# Computation of Inner-Outer Factorizations of Rational Matrices

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*Abstract*— **In this paper we propose a new numerically reliable computational approach to determine the inner-outer factorization of a rational transfer matrix $G$ of a linear descriptor system. In contrast to existing computationally involved "one-shot" methods which require the solution of Riccati or generalized Riccati equations, the new approach relies on an efficient recursive zeros dislocation technique. The resulting inner and outer factors have always minimal order descriptor representations. The proposed approach is completely general being applicable whenever $G$ is proper/strictly proper or not, or of full column/row rank or not.**

*Index Terms*—**Inner-outer factorization, numerical methods, poles and zeros, singular systems, system inversion**

## I. Introduction

Let $G(\lambda)$ be a given $p \times m$ stable rational *transfer-function matrix* (TFM) of a linear time-invariant continuous-time or discrete-time descriptor system and let $G = (E, A, B, C, D)$ denote an equivalent $n$th order *regular* ($\det(\lambda E - A) \not\equiv 0$) irreducible descriptor representation satisfying $G(\lambda) = C(\lambda E - A)^{-1}B + D$, where $\lambda$ is either $s$ or $z$, depending on the type of the system. In this paper we address the problem to compute a factorization of the TFM $G$ as a product of a *square* and *proper* inner factor $G_i$ with a stable, minimum-phase factor $G_o$, that is $G = G_i G_o$. Recall that $G_i$ is *inner* means $G_i$ is stable and $G_i^{\sim} G_i = I$, where $G_i^{\sim}(s) = G_i^T(-s)$ in continuous-time and $G_i^{\sim}(z) = G_i^T(1/z)$ in discrete-time. Notice that for a surjective $G$, the above factorization is the usual *inner-outer* factorization of $G$ [1]. Important applications of this factorization and of the closely related spectral factorization are in the $H_\infty$ control theory [2], [1], [3].

Two classes of methods have been proposed to compute inner-outer factorizations for several particular cases. The *first* class of methods relies on computing stabilizing solutions of certain Riccati or generalized Riccati equations. For *continuous-time systems*, the proper case with full column rank $G(s)$ on the extended imaginary axis (the imaginary axis containing the point at infinity) can be solved by employing the algorithm proposed by Doyle [2]. This algorithm has been extended for the more general full rank case by Tsai and Chen [4]. Here a particular Riccati equation is solved in order to compute the inner denominator of a right coprime factorization of a left or right inverse of $G$. The algorithm for surjective $G$ represents a significant improvement over an earlier algorithm proposed by Chen and Francis [5] which led to a constrained Riccati equation. In contrast with the approach in [4], here the outer factor $G_o$ is determined first by solving a spectral factorization problem. A similar approach for solving the more general factorization problem for a constant rank proper $G$ (possibly of not maximal rank) was proposed by Weiss [6] and is based on linear pencil techniques. The strictly proper case for square systems has been considered by Hara and Sugie [7] by using a descriptor system formulation which involves the solution of a descriptor Riccati equation. The case of injective strictly proper $G$ can be solved by the infinite zeros cancellation technique proposed by Copeland and Safonov [8]. For *discrete-time systems* Gu *et al.* [9] derived explicit formulas for the proper case with $G(z)$ injective on the unit circle. Their method, based on computing the outer factor by solving a discrete spectral factorization problem, is restricted to the case of invertible $A$ as well as on other simplifying assumptions. Katayama [10] extended the results of [9] by using a descriptor system formulation, removing all the above constraining assumptions. Recently a method which parallels that of [6], has been proposed for constant rank proper discrete-time systems by Ionescu and Oara [11]. In both continuous- and discrete-time cases no methods are available for strictly proper non full rank or improper matrices. For all methods in this class the computational burden caused by the need to solve Riccati equations represents the principal disadvantage from numerical point of view. Moreover, in all above methods, the resulting inner factors are usually non-minimal.

The *second* class of methods avoids the solution of Riccati equations and is based on recursive dislocation of unstable zeros of $G$ by premultiplying it with suitable elementary all-pass factors. The general technique of zeros dislocation is described by Van Dooren [12]. Apparently the first algorithm to compute recursively the inner-outer factorization of a square invertible continuous-time system has been proposed by Kimura [13]. His method relies on using recursively a pole conjugation technique to reflect the unstable poles of the inverse system to symmetric positions with respect to the imaginary axis. Algorithms based on a similar idea, but without explicitly forming the inverse system, have been proposed for both continuous-time and discrete-time systems by Zhang and Freudenberg [14], [15]. These algorithms compute elementary inner factors which are used to reflect the unstable zeros of $G$ to stable positions symmetrical with respect to the imaginary axis in continuous-time case or the unit circle in discrete-time case. The computation of the elementary first or second order inner factors which reflect one or two unstable zeros at each step is based on computing state and input zero directions of the system matrix pencil. The approach can be straightforwardly extended to handle descriptor representations too. The need to compute zero directions of the system matrix pencil at each step makes however these algorithms of complexity $0(n^4)$ in the worst case of $n$ unstable zeros.

In this paper we propose a new numerically reliable computational approach to compute the factorization $G = G_i G_o$, where $G_i$ is square, proper and inner, and $G_o$ is stable and minimum-phase. If the given $G$ has poles and/or zeros on the imaginary axis for a continuous-time system or on the unit circle for a discrete-time system, we assume that these poles and zeros are included among those of the factor $G_o$. The proposed approach can be viewed as a recursive zeros dislocation technique. Generalized Lyapunov equations of order at most two are solved repeatedly to compute suitable elementary inner factors to reflect the unstable zeros into the stable region of the complex plane. The proposed approach is completely general being applicable whenever $G$ is proper or not, or of full column/row rank or not. The resulting factors can be always determined as minimal order descriptor system representations. The new method is computationally very efficent because it avoids the solution of any Riccati equation by using instead pole assignment like techniques. In the same time it extends the applicability of existing techniques to arbitrary rational matrices.

The procedure to compute inner-outer factorizations is conceptually similar to that of [4] and has the following main steps:

1. Compute a (1,2)-generalized inverse $G^+$ of $G$ such that the unstable poles of $G^+$ are *exactly* the unstable zeros of $G$.

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2. Compute a *right coprime factorization with inner denominator* (RCFID) of $G^+$ as $G^+ = NG_i^{-1}$, where $N$ and $G_i$ are stable TFMs with $G_i$ inner and of *least* order.
3. Compute $G_o = G_i^{-1}G$.

Recall that, by definition, any (1,2)-generalized inverse $G^+$ of $G$ satisfies the axioms $GG^+G = G$ and $G^+GG^+ = G^+$ [16]. It is easy to see that the computed $G_o$ is stable and minimum-phase. The requirements at steps 1 that $G^+$ has the only unstable poles, the unstable zeros of $G$, and at step 2 that $G_i$ has *least* order, ensure that the order of the resulting inner factor $G_i$ equals the number of unstable zeros of $G$. Thus $G_o$ is stable because a complete unstable poles-zeros cancellation occurs in computing $G_i^{-1}G$ at step 3. $G$ is minimum-phase follows by observing that $N$ is a stable (1,2)-generalized inverse of $G_o$, satisfying $G_oNG_o = G_o$ and $NG_oN = N$.

In the rest of the paper we discuss in detail the main computational problems appearing in the proposed computational approach.

## II. Computation of generalized inverses

In this section we discuss a numerically reliable approach to compute a descriptor representation of a particular (1,2)-generalized inverse $G^+$ fulfilling the requirement to have as unstable poles the unstable zeros of $G$. It is known that for a non-square full rank TFM $G$ the zeros of $G$ (finite and infinite) are always poles of any left or right inverse $G^+$ [17, page 467]. However besides these fixed poles, $G^+$ generally possesses also spurious poles resulting from the particular way the inverse was determined. In order to simplify the procedure at step 2, it is convenient to devise an inversion procedure by which these spurious poles result always stable. This allows an easy computation of a least order inner denominator at step 2.

Consider the *system matrix* pencil

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right] \tag{1}$$

associated to the descriptor representation $G = (E, A, B, C, D)$. The computation of a (1,2)-generalized inverse of an arbitrary rational matrix $G$ can be reduced to the computation of a (1,2)-generalized inverse of the system pencil by using the following formula [18]

$$G^+(\lambda) = \left[\begin{array}{cc} 0 & I_m \end{array}\right] \mathcal{S}(\lambda)^+ \left[\begin{array}{c} 0 \\ I_p \end{array}\right], \tag{2}$$

where $\mathcal{S}(\lambda)^+$ is a (1,2)-generalized inverse of $\mathcal{S}(\lambda)$.

The evaluation of $\mathcal{S}(\lambda)^+$ can be done by reducing the system pencil to a particular form which displays a full rank submatrix of it [16]. Let $Q$ and $Z$ be invertible matrices such that

$$Q\mathcal{S}(\lambda)Z = \left[\begin{array}{cc} \mathcal{S}_{11}(\lambda) & \mathcal{S}_{12}(\lambda) \\ 0 & \mathcal{S}_{22}(\lambda) \end{array}\right] \tag{3}$$

and $\operatorname{rank}\mathcal{S}_{12}(\lambda) = \operatorname{rank}\mathcal{S}(\lambda)$. Then a (1,2)-generalized inverse of $\mathcal{S}(\lambda)$ is given by

$$\mathcal{S}(\lambda)^+ = Z \left[\begin{array}{cc} 0 & 0 \\ \mathcal{S}_{12}(\lambda)^{-1} & 0 \end{array}\right] Q. \tag{4}$$

To compute a descriptor representation of the generalized inverse it is not necessary to evaluate $\mathcal{S}_{12}(\lambda)^{-1}$ explicitly. If we denote

$$A_{12} - \lambda E_{12} = \mathcal{S}_{12}(\lambda), \quad \bar{B} = Q \left[\begin{array}{c} 0 \\ I_p \end{array}\right] = \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right],$$

$$\bar{C} = \left[\begin{array}{cc} 0 & I_m \end{array}\right] Z = \left[\begin{array}{cc} C_1 & C_2 \end{array}\right],$$

where $\bar{B}$ and $\bar{C}$ are partitioned analogously with the column and row partition of $\mathcal{S}(\lambda)$ in (3), respectively, then by combining the above results, the corresponding $G(\lambda)^+$ is given by

$$G(\lambda)^+ = -C_2(\lambda E_{12} - A_{12})^{-1}B_1, \tag{5}$$

and thus $(E_{12}, A_{12}, B_1, -C_2, 0)$ is a descriptor representation of $G(\lambda)^+$.

In what follows we show how to determine the left and right transformation matrices $Q$ and $Z$ in (3) to isolate a suitable full rank sub-pencil of the system matrix. Firstly, with the help of two orthogonal matrices $U_1$ and $V_1$ it is possible to reduce the system pencil (1) to the following Kronecker-like form which exhibits the complete Kronecker structure of $\mathcal{S}(\lambda)$ [19]:

$$U_1\mathcal{S}(\lambda)V_1 = \left[\begin{array}{c|ccc} B_r & A_r - \lambda E_r & * & * \\ 0 & 0 & A_{reg} - \lambda E_{reg} & * \\ 0 & 0 & 0 & A_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array}\right], \tag{6}$$

where: (1) the pencil $[\ B_r\ A_r - \lambda E_r\ ]$, with $E_r$ non-singular, contains the right Kronecker structure of $\mathcal{S}(\lambda)$ and the pair $(B_r, A_r - \lambda E_r)$ is controllable; (2) the *regular* pencil $A_{reg} - \lambda E_{reg}$ has the form

$$A_{reg} - \lambda E_{reg} = \left[\begin{array}{cc} A_\infty - \lambda E_\infty & * \\ 0 & A_f - \lambda E_f \end{array}\right], \tag{7}$$

where the pencil $A_\infty - \lambda E_\infty$, with $A_\infty$ non-singular and $E_\infty$ nilpotent, contains the infinity Kronecker structure of $\mathcal{S}(\lambda)$, while the pencil $A_f - \lambda E_f$, with $E_f$ non-singular, contains the finite Kronecker structure of $\mathcal{S}(\lambda)$; the generalized eigenvalues of the pair $(A_f, E_f)$ are the *finite zeros* of the $G$; (3) the pencil $\left[\begin{array}{c} A_l - \lambda E_l \\ C_l \end{array}\right]$, with $E_l$ non-singular, contains the left Kronecker structure of $\mathcal{S}(\lambda)$ and the pair $(C_l, A_l - \lambda E_l)$ is observable.

By using a second pair of left and right transformation matrices $U_2$ and $V_2$ of the special forms

$$U_2 = \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & K \\ 0 & 0 & 0 & I \end{array}\right], \quad V_2 = \left[\begin{array}{cccc} I & F & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \tag{8}$$

we obtain

$$\left[\begin{array}{c|c} \mathcal{S}_{11}(\lambda) & \mathcal{S}_{12}(\lambda) \\ \hline 0 & \mathcal{S}_{22}(\lambda) \end{array}\right] := Q\mathcal{S}(\lambda)Z =$$

$$\left[\begin{array}{c|ccc} B_r & A_r + B_rF - \lambda E_r & * & * \\ 0 & 0 & A_{reg} - \lambda E_{reg} & * \\ 0 & 0 & 0 & A_l + KC_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array}\right] \tag{9}$$

where $Q = U_2U_1$ and $Z = V_1V_2$. Because the pair $(B_r, A_r - \lambda E_r)$ is controllable and the pair $(C_l, A_l - \lambda E_l)$ is observable, we can arbitrarily assign stable spectrums for the matrix pairs $(A_r + B_rF, E_r)$ and $(A_l + KC_l, E_l)$ by choosing suitable state-feedback and output-injection matrices $F$ and $K$, respectively.

With the partitioning in (9) it follows that for almost all $\lambda$, $\operatorname{rank}\mathcal{S}(\lambda) = \operatorname{rank}\mathcal{S}_{12}(\lambda)$, and thus a generalized (1,2)-inverse of $\mathcal{S}(\lambda)$ can be computed using (4) and a descriptor representation of the generalized inverse $G(\lambda)^+$ is given by (5).

From the above construction it is clear that the poles of $G^+$ include always the zeros of $G$ (the generalized eigenvalues of the regular part (7) ) as the fixed poles of $G^+$. The spurious poles are the union of generalized eigenvalues of the pairs $(A_r + B_rF, E_r)$ and $(A_l + KC_l, E_l)$ and as mentioned before,

can be arbitrarily assigned. If $G$ has only stable zeros, then this approach can be used to compute stable generalized inverses of $G$ by appropriately determining stabilizing matrices $F$ and $K$. Generally with $F$ and $K$ stabilizing, the only unstable poles of the generalized inverse (5) are the unstable zeros of $G$. Thus $G(\lambda)^+$ satisfies the condition imposed at step 1 of the procedure to compute inner-outer factorization given in the previous section.

The reduction techniques to compute (6) is based on structure preserving algorithms similar to those described in [20] to compute the system zeros. A complete algorithm to compute (6) is described in detail in [19] together with the corresponding computational programs. The stabilizing matrices $F$ and $K$ in (9) can be efficiently computed by using either direct stabilization methods or pole assignment techniques for descriptor systems as those proposed in [21]. The overall computational complexity of computing a generalized inverse of $G$ is $0(n^3)$ and all computations can be done with numerically stable or numerically reliable algorithms.

## III. Computation of the inner factor

Let $G = (E, A, B, C, D)$ be a stabilizable descriptor representation of a rational TFM $G$. In this section we present an algorithm to compute a least order inner denominator $G_i$ of a RCFID of $G$, $G = NG_i^{-1}$. This algorithm can be used at step 2 of the proposed approach to determine the least order inner denominator factor of the RCFID of $G^+$. The algorithm to compute $G_i$ relies on several simple facts (see also [22] for more details).

**Fact 1.** [23] *For any $F$ of appropriate dimensions and for any invertible $W$, the descriptor representations*

$$\begin{aligned} N &= (E,\ A+BF,\ BW,\ C+DF,\ DW) \\ M &= (E,\ A+BF,\ BW,\ F,\ W) \end{aligned}$$

*give a fractional representation of $G$ as $G = NM^{-1}$. If $F$ is chosen such that the pair $(E, A+BF)$ is stable, then the fractional representation is a right coprime factorization.*

Notice that in computing the denominator factor $M$ the output matrices $C$ and $D$ play no role. As it will be later apparent this fact can be exploited to simplify sensibly the first two steps of the overall procedure presented in section 1.

The algorithm to compute inner denominators uses recursively the following explicit updating technique of fractional representations.

**Fact 2.** *If $G = N_1M_1^{-1}$ and $N_1 = N_2M_2^{-1}$ are fractional representations with inner denominators, then $G$ has the fractional representation $G = NM^{-1}$, where $N = N_2$ and $M = M_1M_2$ is inner.*

This simple fact allows us to obtain explicit formulas to update partial factorizations by using simple state space formulas. Let $N_1$ and $M_1$ be the factors computed as

$$\begin{aligned} N_1 &= (E,\ A+BF_1,\ BW_1,\ C+DF_1,\ DW_1) \\ M_1 &= (E,\ A+BF_1,\ BW_1,\ F_1,\ W_1) \end{aligned}$$

and let $N_2$ and $M_2$ be the factors of $N_1$ computed as

$$\begin{aligned} N_2 &= (E,\ A+BF,\ BW,\ C+DF,\ DW) \\ M_2 &= (E,\ A+BF,\ BW,\ F_2,\ W_2) \end{aligned}$$

where

$$\begin{aligned} F &= F_1 + W_1F_2 \\ W &= W_1W_2 \end{aligned} \tag{10}$$

It easy to verify that the product $M_1M_2$ is given by

$$M_1M_2 = (E,\ A+BF,\ BW,\ F,\ W) \tag{11}$$

and thus the equations (10) and (11) serve as explicit updating formulas of fractional representations.

Elementary first or second order inner factors can be used to reflect the unstable poles of $G$ to symmetric positions with respect to the imaginary axis in case of continuous-time systems or with respect of the unit circle in case of discrete-time systems. This pole dislocation technique is used in our algorithm and is conceptually similar to the conjugation technique proposed in [13]. Formulas to compute elementary inner denominators are established below.

**Fact 3.** *Let $G = (E, A, B, *, *)$ a controllable descriptor representation with $E$ non-singular and $\Lambda(E, A) \in \mathbb{C}^+$ (the appropriate unstable region of the complex plane $\mathbb{C}$). Then the denominator factor $G_i = (E, A+BF, BW, F, W)$ is* inner *by choosing $F$ and $W$ as:*

$$\begin{aligned} &AYE^T + EYA^T - BB^T = 0 \\ &F = -B^T(YE^T)^{-1}, \quad W = I, \end{aligned}$$

*for a continuous-time system, and*

$$\begin{aligned} &AYA^T - BB^T = EYE^T \\ &F = -B^T(EYE^T + BB^T)^{-1}A \\ &W = (I + B^T(EYE^T)^{-1}B)^{-\frac{1}{2}}, \end{aligned}$$

*for a discrete-time system.*

A general algorithm to compute RCFIDs, based on the above facts, is given in [22]. The following algorithm is a specially tailored efficient variant of it devised to compute exclusively the inner denominator $G_i$ of a RCFID of $G$.

**RCFID Algorithm.**

1. Find orthogonal matrices $Q$ and $Z$ to reduce the pair $(E, A)$ to the ordered *generalized real Schur form* (GRSF) $(QEZ, QAZ)$ such that

$$QEZ = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix},\ QAZ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},\ QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

   where $E_{22}, A_{22} \in \mathbb{R}^{q\times q}$, $\Lambda(A_{11}, E_{11}) \cap \mathbb{C}^+ = \emptyset$ and $\Lambda(A_{22}, E_{22}) \subset \mathbb{C}^+$.
   Set $(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{F}, \widehat{W}) = (E_{22}, A_{22}, B_2, 0, I)$ and $j = 1$.

2. If $q = 0$, go to 6.

3. Let $(\delta, \alpha)$ be the last pair of elementary diagonal blocks of order $k$ ($k = 1$ or 2) of the pair $(\widehat{E}, \widehat{A})$ in GRSF and let $\beta$ be the $k \times m$ matrix formed from the last $k$ rows of $\widehat{B}$. For the system $(\delta, \alpha, \beta, *, *)$ compute $\varphi$ and $V$ such that $(\delta, \alpha + \beta\varphi,\ \beta V, \varphi, V)$ is inner. Set $K = [\,0\,\varphi\,]$.

4. Compute $\widehat{A} \leftarrow \widehat{A} + \widehat{B}K$, $\widehat{F} \leftarrow \widehat{F} + \widehat{W}K$, $\widehat{B} \leftarrow \widehat{B}V$, $\widehat{W} \leftarrow \widehat{W}V$.

5. Compute the orthogonal matrices $\widehat{Q}$ and $\widehat{Z}$ to reorder the diagonals blocks of the pair $(\widehat{E}, \widehat{A})$ in GRSF, such that the last block of $(\widehat{E}, \widehat{A})$ is moved by successive interchanging of diagonal blocks to row position $j$. Compute $\widehat{E} \leftarrow \widehat{Q}\widehat{E}\widehat{Z}$, $\widehat{A} \leftarrow \widehat{Q}\widehat{A}\widehat{Z}$, $\widehat{B} \leftarrow \widehat{Q}\widehat{B}$, $\widehat{F} \leftarrow \widehat{F}\widehat{Z}$. Put $q \leftarrow q - k$, $j \leftarrow j + k$ and go to 2.

6. Set $G_i = (\widehat{E}, \widehat{A}, \widehat{B}, \widehat{F}, \widehat{W})$.

The RCFID algorithm uses extensively orthogonal transformations and thus relies on reliable numerical techniques. It can be viewed as a generalized Schur algorithm for pole assignment (similar to that of [21]) which assigns the unstable poles in symmetrical positions with respect to the imaginary axis in

the continuous-time case or the unit circle in the discrete-time case.

If the RCFID Algorithm is applied to the TFM $G^+$ computed in the previous section, then the order of the inner factor is precisely the number of unstable zeros of $G$. In this case the RCFID algorithm can be viewed as a recursive zeros dislocation algorithm where at each step the unstable zeros of $G$ are reflected to stable symmetrical positions by premultiplying $G$ with the inverses of elementary inner factors. The resulting pair $(\widehat{E}, \widehat{A})$ is in a generalized real Schur form having as stable eigenvalues the reflected unstable zeros of $G$.

**Remark.** For the computation of the inner denominator $G_i$ of the RCFID of $G^+$ it is actually not necessary to form the complete descriptor representation (5) of the generalized inverse $G^+$. Because the output matrix $-C_2$ of $G^+$ plays no role in the computations, to compute $G_i$ it is sufficient to determine $\bar{A}_{12} - \lambda\bar{E}_{12}$, the lower right corner of $A_{12} - \lambda E_{12}$ given by

$$\bar{A}_{12} - \lambda\bar{E}_{12} = \left[\begin{array}{cc} A_f - \lambda E_f & * \\ 0 & A_l + KC_l - \lambda E_l \end{array}\right], \qquad (12)$$

which contains the unstable zeros of $G$, and $\bar{B}_1$, the corresponding rows of $B_1$, given by

$$\bar{B}_1 = [\,0\ I_{\bar{n}}\,]B_1, \qquad (13)$$

where $\bar{n}$ is the order of $\bar{A}_{12} - \lambda\bar{E}_{12}$. The system matrices $\bar{A}_{12}$, $\bar{E}_{12}$ and $\bar{B}_1$ can be then used as input data for the RCFID Algorithm. The matrices in (12) and (13) can be determined by computing (without accumulating $Z$) the following simpler Kronecker-like form

$$\widehat{\mathcal{S}}(\lambda) := Q\mathcal{S}(\lambda)Z = \left[\begin{array}{ccc} A_r - \lambda E_r & * & * \\ 0 & A_f - \lambda E_f & * \\ \hline 0 & 0 & A_l - \lambda E_l \\ 0 & 0 & C_l \end{array}\right], \qquad (14)$$

where this time the pencil $A_r - \lambda E_r$ contains the right and the infinity Kronecker structures of the system pencil. A specially devised procedure presented in [24] can be employed for this purpose. These algorithmic refinements lead to substantial savings of the computational effort for the overall factorization algorithm.

## IV. Computation of the outer factor

The outer factor $G_o$ can be computed by removing the uncontrollable unstable eigenvalues from the descriptor representation of $G_i^{-1}G$:

$$\left(\left[\begin{array}{cc} \widehat{E} & 0 \\ 0 & E \end{array}\right], \left[\begin{array}{cc} \widehat{A} - \widehat{B}\widehat{F} & \widehat{B}C \\ 0 & A \end{array}\right], \left[\begin{array}{c} \widehat{B}D \\ B \end{array}\right], -\widehat{W}^{-1}[\,\widehat{F}\ \ C\,], \widehat{W}^{-1}D\right)$$

These eigenvalues can be removed in a numerically sound way by using the following approach. First reduce the pair

$$\left(\left[\begin{array}{cc} \widehat{E} & 0 \\ 0 & E \end{array}\right], \left[\begin{array}{cc} \widehat{A} - \widehat{B}\widehat{F} & \widehat{B}C \\ 0 & A \end{array}\right]\right)$$

by using orthogonal similarity transformations, to an ordered GRSF where the unstable diagonal blocks are situated in the bottom right corner of the resulting pair. Because the unstable eigenvalues are uncontrollable, the corresponding rows in the transformed input matrix should be zero. Thus, after applying the transformations to the input and output matrices of $G_i^{-1}G$ , the $n$th order outer factor results by simply retaining the subsystem

$$G_o = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$$

corresponding to the stable eigenvalues. Notice that the pair $(\widetilde{E}, \widetilde{A})$ is in a GRSF.

The computed descriptor representation of the outer factor is minimal unless $G$ contains a so-called *free* inner factor [25]. Because we assumed the original descriptor representation minimal and no stable pole-zero cancelations took place in computing $G_i^{-1}G$, it follows that the descriptor representation of $G_o$ has no finite uncontrollable eigenvalues but could have finite unobservable eigenvalues if $G$ contains a free inner factor. To remove the unobservable eigenvalues either the general method of [26] or the special technique of [24] can be employed.

## V. An example

Consider the proper TFM

$$G = \left[\begin{array}{ccc} \dfrac{s-2}{(s+1)^2} & \dfrac{s-2}{s+1} & 0 \\ \dfrac{1}{(s+1)^2} & \dfrac{s^2+s+1}{(s+1)(s+2)} & \dfrac{s-1}{(s+1)(s+2)} \end{array}\right]$$

having two unstable zeros at $z_1 = 1$ and $z_2 = 2$. $G$ has full normal rank, but is not surjective at infinity. The methods of [4] or [6] require that $G$ has full row rank or constant rank, respectively, over the extended imaginary axis. Thus to compute the inner-outer factorization of $G$, none of these methods can be employed. In what follows we show how to compute this factorization by using the algorithm proposed in this paper.

A minimal order standard state-space realization of $G$ is given by the matrices

$$A = \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right], \quad B = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & -2 \\ 0 & -3 & 3 \end{array}\right],$$

$$C = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right], \quad D = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

With the transformation matrices $Q$ and $Z$

$$Q = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 2 & -3 & 6 \\ -1 & 1 & 0 & 0 & 3 & 0 \end{array}\right], \quad Z = \left[\begin{array}{ccccccc} 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & 0 & 3 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we obtain the Kronecker-like form (14) of the system matrix. Because there is no left Kronecker structure, it is not necessary to determine an output injection matrix $K$ in (12). Thus we obtain without additional computations the matrices

$$\bar{A}_{12} - \lambda\bar{E}_{12} = \left[\begin{array}{cc} 1 & -3 \\ 0 & 2 \end{array}\right] - \lambda\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \quad \bar{B}_1 = \left[\begin{array}{cc} -3 & 6 \\ 3 & 0 \end{array}\right]$$

in (12) and (13).

The inner factor determined with the recursive RCFID algorithm has the state-space realization

$$A_i = \left[\begin{array}{cc} -2 & 0 \\ 0 & -1 \end{array}\right], \quad B_i = \left[\begin{array}{cc} -3.7947 & 1.8974 \\ -1.8974 & 5.6921 \end{array}\right],$$

$$C_i = \left[\begin{array}{cc} 1.2649 & -0.3162 \\ 0 & -0.3162 \end{array}\right], \quad D_i = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

and the corresponding TFM $G_i$ is

$$G_i = \begin{bmatrix} \dfrac{s^2-1.2s-1.6}{(s+1)(s+2)} & \dfrac{0.6s-1.2}{(s+1)(s+2)} \\ \dfrac{0.6}{s+1} & \dfrac{s-0.8}{s+1} \end{bmatrix}.$$

Because its recursive nature, the RCFID algorithm determines $G_i$ essentially in a factored form $G_i = G_{i1}G_{i2}$, where the elementary first order inner factors used for zeros dislocations (one-zero-at-time) can be explicitly computed as

$$G_{i1} = \begin{bmatrix} \dfrac{s-2}{s+2} & 0 \\ 0 & 1 \end{bmatrix}, \quad G_{i2} = \begin{bmatrix} \dfrac{s+0.8}{s+1} & \dfrac{0.6}{s+1} \\ \dfrac{0.6}{s+1} & \dfrac{s-0.8}{s+1} \end{bmatrix}.$$

After removing the uncontrollable eigenvalues from the state-space realization of $G_i^{-1}G$ we obtain the following state-space matrices for $G_o$ (rounded to five decimal figures):

$$\widetilde{A} = \begin{bmatrix} -1.0000 & -1.2989 & 2.3795 & 3.4980 \\ 0 & -1.0000 & -1.1438 & 0.4421 \\ 0 & 0 & -2.0000 & 0.3865 \\ 0 & 0 & 0 & -1.0000 \end{bmatrix},$$

$$\widetilde{B} = \begin{bmatrix} -0.5669 & 1.8898 & -0.3780 \\ -0.7712 & 4.3527 & 0.4836 \\ 0.3753 & 3.8056 & -3.2559 \\ 0.9710 & 0.0000 & 1.4220 \end{bmatrix},$$

$$\widetilde{C} = \begin{bmatrix} 0.2268 & -0.2339 & 0.3126 & 0.8556 \\ 0.0756 & 0.0038 & -0.2522 & 0.1447 \end{bmatrix},$$

$$\widetilde{D} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This state-space model is not minimal (not observable) and therefore $G$ contains a free inner factor. After removing the unobservable part from the above model we obtain the minimal order state-space realization of $G_o$

$$A_o = \begin{bmatrix} -1.4811 & -0.3818 & 3.9043 \\ 0.1761 & -2.1326 & 1.3723 \\ -0.0660 & -0.0879 & -0.3863 \end{bmatrix},$$

$$B_o = \begin{bmatrix} -0.7819 & 3.8042 & -1.3904 \\ -0.2425 & -2.4397 & 3.2742 \\ 1.1164 & 0.4923 & -0.1133 \end{bmatrix},$$

$$C_o = \begin{bmatrix} 0.0749 & 0.0651 & 0.9623 \\ -0.0265 & 0.2957 & 0.0456 \end{bmatrix},$$

$$D_o = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

and the corresponding TFM

$$G_o = \begin{bmatrix} \dfrac{s+2.2}{(s+1)^2} & \dfrac{s^2+3.6s+3.8}{(s+1)(s+2)} & \dfrac{-0.6}{(s+1)(s+2)} \\ \dfrac{0.4}{(s+1)^2} & \dfrac{s^2+2.2s+1.6}{(s+1)(s+2)} & \dfrac{s+0.8}{(s+1)(s+2)} \end{bmatrix}.$$

## VI. Conclusions

A completely general implementable procedure to compute inner-outer factorizations of rational matrices has been proposed. All computational steps of this procedure can be performed by using exclusively numerically reliable algorithms. The procedure is well suited for robust and modular software implementation. It is worth to mention that the proposed procedure is applicable even when the given rational matrix has zeros on the imaginary axis for a continuous-time system or on the unit circle for a discrete-time system. With appropriate modifications of the RCFID algorithm a similar procedure can be devised for the computation of J-inner-outer factorizations [24].

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