is fulfilled. Then

$$\frac{\mathrm{d}J}{\mathrm{d}k} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}k} = \frac{\partial J}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}k} + \psi^T \left\{ \frac{\partial R}{\partial k} + \frac{\partial R}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}k} \right\} = \left\{ \frac{\partial J}{\partial w} + \psi^T \frac{\partial R}{\partial w} \right\} \frac{\mathrm{d}w}{\mathrm{d}k} + \psi^T \frac{\partial R}{\partial k} = \psi^T \frac{\partial R}{\partial k},$$

whereby the final equality holds if ψ satisfies the *adjoint equation*

$$\frac{\partial R}{\partial w}^T \psi = -\frac{\partial J}{\partial w}^T.$$

For a given cost function ψ must be evaluated only once in order to calculate the sensitivity of J with respect to any number of parameters. In our case the remaining term in the expression for the gradient may be written down explicitly. For $k^{(4)}$ for example it is

$$\frac{\partial R_i}{\partial k_j^{(4)}} = \begin{cases} \sum_{m \in \mathcal{N}(i)} -\frac{1}{4}|\lambda_{im}| \{L_m(w) - L_i(w)\} & j = i \\ -\frac{1}{4}|\lambda_{ij}| \{L_j(w) - L_i(w)\} & j \in \mathcal{N}(i) \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

and is always of the same order as the original dissipation term. Note that the relations

$$\sum_i \frac{\partial R_i}{\partial k_j^{(4)}} = 0, \quad \sum_j \frac{\partial R_i}{\partial k_j^{(4)}} = \frac{\partial R_i}{\partial k^{(4)}},$$

hold, so that the sum of all local error indicators is the total error estimator as expected.

Before this estimator is studied numerically, it is useful to examine its relationship to the wider theory of error representation formulas [1, 8]. Moving to a continuous perspective, let the governing equations be written $\mathcal{R}(w) = 0$, where w is now the exact solution of the continuous problem, and the solution of the discrete problem is written w_h. Assume there exist Fréchet derivatives $\mathcal{R}'_w\tilde{w}$ and $(J'_w, \tilde{w})$ of $\mathcal{R}$ and J respectively about w for general $\tilde{w}$. Then the linear adjoint problem and corresponding equivalence relation may be written

$$\mathcal{R}'^*_w\psi = J'_w, \qquad (\mathcal{R}'^*_w\psi, \tilde{u}) = (\psi, \mathcal{R}'_w\tilde{u}),$$

where the inner product is a volume integral over Ω. Also let ψ_h be the solution of the discretized adjoint problem, and define $J'_h := \mathcal{R}'^*_{w_h}\psi_h$. In order to handle the non-linearity of $\mathcal{R}$ introduce its mean-value linearization and the corresponding relation:

$$\bar{\mathcal{R}}' = \int_0^1 \mathcal{R}'_{w+\theta(w_h-w)}\,\mathrm{d}\theta, \quad \mathcal{R}(w_h) - \mathcal{R}(w) = \int_0^1 \frac{\partial}{\partial\theta}\mathcal{R}(w + \theta(w_h - w))\,\mathrm{d}\theta = \bar{\mathcal{R}}'(w_h - w),$$

and similarly for J. Using these relations we obtain

$$\begin{aligned} J(w_h) - J(w) &= (\bar{J}'_w, w_h - w) \simeq (J'_h, w_h - w) = (\mathcal{R}'^*_{w_h}\psi_h, w_h - w) \\ &= (\psi_h, \mathcal{R}'_{w_h}(w_h - w)) \simeq (\psi_h, \bar{\mathcal{R}}'(w_h - w)) = (\psi_h, \mathcal{R}(w_h)), \end{aligned} \tag{4}$$

where terms neglected in the approximate equalities above are of the order of $\|w_h - w\|^2$ and $\|w_h - w\|\|\psi_h - \psi\|$. Thus the error in J is recast in terms of solutions of the discrete problem and its adjoint, without reference to the solution of the original continuous problem.

A computable estimate of the error may be obtained by approximating $\mathcal{R}(\cdot)$ in the final term of (4), however the original discretization is not acceptable because $R(w_h) \equiv 0$. The response is either to evaluate R on a globally refined mesh [9], at a higher-order [6], or to apply a modified discretization. For example Müller and Giles [8] evaluated the integral of the analytic flux about each control volume rather than a numerical flux. This latter approach is almost equivalent to the indicator already described. To see this write the inner flux discretization as two additively separated parts $R(w_h) = F(w_h) + D(w_h) = 0$, where F is the central difference and D the dissipation. Clearly the two parts are identical in value up to a sign difference. Noting further that if the averaging of the dissipation coefficients were not performed, then the dissipation derivative (3) would take the same form as the dissipation itself, we can conclude that $k\,\partial R/\partial k$ is approximately $-F$, and therefore η_J may also be regarded as a computable approximation to an error representation formula (4). To determine the validity of the approximation numerical investigations follow.

4. NUMERICAL RESULTS

To quantify the effectiveness of the refinement indicator we consider the NACA0012 aerofoil at sub-, trans- and supersonic onflow conditions, namely $(M_\infty, \alpha) = (0.5, 0.0)$, $(0.85, 2.0)$ and $(1.5, 1.0)$. Otherwise all parameters are identical, in particular the levels of second- and fourth-order dissipation, and we regard the coefficients of lift c_L, and drag c_D as the functionals of interest. For the subsonic case the analytic values of the coefficients are known; to obtain reliable estimates in the other cases calculations on globally refined meshes were performed and Richardson extrapolation applied. All errors shown are with respect to these results.

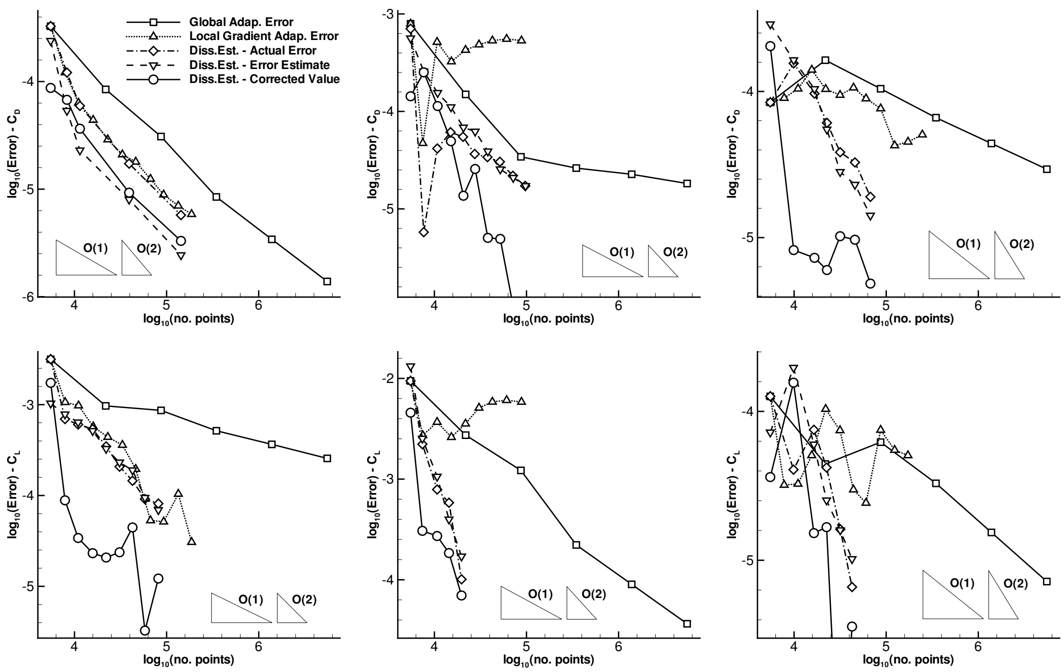

Figure 1. Convergence of c_D (top row) and c_L (bottom row) errors for global, local-gradient and dissipation-error refinement, for sub-, trans- and supersonic cases (left to right).

The convergence of all cases is displayed in Figure 1 for global refinement, refinement based on local solution gradients, and refinement using the dissipation-error estimator. For the latter calculations the error estimate (2), and the resulting corrected values of c_L and c_D are also plotted. Immediately evident is that the solution gradient indicator only converges in the subsonic case, where the problem has an elliptic character — in the transonic case it even approaches an incorrect value. Several modifications of the indicator have had no effect on the character of the results. In contrast the dissipation-error adaptation performs consistently well, resulting in a factor 100 reduction in grid size for the highest achieved accuracy in comparison to global refinement. The related error estimator agrees well with the actual error in all cases, and the corrected value is therefore consistently more accurate then the uncorrected value. Some example grids are shown in Figure 2, whereby the concentration of points in regions with a strong causal connection to the surface of the aerofoil is apparent for the estimator refined grids.

5. CONCLUSIONS

A novel error estimator has been developed on the basis of sensitivity of functionals to artificial viscosity in the flux function. The numerical overhead of the error estimator is purely the cost of solving the adjoint equation on the existing grid, making it substantially cheaper than

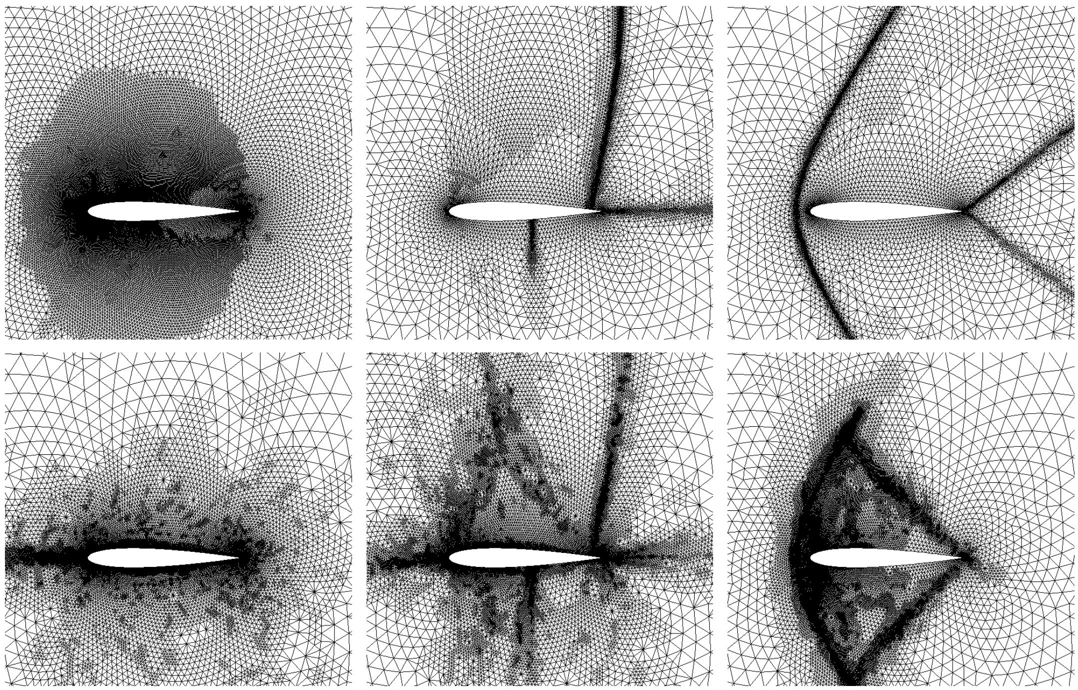

Figure 2. Meshes for sub-, trans- and supersonic cases using solution gradient adaptation (top), and dissipation-error adaptation for c_L (bottom), with similar grid size.

related approaches. Further the method can be regarded as an approximation to a true error representation formula, and numerical tests have demonstrated excellent accuracy for a range of conditions. Future work will be the extension of the scheme to viscous flows.

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