Mode Transition Behavior
in Hybrid Dynamic Systems

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Mode Transition Behavior in Hybrid Dynamic Systems

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Introduction

Mode Transitions in Hybrid Models of Physical Systems

- hybrid because
  - continuous, differential equations
  - discrete, finite state machine
- overview of phenomena involved

Illustrated by Hydraulic Actuator Used for Aircraft Attitude Control Surfaces
Modeling of Physical Systems

Ideal Picture Model (Schematic)
Identify Behavioral Phenomena
For Example, A Hydraulic Actuator

Equation Generation

Compile Constituent Equations

- \( R_{in} \)
- \( R_{rel} \)
- \( C_{cil} \)
- \( m_p \)
- \( R_p \)
- \( I_{rel} \)
- \( 0, \) cylinder chamber
- \( l, \) relief flow pipe
- \( l, \) intake pipe
- \( l, \) oil compression

\[
\begin{align*}
\text{for } m \text{ in } & R_{in} = P_{Rin} - f_p R_{in} \\
\text{for } p \text{ in } & R_{rel} = P_{Rrel} - f_p R_{rel} \\
\text{for } C_{cil} \text{ in } & C_{cil} = f_R \\
\text{for } m_p \text{ in } & p = \dot{m}_p = A_p P_{cyl} \\
\text{for } R_p \text{ in } & f_p R_{rel} = P_{Prel} \\
\text{for } I_{rel} \text{ in } & I_{rel} = P_{Prel} \\
\text{for } 0 \text{, cylinder chamber } & v_p = \dot{f}_{in} - \dot{f}_{rel} \\
\text{for } l \text{, relief flow pipe } & P_{Prel} = P_{smp} - f_{rel} R_{rel} + P_{cyl} \\
\text{for } l \text{, intake pipe } & P_{Rin} = P_{in} - P_{cyl} \\
\text{for } l \text{, oil compression } & P_{Rrel} = P_{rel} - P_C 
\end{align*}
\]
Equation Processing

Before Simulation

- the number of equations is reduced
  - substitution/elimination
- equations are sorted
  - each equation computes one variable
- equations are solved
  - high index problems may require differentiation of certain equations

Hybrid Behavior

Introduce Valves

- make highly nonlinear behavior piecewise linear
  - intake valve if \( v_{in} \) then \( p_{in} = p_{in} - p_{cyl} \) else \( f_{in} = 0 \)
  - relief valve if \( v_{rel} \) then \( p_{rel} = p_{smp} - f_{rel}R_{rel} + p_{cyl} \) else \( f_{rel} = 0 \)

Switching Between Modes of Continuous Behavior

- intake valve, \( v_{in} \) external switch (control law)
- relief valve, \( v_{rel} \) autonomous switch triggered by physical quantities
  \[ v_{rel} = p_{cyl} > P_{th} \]
- different sets of equations
Computational Causality

When Switching Equations

- computational causality may change
  - re-ordering
  - re-solving

Example

- when the intake valve closes, equations change
  - from $P = P_{r} - P_{cyl}$
  - to $f_{in} = 0$

- therefore, in this equation
  - $P_{r}$ becomes unknown
  - $f_{in}$ becomes known

Implicit Modeling

Deal With Causal Changes Numerically

Valve behavior

- residue on $f_{in}$: $0 = \text{if } v_{in} \text{ then } -P_{rm} + P_{in} - P_{cyl} \text{ else } f_{in}$
- residue on $f_{rel}$: $0 = \text{if } v_{rel} \text{ then } -P_{rel} + P_{mp} - f_{rel}R_{rel} + P_{cyl} \text{ else } f_{rel}$

Implicit Numerical Solver (e.g., DASSL)

- designed to handle this formulation
Hybrid Dynamic Behavior

Geometric View

- modes of continuous, smooth, behavior
- patches of admissible state variable values

Specification Parts

Hybrid Behavior Specification

- a function, $f$, that defines continuous, smooth, behavior for each mode
  \[ f_{\alpha_i} E_{\alpha_i} \dot{x} + A_{\alpha_i} x + B_{\alpha_i} u = 0 \]
- an inequality, $\gamma$, that defines admissible state variable values
  \[ \gamma_{\alpha_i} C_{\alpha_i} x + D_{\alpha_i} u \geq 0 \]
Dynamics

Behavior Characteristics

- $C^0$, i.e., no jumps in state variables
- steep gradients

Example

- when the intake valve closes, piston velocity quickly reduces to 0

The Next Step

Remove Steep Gradients

- e.g., singular perturbation

Algebraic Constraints Between State Variables

- high index systems
- subspace with admissible (continuous) dynamic behavior
- discontinuities (jumps) in state behavior
Hybrid Dynamic Behavior - Refined

Geometric View

- modes of continuous, smooth, behavior
- patches of admissible state variable values
- manifold of dynamic behavior

\[ x_{\alpha_2} \quad x_{\alpha_1} \]
\[ \alpha_2 \quad \alpha_1 \]
\[ \alpha_{\gamma_{\alpha_2}} \quad \alpha_{\gamma_{\alpha_1}} \]

Specification Parts

Hybrid Behavior Specification

- a function, \( f \), that implicitly defines for each mode
  - continuous, smooth, behavior
  - state variable value jumps
    \[ f_{\alpha_i} : E_{\alpha_i} \dot{x} + A_{\alpha_i} x + B_{\alpha_i} u = 0 \]
- an inequality, \( \gamma \), that defines admissible generalized state variable values
  \[ \gamma_{\alpha_{i+1}} : C_{\alpha_i} x + D_{\alpha_i} u \geq 0 \]
- for explicit reinitialization (semantics of \( x \))
  \[ f_{\alpha_i} : E_{\alpha_i} \dot{x} + A_{\alpha_i} x + B_{\alpha_i} u + B_{\alpha_i}^2 x^2 = 0 \]
Handling of Systems With High Index

DASSL Handles Index 2 Systems

- implicit formulation for continuous behavior

Requires Consistent Initial Conditions When Mode Changes Occur

- compute from implicit formulation to make jump space (projection) explicit
- for example, sequences of subspace iteration
  - space of dynamic behavior: \( \dot{V}^{-1} = A^T E V^o, V^o - R^o \)
  - jump space: \( T^o = E - A^T P^o, P^o = 0 \)

- or, decomposition in (pseudo) Kronecker Normal Form

Projections

Linear Time Invariant Index 2 System

- derive pseudo Kronecker Normal Form (numerically stable)

\[
\begin{bmatrix}
E_{11} & 0 & 0 \\
0 & E_{22,12} & \dot{x}_{i,1} \\
0 & 0 & \dot{x}_{i,2}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_f \\
x_{i,1} \\
x_{i,2}
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12,1} & A_{12,2} \\
0 & A_{22,11} & A_{22,12} \\
0 & 0 & A_{22,22}
\end{bmatrix}
\begin{bmatrix}
x_f \\
x_{i,1} \\
x_{i,2}
\end{bmatrix}
+ \begin{bmatrix}
B_{1} \\
B_{2,1} \\
B_{2,2}
\end{bmatrix} u = 0
\]

- after integration (no impulsive input behavior), consistent values are:

\[
\begin{align*}
x_f &= x_f^o - E_{11}^{-1} A_{12,1} A_{22,11} E_{22,12} (x_{i,2} - x_{i,2}^o) \\
x_{i,1} &= A_{22,11} (\frac{1}{-B_{2,1}}+E_{22,12}\dot{x}_{i,2}) - A_{22,12} x_{i,2} \\
x_{i,2} &= -A_{22,22} B_{2,2} u
\end{align*}
\]
The Hydraulic Actuator

Generalized State Jumps for Each Mode

<table>
<thead>
<tr>
<th>Mode</th>
<th>Projection</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{00}$</td>
<td>$f_{rel} = 0$</td>
</tr>
<tr>
<td></td>
<td>$v_p = 0$</td>
</tr>
<tr>
<td>$\alpha_{11}$</td>
<td>$v_p = (m_p v_p - f_{rel ref}) / (m_{rel} + m_p)$</td>
</tr>
<tr>
<td></td>
<td>$f_{rel} = (m_p v_p - f_{rel ref}) / (m_{rel} + m_p)$</td>
</tr>
<tr>
<td>$\alpha_{1}$</td>
<td>$v_p = v_p$</td>
</tr>
<tr>
<td></td>
<td>$f_{rel} = 0$</td>
</tr>
<tr>
<td>$\alpha_{+1}$</td>
<td>$v_p = v_p$</td>
</tr>
<tr>
<td></td>
<td>$f_{rel} = f_{rel}$</td>
</tr>
</tbody>
</table>

A Scenario

Intake Valve is Open
- piston starts to move

Intake Valve Closes
- piston inertia causes pressure build-up
- pressure reaches critical value

Relief Valve Opens
- cylinder pressure decreases

$\Rightarrow$ Interaction Between Mode Transition Behavior
Phase Space of Cylinder Scenario

Projection Is Aborted
- immediately
- after partial completion

Sequences of Mode Changes
(a) State Outside of a Patch in the New Mode
(b) During Projection State Values are Reached Outside of a Patch in the New Mode
Impulses

High Index Systems May Contain Impulsive Behavior

- in case of the hydraulic cylinder, $p > p_{cr}$ would always hold if not $v_p^{-} = v_p^{+}$
- unknown where the patch is abandoned

In-Depth Analysis of Switching Conditions

- solve for required $x(t)$
- compute earliest $t = t_0$ at which $\gamma(x(t), u(t), t) \geq 0$
- substitute $t_0$ to compute $x(t_0)$

Complex Switching Structure

Additional Difficulty When Interacting Fast Transients (e.g., collision)

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Detailed Analysis of the Projection

Cylinder Example (Imaginary Eigenvalues, $\lambda = \lambda_+ + i \lambda_-$)

- from detailed model
  
  $p(t) = e^{\lambda t} \left( p^- \cos(\lambda t) - \frac{1}{\lambda_i} \left( \frac{1}{C_i} v_p^- + \lambda_r p^- \right) \sin(\lambda t) \right)$

- substitute $t$ at which $p(t) > p_{cr}$
  
  $v_p = e^{\lambda t} \left( v_p^- \cos(\lambda t) - \left( \frac{R_p v_p^-}{I_1} + \lambda_r v_p^- \right) \sin(\lambda t) \right)$

---
Complex Switching Structure

Explicit Re-initialization

Chattering

What if the new mode switches back
- immediately => inconsistent model, no solution
- after infinitesimal period of time => chattering behavior, solve with
  - equivalent control
  - equivalent dynamics
Equivalent Dynamics

Chattering
- fast component
  - remove
- slow component
  - weighted mean of instantaneous vector fields (Filippov Construction)
- sliding behavior

Ontology

Phase Space Transition Behavior Classification
- mythical (state invariant)
- pinnacle (state projection aborted)
- continuous
  - interior (continuous behavior)
  - boundary (further transition after infinitesimal time advance)
  - sliding (repeated transitions after each infinitesimal time advance)

Combinations of Behavior Classes
Conclusions

Mode Transition Behavior

- Rich
- Complex

Requires

- special algorithms/computations
- model verification analyses

How to Efficiently Generate Behavior (e.g., for Real-time Applications)?