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Solutions to the General Inner–Outer and Spectral Factorization Problems

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Abstract

In this paper we solve two open problems in linear system theory: the computation of the inner–outer and spectral factorizations of a descriptor continuous–time system considered in the most general setting. Our method is based on descriptor state–space computations and relies on three techniques developed for rational matrices: compression of G to a surjective (full row rank) matrix, and dislocation of the unstable zeros/poles of G to stable locations, all achieved by left multiplication with all–pass factors. The proposed procedures are completely general being applicable for G polynomial/proper/improper, of arbitrary rank, with poles/zeros on the imaginary axis.

1 Introduction

In this paper we address two related problems in linear system theory: the computation of the inner–outer factorization and of the spectral factorization of a rational matrix. A large number of quite different types of factorizations are covered in the literature under these names but most of them impose additional restrictive assumptions that rule out the difficult cases. The two general factorization problems that we solve in this paper are now stated.

1.1 Problem statement

Throughout the paper we consider matrices with real coefficients as this is the leading case in control applications. Moreover, from a numerical viwepoint the real case is slightly more difficult than the complex case, and the latter follows by minor modifications of notation and definitions from the former. We start with some notation and definitions. By $\mathbb{C}$, $\mathbb{C}^-$, $\mathbb{C}^+$, and $\mathbb{C}_0$, we denote the complex plane, the open left half plane, the open right half plane, and the imaginary axis, respectively, and let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\overline{\mathbb{C}}^- := \mathbb{C}^- \cup \mathbb{C}_0 \cup \{\infty\}$, $\overline{\mathbb{C}}^+ := \mathbb{C}^+ \cup \mathbb{C}_0 \cup \{\infty\}$. Here "overbar" denotes closure. We call a real rational matrix $G(s)$ injective if it has full column rank for almost all s and we call it surjective if it has full row rank for almost all s. For a rational matrix with real coefficients $G(s)$ we define its conjugate $G^*(s) := G^T(-s)$. We call a real rational matrix $M(s)$ *all–pass* if it is square and satisfies $M^*(s)M(s) = I$, and we call it *inner* if it is analytic in $\overline{\mathbb{C}}^+$ and satisfies $M^*(s)M(s) = I$.

Inner–Outer Factorization Problem. Given $G(s)$ an arbitrary real rational matrix analytic in $\overline{\mathbb{C}}_+$ (i.e., proper and stable), determine two real rational matrices $G_i(s)$ and $G_o(s)$ such that

$$G(s) = G_i(s)G_o(s), \tag{1}$$

where $G_i(s)$ and $G_o(s)$ are analytic in $\overline{\mathbb{C}}^+$, $G_i(s)$ is inner, and $G_o(s)$ is surjective and has a right inverse analytic in $\mathbb{C}^+$. $G_o(s)$ is called an *outer* factor, and (1) defines an inner–outer factorization of $G(s)$. By a slight abuse of terminology, we shall consider in this paper also inner–outer factorizations for systems which are analytic in $\mathbb{C}^+$ (also called "weak–stable", allowing for $G(s)$ to be improper and to have poles on $\mathbb{C}_0$). In this case we require for the inner factor to be further analytic in $\overline{\mathbb{C}}^+$, and for the "outer" factor to have a right inverse analytic in $\mathbb{C}^+$, but we impose correspondingly analyticity of $G_o(s)$ only in $\mathbb{C}^+$. □

Spectral Factorization Problem. Given $G(s)$ an arbitrary real rational matrix, determine a real rational matrix $G_o(s)$ such that

$$G^*(s)G(s) = G_o^*(s)G_o(s), \tag{2}$$

where $G_o(s)$ is surjective and analytic in $\mathbb{C}^+$ and has a right inverse analytic in $\mathbb{C}^+$. $G_o(s)$ is called a *spectral factor* of $G(s)$ and (2) defines a spectral factorization of $G(s)$. □

We should mention at this point that the spectral factorization problem as formulated in [19], [1], is more

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general. There, a real rational matrix $\Phi(s)$ is given – also called spectral density –, satisfying $\Phi(s) = \Phi^T(-s) \geq 0$, $\forall s \in \overline{\mathbb{C}}_0$, and one seeks a real rational matrix $G_o(s)$ such that

$$\Phi(s) = G_o^*(s)G_o(s),$$

where the spectral factor $G_o(s)$ must satisfy the same requirements as in our formulation. The difference is that in our formulation $\Phi(s)$ is already given in a pre–factorized form

$$\Phi(s) = G^*(s)G(s), \qquad (3)$$

although no restrictive assumptions are imposed on $G(s)$. Nevertheless, in most control related applications $\Phi(s)$ is already in a prefactorized form [20].

The inner–outer and spectral factorizations appear throughout in control systems, identification, signal processing, and circuit theory, and it is surely hopeless to give here a short but still comprehensive account on all applications in which they occur. The reader is refered to [2, 19, 1, 4, 9, 8, 4, 5, 6, 15] and the references therein. However, in all the above papers one or several restrictive assumptions reduce the generality of the proposed method. Briefly, no reliable methods are available for arbitrary rank matrices featuring zeros on the imaginary axis and at infinity, although this situation occurs frequently in practice.

1.2 Outline of the proposed approach

Let $G(s)$ be an arbitrary $p \times m$ real rational matrix of rank r. Our approach to the inner–outer and spectral factorization problems rely on two basic factorizations.

1. Row compression by all–pass factors. We factorize an arbitrary $G(s)$ as

$$G(s) := G_a(s)\widetilde{G}(s), \qquad (4)$$

where $G_a(s)$ is all–pass with all poles in $\mathbb{C}^-$, and $\widetilde{G}(s)$ is row compressed, i.e., the trailing $p-r$ rows of $\widetilde{G}(s)$ are zero

$$\widetilde{G}(s) = \left[\begin{array}{c} \widetilde{G}_1(s) \\ O \end{array}\right] \begin{array}{l} \}r \\ \}p-r \end{array}. \qquad (5)$$

This comes up to computing $G_a^{-1}(s)$ such that in the product $G_a^{-1}(s)G(s)$ all minimal indices to the left of $G(s)$ are cancelled (are all made zero) while zeros in $\mathbb{C}^-$ are introduced instead. We chose $G_a(s)$ to have the smallest possible McMillan degree n_ℓ which is equal to the sum of all left minimal indices of $G(s)$. The computation of $G_a(s)$ amounts to solving for the stabilizing solution a standard Riccati equation of order n_ℓ. Combining (5) and (4) we get

$$G(s) = G_{i1}(s)\widetilde{G}_1(s)$$

where $G_a(s) = \left[\begin{array}{cc} G_{i1}(s) & G_{i2}(s) \end{array}\right]$, $G_{i1}(s)$ is inner, $\widetilde{G}_1(s)$ is surjective and has the same zeros in $\overline{\mathbb{C}}^+$ as $G(s)$, and its zeros in $\mathbb{C}^-$ are the union of the zeros in $\mathbb{C}^-$ of $\widetilde{G}(s)$ with the zeros of $G_a^{-1}(s)$.

2. Dislocation of zeros by all–pass factors. We factorize a surjective $G(s)$ as

$$G(s) = G_a(s)\widetilde{G}(s) \qquad (6)$$

where $G_a(s)$ is all–pass with all poles in $\mathbb{C}^-$, and $\widetilde{G}(s)$ is surjective and has no zeros in $\mathbb{C}^+$ (i.e., $\widetilde{G}(s)$ is outer). This comes up to computing $G_a^{-1}(s)$ such that in the product $G_a^{-1}(s)G(s)$ all $\mathbb{C}^+$ zeros of $G(s)$ are cancelled and reflected into symmetric positions in $\mathbb{C}^-$ (with respect to the imaginary axis). Again, we chose $G_a(s)$ to have the smallest possible McMillan degree which is equal to the number n_b of zeros of $G(s)$ in $\mathbb{C}^+$. The computation of $G_a(s)$ is achieved by solving a Lyapunov equation of order n_b.

These two basic factorizations are then performed succesively to get the inner–outer and spectral factorizations as explained in Section 5.

2 Preliminaries

Let $G(s)$ be an arbitrary (possibly improper) real rational matrix. Throughout the paper we use the following notation for the structural elements of $G(s)$: $p \times m$ are its dimensions, r stands for the rank over rationals, n denotes the McMillan degree (which equals the number of poles counting multiplicities and including infinity), $n_z = n_b + n_g$ is the number of zeros (counting multiplicities and including infinity), where n_g is the number of "good" zeros in $\overline{\mathbb{C}}^-$ and n_b is the number of "bad" zeros in $\mathbb{C}^+$, $\mathcal{Z}(G(s))$ denotes the union of zeros (with multiplicities), n_ℓ is the sum of degrees of any minimal basis of the left null space (these degrees are called left minimal indices), n_r is the sum of degrees of any minimal basis of the right null space (these degrees are called right minimal indices). For a rational matrix there is an interesting relation [17] among its structural elements that will be insightfull for the problems treated in the sequel:

$$n = n_z + n_r + n_\ell. \qquad (7)$$

It is well known that any rational matrix $G(s)$ (even improper or polynomial) has a descriptor realization

$$G(s) = \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right] := C(sE-A)^{-1}B + D, \quad (8)$$

where the so called *pole pencil* $A - sE$ is *regular*, i.e., it is square and $\det(A - sE) \not\equiv 0$. A pencil that is not regular is called singular. By $\Lambda(A - sE)$ we shall denote the union of generalized eigenvalues of an arbitrary (possibly singular) pencil $A - sE$ (finite and infinite, multiplicities counting).

The descriptor representation (8) of G is called *minimal* if the dimension k of the square matrices E and A is as small as possible. Note that for a minimal descriptor realization (8) of order k we have $k = n + \kappa$, where κ is the number of infinite elementary divisors of $A - sE$ [17] and n is the McMillan degree of $G(s)$. Moreover, we have $n = \text{rank}(E)$. With a particular realization (8) of $G(s)$ we associate also the *system pencil*

$$\mathcal{S}_G(s) = \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}. \tag{9}$$

The *pole pencil* and the *system pencil* play a fundamental role as their Weierstrass and Kronecker canonical forms, respectively, are in one–to–one correspondence with the structural elements of the rational matrix (see [14] and [17]).

3 Spectral decompositions of the system pencil

In this section we give two spectral decompositions of the system pencil $\mathcal{S}_G(s)$ which correspond to the two basic factorizations described in Section 1.2. We start with a spectral decomposition that outlines in an appropriate form the left Kronecker indices of $\mathcal{S}_G(s)$.

Theorem 3.1 *Let $G(s)$ be a $p \times m$ real rational matrix of McMillan degree n, of rank r, having n_z zeros, and the sums of minimal indices to the left and right n_ℓ and n_r, respectively. Then there exists a k-dimensional minimal realization (8) of $G(s)$ and two orthogonal matrices Q and Z such that*

$$\begin{bmatrix} I & O \\ O & Q^T \end{bmatrix} \left[\begin{array}{c} A - sE \quad B \\ \hline C \quad D \end{array}\right] Z =$$

$$\left[\begin{array}{cccc} A_{rz} - sE_{rz} & B_1 - sF_1 & B_2 - sF_2 & B_3 - sF_3 \\ O & A_\ell - sE_\ell & B_\ell & B_{\ell n} - sF_{\ell n} \\ O & O & O & B_n \\ \hline O & C_{\ell 1} & D_\ell & D_1 \\ O & C_{\ell 2} & O & D_2 \end{array}\right] \begin{array}{l} \}n_r + n_z \\ \}n_\ell \\ \}k - n \\ \}r \\ \}p - r \end{array}$$

$$\underbrace{\qquad}_{n_r + n_z + \\ +m - r} \quad \underbrace{\qquad}_{n_\ell} \quad \underbrace{\qquad}_{r} \quad \underbrace{\qquad}_{k - n} \tag{10}$$

where

(a) *$\mathcal{Z}(G(s)) = \Lambda(A_{rz} - sE_{rz})$ and $A_{rz} - sE_{rz}$ is surjective for all $s \notin \mathcal{Z}(G(s))$.*

(b) *E_ℓ, D_ℓ, B_n are invertible and $\forall s \in \mathbb{C}$ we have*

$$\textbf{(bi)} \quad \text{rank} \begin{bmatrix} A_\ell - sE_\ell & B_\ell \end{bmatrix} = n_\ell,$$

$$\textbf{(bii)} \quad \text{rank} \begin{bmatrix} A_\ell - B_\ell D_\ell^{-1} C_{\ell 1} - sE_\ell \\ C_{\ell 2} \end{bmatrix} = n_\ell. \tag{11}$$

The minimal realization (8) and the matrices Q and Z satisfying (10) can effectively be constructed by using a sequence of solely orthogonal transformations.

We assume now that $G(s)$ has no left minimal indices, i.e., it is surjective. As we shall see further, this is always possible after we have performed the first step of the factorization process. The theorem below provides a special spectral decomposition of the system pencil $\mathcal{S}_G(s)$ that is key to understand the conditions for dislocating the zeros in $\mathbb{C}_+$ of $G(s)$.

Theorem 3.2 *Let $G(s)$ be a $p \times m$ real rational matrix of McMillan degree n, of rank p (i.e., $G(s)$ is surjective), having the sum of minimal indices to the right n_r, and a number of $n_z = n_b + n_g$ zeros, where n_b and n_g are the number of zeros in $\mathbb{C}^+$ and $\overline{\mathbb{C}}^-$, respectively. Then there exists a k-dimensional minimal realization (8) of $G(s)$ and an orthogonal matrix Z such that*

$$\left[\begin{array}{c} A - sE \quad B \\ \hline C \quad D \end{array}\right] Z =$$

$$\left[\begin{array}{cccc} A_{rg} - sE_{rg} & B_1 - sF_1 & B_2 - sF_2 & B_3 - sF_3 \\ O & A_b - sE_b & B_b & B_{bn} - sF_{bn} \\ O & O & O & B_n \\ \hline O & C_b & D_b & D_1 \end{array}\right] \begin{array}{l} \}n_r + n_g \\ \}n_b \\ \}k - n \\ \}p \end{array}$$

$$\underbrace{\qquad}_{n_r + n_b + \\ +m - p} \quad \underbrace{\qquad}_{n_b} \quad \underbrace{\qquad}_{p} \quad \underbrace{\qquad}_{k - n} \tag{12}$$

where

(a) *$\mathcal{Z}(G(s)) \cap \overline{\mathbb{C}}^- = \Lambda(A_{rg} - sE_{rg})$ and $A_{rg} - sE_{rg}$ is surjective for all $s \notin \mathcal{Z}(G(s)) \cap \overline{\mathbb{C}}^-$.*

(b) *E_b, D_b, B_n are invertible and*

$$\textbf{(bi)} \ \Lambda(A_b - B_b D_b^{-1} C_b - sE_b) = \mathcal{Z}(G(s)) \cap \mathbb{C}_+,$$

$$\textbf{(bii)} \ \text{rank} \begin{bmatrix} A_b - sE_b & B_b \end{bmatrix} = n_b, \quad \forall s \in \mathbb{C}. \tag{13}$$

The minimal realization (8) and the matrix Z satisfying (12) can effectively be constructed by using a sequence of solely orthogonal transformations.

4 The basic factorization steps

In this section we describe the factorization steps introduced in Section 1.2.

4.1 Row compression by all–pass factors

Theorem 4.1 *Let $G(s)$ be a real rational matrix with the minimal realization (8), and let Q and Z be orthogonal matrices as in Theorem 3.1 such that (10) holds. Then we have:*

1. *The continuous-time algebraic Riccati equation*

$$A_\ell^T X E_\ell + E_\ell^T X A_\ell - (E_\ell^T X B_\ell + C_{\ell 1}^T D_\ell)(D_\ell^T D_\ell)^{-1} (B_\ell^T X E_\ell + D_\ell^T C_{\ell 1}) + C_\ell^T C_\ell = 0$$

has a stabilizing symmetric positive definite solution X_s such that $\Lambda(A_\ell + B_\ell F_s - sE_\ell) \subset \mathbb{C}^-$, where $F_s :=$

$-(D_\ell^T D_\ell)^{-1}(B_\ell^T X_s E_\ell + D_\ell^T C_{\ell 1})$ is the stabilizing Riccati feedback and $C_\ell^T := \left[\begin{array}{cc} C_{\ell 1}^T & C_{\ell 2}^T \end{array}\right]$.

2. *Let $G_i(s) = \left[\begin{array}{cc} G_{i1}(s) & G_{i2}(s) \end{array}\right] =$*

$$= Q \left[\begin{array}{c|cc} A_\ell + B_\ell F_s - sE_\ell & B_\ell D_\ell^{-1} & -X_s^{-1}E_\ell^{-T}C_{\ell 2}^T \\ \hline C_{\ell 1} + D_\ell F_s & I & O \\ C_{\ell 2} & O & I \end{array}\right] \quad (14)$$

and

$$G_o(s) = \left[\begin{array}{c|c} A - sE & B \\ \hline D_\ell H_1 & D_\ell H_2 \\ O & O \end{array}\right] = \left[\begin{array}{c} G_{o1}(s) \\ O \end{array}\right] \quad (15)$$

where $\left[\begin{array}{cc} H_1 & H_2 \end{array}\right] := \left[\begin{array}{cccc} O & -F_s & I & O \end{array}\right] Z^T$. Then $G_i(s)$ is a $p \times p$ inner matrix, the realization (14) is minimal, $G(s) = G_i(s)G_o(s)$, $G_{o1}(s)$ has no left minimal indices, and $\mathcal{Z}(G_{o1}(s))$ has n_b elements in $\mathbb{C}^+$.

Remark 4.2 The above theorem gives in fact a solution to the following problem: Given an arbitrary transfer matrix $G(s)$ determine an all–pass rational matrix $G_a(s)$ $(= G_i(s)^{-1})$, of minimal McMillan degree, that dislocates, by left multiplication all minimal indices of $G(s)$ and makes them zero, while it introduces "good" zeros (in $\mathbb{C}^-$) instead. The resulting $G_o(s) = G_a(s)G(s)$ has the same structural elements as $G(s)$ excepting some additional zeros in $\mathbb{C}^-$ and the minimal basis for the left null space which are built up from constant elements (polynomial degree 0). If $G(s)$ has "good" zeros (in $\overline{\mathbb{C}}^+$) then $G_o(s)$ is already a solution to the inner–outer factorization problem. Comparing our formulas for the Riccati equation and for the inner and outer factors with the formulas in [20] (on page 367), we see that our method has extracted from the original realization of $G(s)$ a proper subsystem

$$G_\ell(s) = \left[\begin{array}{c|c} A_\ell - sE_\ell & B_\ell \\ \hline C_{\ell 1} & D_\ell \\ C_{\ell 2} & O \end{array}\right] \quad (16)$$

which is left invertible and without zeros, and it has solved the corresponding inner–outer factorization problem for this subsystem. Then the solution to the original inner–outer factorization problem for $G(s)$ follows immediately. Moreover, for extracting (16) from the original realization of $G(s)$ we have performed exclusively orthogonal transformations and the subsystem (16) is the smallest one possible.

We switch now to the problem of dislocating zeros.

4.2 Dislocation of zeros by all–pass factors

Theorem 4.3 *Let $G(s)$ be a surjective real rational matrix with the minimal realization (8) and let Z be an orthogonal matrix as in Theorem 3.2 satisfying (12). Then:*

1. *The Lyapunov equation $(A_b - B_b D_b^{-1} C_b) Y E_b^T + E_b Y (A_b - B_b D_b^{-1} C_b)^T - B_b (D_b^T D_b)^{-1} B_b^T = 0$ has a unique syymmetric positive definite solution such that $\Lambda(A_b + B_b F_s - sE_b) \subset \mathbb{C}^-$, where $F_s := -(D_b^T D_b)^{-1}(B_b^T E_b^{-T} Y^{-1} + D_b^T C_b)$.*

2. *Let*

$$G_i(s) = \left[\begin{array}{c|c} A_b + B_b F_s - sE_b & B_b D_b^{-1} \\ \hline C_b + D_b F_s & I \end{array}\right] \quad (17)$$

and

$$G_o(s) = \left[\begin{array}{c|c} A - sE & B \\ \hline D_b H_1 & D_b H_2 \end{array}\right] \quad (18)$$

where $\left[\begin{array}{cc} H_1 & H_2 \end{array}\right] := \left[\begin{array}{cccc} O & -F_s & I & O \end{array}\right] Z^T$. Then $G_i(s)$ is square inner, the realization (17) is minimal, $G(s) = G_i(s)G_o(s)$, and $\mathcal{Z}(G_o) \subset \overline{\mathbb{C}}^-$.

We comment now on alternative existing methods of dislocating zeros and their applicability. Dislocation of zeros by using square inner factors was first performed in [15], but the method is limited to proper rational matrices $G(s)$. The first paper that reports on zero dislocation of an improper $G(s)$ by square inner factors is [16]. In both [15], [16], zeros are dislocated only one–by–one (or in conjugated pairs), in a sequential way, and the algorithms could perform on a non–surjective $G(s)$ as well. However, for those seeking to apply the methods of [15], [16] we make the cautionary remark that dislocating zeros without first dislocating the left minimal indices (i.e., performing on a non–surjective $G(s)$) could have perverse effects as instead of dislocating "bad" zeros and replacing them with "good" ones we may get in turn an increase in the sum of left minimal indices. Loosely speaking, if $G(s)$ is not surjective and no special care is taken, we may replace *bad* zeros with minimal indices *which are even worse* for the problems at hand. This is now illustrated by an example. Let $G(s) = \left[\begin{array}{c} \frac{s-1}{s+2} \\ \frac{s-1}{s+2} \end{array}\right]$ which has one "bad" zero at $s = 1$, one pole at $s = -2$, and $n_r = 0$ and $n_\ell = 0$. In fact, it is easy to see that a row minimal basis of the left null space is $\left[\begin{array}{cc} 1 & -1 \end{array}\right]$. Then $G_1(s) = \left[\begin{array}{cc} \frac{s-1}{s+1} & 0 \\ 0 & 1 \end{array}\right]$ is inner and let $G_2(s) := G_1^{-1}(s)G(s) = \left[\begin{array}{c} \frac{s+1}{s+2} \\ \frac{s-1}{s+2} \end{array}\right]$ which has no "bad" zeros, has one pole at $s = -2$, has no right minimal indices and has a minimal index to the left equal to 1. It is easy to figure out that $\left[\begin{array}{cc} s-1 & -(s+1) \end{array}\right]$ is a minimal basis for the left null space. Hence form $G(s) = G_1(s)G_2(s)$ we see that the "bad" zero at 1 was dislocated by left multiplication with a square inner factor, but in an completely unfortunate way as it was replaced with a minimal index to the left.

The method of zero–dislocation described in this paper is closer to [15] rather than [10], [16]. Similarly to [15], our entire reasoning is made on the zeros of the original

$G(s)$ and not on the poles of a generalized inverse of $G(s)$. We believe that this brings more insight in the theoretical aspects of the problem.

5 Solution to the factorization problems

In this section we explain briefly how we can apply the already obtained results to compute the inner–outer factorization and the spectral factorization in the most general setting.

5.1 Solution to the inner–outer factorization

Let $G(s)$ be an arbitrary rational matrix analytic in $\mathbb{C}^+$ (i.e., $G(s)$ is weakly stable).

Step 1. Use Theorem 4.1 to determine a factorization

$$G(s) = G_i^{(1)}(s)G_o^{(1)}(s)$$

where

$$G_i^{(1)}(s) = \left[G_{i1}^{(1)}(s)\; G_{i2}^{(1)}(s) \right], G_o^{(1)}(s) = \begin{bmatrix} G_{o1}^{(1)}(s) \\ O \end{bmatrix}.$$

The resulting $G_i^{(1)}(s)$ has all poles in $\mathbb{C}^-$, is inner and square, while $G_{o1}^{(1)}(s)$ is surjective with n_b zeros in $\mathbb{C}^+$.

Step 2. Use Theorem 4.3 to determine a factorization

$$G_o^{(1)}(s) = G_i^{(2)}(s)G_o^{(2)}(s)$$

where $G_i^{(2)}(s)$ has all poles in $\mathbb{C}^-$ and it is inner (and square), while $G_o^{(2)}(s)$ is surjective with all zeros in $\overline{\mathbb{C}}^-$ (i.e., it is outer).

Result. The inner–outer factorization results as

$$G(s) = G_i(s)G_o(s), G_i(s) := G_{i1}^{(1)}(s)G_i^{(2)}(s),$$
$$G_o(s) := G_o^{(2)}(s).$$

5.2 Solution to the spectral factorization

Let $G(s)$ be an arbitrary rational matrix.

Step 0. Use a particular version of Theorem 5.2 in [13] to determine a minimal degree coprime factorization of $G(s)$ in the form

$$G(s) = M^{-1}(s)N(s)$$

where $N(s)$ has all its poles in $\overline{\mathbb{C}}^-$ and $M(s)$ is inner.

Steps 1–2. Apply Steps 1–2 above to determine an inner–outer factorization of $N(s)$ in the form

$$N(s) = G_i^N(s)G_o^N(s).$$

The spectral factorization of $G(s)$ results as

$$G^*(s)G(s) = N^*(s)N(s) = G_o^*(s)G_o(s) \qquad (19)$$

where $G_o(s) := G_o^N(s)$ is the spectral factor.

6 Pseudoinverses of rational matrices

A straightforward application of the inner-outer factorization is the computation of a generalized (Moore–Penrose type) (pseudo)–inverse $G^{\#}(s)$ of an arbitrary rational matrix $G(s)$. $G^{\#}(s)$ is a generalized inverse of $G(s)$ if it satisfies the four axioms: (i) $G(s)G^{\#}(s)G(s) = G(s)$; (ii) $G^{\#}(s)G(s)G^{\#}(s) = G^{\#}(s)$; (iii) $G(s)G^{\#}(s) = (G(s)G^{\#}(s))^*$; (iv) $G^{\#}(s)G(s) = (G^{\#}(s)G(s))^*$. Depending on the interpretation of the conjugation operator $(\cdot)^*$, we get different pseudoinverses for the same rational matrix $G(s)$. We exemplify for the case in which the operator is conjugation in continuous–time.

To compute the generalized inverse, we perform successively a row and a column compression of $G(s)$ as follows: $G(s) = U(s)\begin{bmatrix} G_1(s) \\ O \end{bmatrix}$ where $G_1(s)$ is surjective and $U(s)$ is square inner, and $G_1^T(s) = V^T(s)\begin{bmatrix} G_2^T(s) \\ O \end{bmatrix}$ where $G_2(s)$ is invertible and $V^T(s)$ is square inner. We get the overall decomposition

$$G(s) = U(s)\begin{bmatrix} G_2(s) & O \\ O & O \end{bmatrix} V(s). \qquad (20)$$

Define $G^{\#}(s) := V^*(s)\begin{bmatrix} G_2^{-1}(s) & O \\ O & O \end{bmatrix} U^*(s)$. Then $G^{\#}(s)$ fulfills all four axioms (i)–(iv) above, and thus, it is the unique generalized inverse of $G(s)$ (with respect to the conjugation operator in continuous–time). Notice that the rational decomposition (20) of $G(s)$ generalizes the complete orthogonal decomposition of a real matrix.

7 Numerical example

Example 1. Consider the proper rational matrix

$$G(s) = \begin{bmatrix} \dfrac{s-1}{(s+2)} & 0 & \dfrac{s-1}{s+2} \\ \dfrac{s}{(s+2)} & \dfrac{s-2}{(s+1)^2} & \dfrac{s^2+2s-2}{(s+1)(s+2)} \\ \dfrac{1}{(s+2)} & \dfrac{s-2}{(s+1)^2} & \dfrac{2s-1}{(s+1)(s+2)} \end{bmatrix}.$$

The structural elements of $G(s)$ are: zeros at 1, 2, and ∞, all of order 1; poles at -1, -2 both of order 2; normal rank $r = 2$; one left minimal index equal to 0; one right minimal index equal to 1. Thus $n_z = 3$, $n_b = 2$, $n_g = 1$, $n_\ell = 0$, $n_r = 1$. Therefore, none of known methods is applicable to compute the inner–outer factorization of $G(s)$. We start with a minimal order standard state space realization for $G(s)$ given by

$$A=\begin{bmatrix}-2&0&0&0\\-1&-4&-9&2\\0&1&2&-1\\0&0&0&-2\end{bmatrix},\quad B=\begin{bmatrix}0&0&1\\1&1&2\\0&0&0\\1&0&1\end{bmatrix},$$

$$C=\begin{bmatrix}0&0&0&-3\\0&1&0&-3\\0&1&0&0\end{bmatrix},\qquad D=\begin{bmatrix}1&0&1\\1&0&1\\0&0&0\end{bmatrix}.$$

By applying the methods in the paper we obtain the inner and the outer factors as

$$G_i=\left[\begin{array}{cc|c}\frac{\sqrt{2}(s-1)}{(s+1)(s+2)} & -\frac{\sqrt{6}(s-1)}{3(s+2)} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{2}(s-1)}{2(s+1)} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{2}s(s-1)}{2(s+1)(s+2)} & \frac{\sqrt{6}(s-4)}{6(s+2)} & \frac{\sqrt{3}}{3}\end{array}\right],$$

$$G_o=\begin{bmatrix}\frac{\sqrt{2}(s+1)}{2(s+2)} & \frac{\sqrt{2}}{s+1} & \frac{\sqrt{2}(s+3)}{2(s+2)}\\ -\frac{\sqrt{6}(s+1)}{2(s+2)} & \frac{\sqrt{6}}{(s+1)^2} & -\frac{\sqrt{6}(s^2+2s-1)}{2(s+1)(s+2)}\end{bmatrix}.$$

8 Conclusions

We have given complete solutions to the inner–outer and spectral factorization problems formulated for a continuous–time system in the most general setting possible. We have provided both theoretical solutions and numerically reliable procedures. The numerical algorithms for computing these factorizations rely on staircase algorithms as those presented in [11].

Our approach can be viewed as a *divide et impera* procedure as we isolate from the original system only that subsystem which is really needed and for which we can actually solve the factorization problems. This feature of our method leads to the avoidance of the unnecessary redundancy and recommends our procedure also in "standard" cases in which $G(s)$ has no zeros on the imaginary axis and it is injective. Another important feature is that the subsystem in terms of which we write the Lyapunov and Riccati equations is obtained from a realization of the original system by using exclusively orthogonal transformations.

With changes to a certain extent, we can apply this method to obtain the inner–outer and spectral factorizations for a discrete–time system. In this case we rely on a new descriptor realization of an arbitrary rational matrix introduced recently in [12].

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