Robust and minimum norm pole assignment with periodic state feedback

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Abstract

A computational approach is proposed to solve the minimum norm and/or robust pole assignment problem for linear periodic discrete-time systems. The proposed approach uses a periodic Sylvester equation based parametrization of the periodic pole assignment problem and exploits the non-uniqueness of the problem by imposing conditions on the norm of the resulting periodic state feedback and/or on the condition numbers of the periodic eigenvector matrices of the closed-loop system. The solution method relies on using gradient search methods on suitably defined cost functions. Explicit expression of the gradients of cost functions are derived and the efficient evaluation of the cost functions and gradients is discussed. Numerical examples illustrate the effectiveness of the proposed approach.

1 Introduction

Consider the linear discrete-time periodic system of the form

$$x_{k+1} = A_k x_k + B_k u_k, \qquad (1)$$

where the matrices $A_k \in \mathbb{R}^{n\times n}$ and $B_k \in \mathbb{R}^{n\times m}$ are periodic with period $K \geq 1$. We consider the following *periodic eigenvalue assignment problem* (PEAP): given the completely reachable periodic matrix pair (A_k, B_k), determine the periodic feedback matrix $F_k \in \mathbb{R}^{m\times n}$ such that the eigenvalues of the closed-loop monodromy matrix $\Phi_{A+BF}(K,0) = (A_{K-1} + B_{K-1}F_{K-1})\cdots(A_0 + B_0F_0)$ are at desired locations $\Gamma = \{\lambda_1, \ldots, \lambda_n\}$ in the complex plane. We assume that Γ is symmetric with respect to the real axis. This assumption guarantees that the resulting periodic matrix F_k can be chosen real.

In the multi-input case or when $K > 1$, the PEAP has a non-unique solution. Therefore it is reasonable to exploit the non-uniqueness by imposing additional conditions. One aspect which is desirable from a practical point of view is to determine feedback matrices with small gains. Intuitively this must be advantageous since small feedback gains lead to smaller control signals, and thus to less energy consumption. Small gains are also beneficial to reduce noise amplification. The robust pole assignment [1, 2] tries to achieve small condition numbers for the eigenvector matrices of the closed-loop system. Both these aspects are also relevant from the numerical point of view. It was argued in the standard case [3] that both high gains as well as high condition numbers lead to an increased sensitivity of the closed-loop eigenvalues. It appears thus that the simultaneous minimization of these quantities is a desirable goal for solving the PEAP.

In this paper we address the problem to determine the minimum Frobenius-norm periodic state feedback which solves the PAEP and simultaneously minimizes the sensitivity of the closed-loop eigenvalues. The norm minimization problem for standard pole assignment problem has been considered in [4]. Here, a Sylvester equation based parametrization of the eigenvalue assignment problem is used and the minimum norm feedback is computed by performing a gradient search on the free problem parameters. This approach has been improved in [5], by eliminating artificial constraints on the problem and by improving tremendously the computational efficiency. An enhanced approach for the minimum norm periodic pole assignement has been recently developed in [6].

In this paper we propose a computational approach which extends the method of [6] to address simultaneously the robust pole assignment aspect by defining new cost functions to be minimized. By using a periodic Sylvester equation based parametrization of the PEAP we derive explicit expressions of the gradients of the cost functions in terms of the free problem parameters. The availability of analytic gradients allows an efficient use of powerful minimization methods based

on gradient search techniques. Further we discuss the numerical aspects of evaluating the cost functions and their gradients. Each function/gradient evaluation involves the solution of two periodic Sylvester equations. Efficient, numerically stable algorithms to solve such equations are based on the use of the periodic Schur form and are described in [7, 8]. We show how to use these algorithms to make cost and gradient evaluations highly efficient. Finally, we present some numerical examples to illustrate the effectiveness of the proposed approach to solve minimum norm robust pole assignment problems.

Notation and notational conventions. For a square time-varying matrix A_k, $k = 0, 1, \ldots$, we denote $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$ for $j > i$ and $\Phi_A(i,i) := I$. If A_k is periodic with period K, then the *monodromy matrix* of the system (1) at time τ is $\Phi_A(\tau+K,\tau)$. Its eigenvalues are independent of τ and are called *characteristic multipliers*. For an arbitrary periodic matrix X_k of period K we use alternatively the *script* notation $\mathcal{X}$ which associates the block-diagonal matrix $\mathcal{X} = \mathrm{diag}\,(X_0, X_1, \ldots, X_{K-1})$ to the cyclic sequence of matrices X_k, $k = 0, \ldots, K-1$. This notation is consistent with the standard matrix operations like addition, multiplication, transposing, or inversion. Further we denote with $\sigma\mathcal{X}$ the K-cyclic shift $\sigma\mathcal{X} = \mathrm{diag}\,(X_1, \ldots, X_{K-1}, X_0)$ applied to the cyclic sequence X_k, $k = 0, \ldots, K$–1. The notation $\mathcal{X}_{ij}$ is used to refer simultaneously to *all* (i,j) elements or to *all* (i,j) blocks of the cyclic sequence X_k, $k = 0, \ldots, K$–1. With a little abuse, we will also use this notation to denote the submatrices of partitioned periodic matrices.

2 Computation of Robust and Minimum Norm Feedback

To solve the PEAP we use a straightforward parameterization of the pole assignment problem. Let $G_k \in \mathbb{R}^{m\times n}$ be a given periodic parameter matrix and let $\widetilde{A}_k \in \mathbb{R}^{n\times n}$ be a given periodic matrix such that $\Lambda(\Phi_{\widetilde{A}}(K,0)) = \Gamma$, where Γ is a given set of characteristic values. If we determine F_k as $F_k = G_k X_k^{-1}$, where the periodic matrix X_k satisfies the *periodic Sylvester equation* (PSE)

$$A_k X_k - X_{k+1}\widetilde{A}_k + B_k G_k = 0,\ k = 0, \ldots, K-1, \quad (2)$$

then we have $X_0^{-1}\Phi_{A+BF}(K,0)X_0 = \Phi_{\widetilde{A}}(K,0)$, and thus, F_k is a periodic matrix which solves the PEAP. Usual restrictions on choosing $\widetilde{A}_k$ and G_k are similar to those in the standard case [4]: (1) the periodic pair $(\widetilde{A}_k, G_k)$ is completely observable; and (2) $\Lambda(\Phi_A(K,0)) \cap \Lambda(\Phi_{\widetilde{A}}(K,0)) = \emptyset$. If additionally the periodic pair (A_k, B_k) is completely reachable, then X_k satisfying (2) is generically nonsingular and for the above F_k, we have $X_{k+1}^{-1}(A_k + B_kF_k)X_k = \widetilde{A}_k$. To solve the robust pole assignment problem we also assume that the resulting closed-loop monodromy matrix $\Phi_{A+BF}(K,0)$ is *non-defective*. This can be achieved by choosing, for instance, $\widetilde{A}_0$ in a real Jordan form with only 1×1 or 2×2 diagonal blocks and $\widetilde{A}_k$, $k = 1, \ldots, K-1$, diagonal. In this case, each X_k is also an eigenvector matrix for the corresponding monodromy matrix $\Phi_{A+BF}(k+K,k)$. Although redundant, the above parameterization has the main advantage to allow the derivation of analytic expressions of gradients of many useful cost functions which can be formulated to eliminate the intrinsic non-uniqueness of the PEAP.

To solve the minimum norm robust PEAP consider the following cost function to be minimized

$$J = \alpha\frac{1}{2}\sum_{k=0}^{K-1}\kappa_F^2(X_k) + (1-\alpha)\frac{1}{2}\sum_{k=0}^{K-1}\|F_k\|_F^2,$$

where $\kappa_F(X_k) := \|X_k\|_F \cdot \|X_k^{-1}\|_F$ is the Frobenius-norm condition number of X_k and $0 \le \alpha \le 1$ is a weighting factor. Notice that with $\alpha = 0$, J defines a norm minimization problem, while with $\alpha = 1$ we get a pure robust pole assignment problem. Intermediary values of α lead to a combination of both aspects.

J can be expressed alternatively as

$$\begin{aligned} J &= \alpha\frac{1}{2}\sum_{k=0}^{K-1}\mathrm{tr}\,[X_k^T X_k]\mathrm{tr}\,[X_k^{-T}X_k^{-1}] \\ &\quad +(1-\alpha)\frac{1}{2}\sum_{k=0}^{K-1}\mathrm{tr}\,[F_k^T F_k], \end{aligned} \quad (3)$$

or, by using the script notation, as

$$\begin{aligned} J &= \alpha\frac{1}{2}\sum_{k=0}^{K-1}\mathrm{tr}\,[\mathcal{I}_k\mathcal{X}^T\mathcal{X}]\mathrm{tr}\,[\mathcal{I}_k\mathcal{X}^{-T}\mathcal{X}^{-1}] \\ &\quad +(1-\alpha)\frac{1}{2}\mathrm{tr}\,[\mathcal{F}^T\mathcal{F}], \end{aligned} \quad (4)$$

where $\mathcal{I}_k$ is a block diagonal matrix with all diagonal blocks equal to zero with the exception of k-th block which is equal to I_n. Using the proposed parametrization $\mathcal{F}$ is computed as

$$\mathcal{F} = \mathcal{G}\mathcal{X}^{-1}, \quad (5)$$

where $\mathcal{X}$ satisfies the PSE

$$\mathcal{A}\mathcal{X} - \sigma\mathcal{X}\widetilde{\mathcal{A}} + \mathcal{B}\mathcal{G} = 0. \quad (6)$$

The gradient of J with respect to $\mathcal{G}$ can be computed by employing the following result:

Proposition 1 *Let $\mathcal{F}$ be the periodic feedback computed as in (5), assigning the desired characteristic values Γ for given $\widetilde{\mathcal{A}}$ and $\mathcal{G}$. Then, the gradient of J with respect to $\mathcal{G}$ is given by*

$$\nabla_{\mathcal{G}} J = (1-\alpha)(\mathcal{H}^T - \mathcal{B}^T\mathcal{U}^T) + \alpha \sum_{k=0}^{K-1} \mathrm{tr}\,[X_k^T X_k]\mathcal{B}^T\mathcal{V}_k^T + \alpha \sum_{k=0}^{K-1} \mathrm{tr}\,[X_k^{-T} X_k^{-1}]\mathcal{B}^T\mathcal{W}_k^T, \quad (7)$$

where $\mathcal{H} = \mathcal{X}^{-1}\mathcal{F}^T$, $\mathcal{U}$ satisfies the PSE

$$\widetilde{\mathcal{A}}\mathcal{U} - \sigma\mathcal{U}\sigma\mathcal{A} - \sigma\mathcal{H}\sigma\mathcal{F} = 0 \quad (8)$$

and $\mathcal{V}_k^T$, $\mathcal{W}_k^T$, $k = 0, \ldots, K-1$ satisfy the PSEs

$$\widetilde{\mathcal{A}}\mathcal{V}_k - \sigma\mathcal{V}_k\sigma\mathcal{A} - \sigma\mathcal{R}_k = 0 \quad (9)$$

$$\widetilde{\mathcal{A}}\mathcal{W}_k - \sigma\mathcal{W}_k\sigma\mathcal{A} - \sigma\mathcal{Q}_k = 0\,, \quad (10)$$

with $\mathcal{R}_k = \mathcal{I}_k\mathcal{X}$ and $\mathcal{Q}_k = \mathcal{I}_k\mathcal{X}^{-1}\mathcal{X}^{-T}\mathcal{X}^{-1}$.

Proof. See Appendix A. □

To compute for a given $\mathcal{G}$ the function J and its gradient, we have to solve apparently $2(K+1)$ PSEs of essentially the same form as equation (6). By exploiting the linearity of the PSE it is easy to observe that we can solve directly for

$$\mathcal{Z} = \sum_{k=0}^{K-1} (\mathrm{tr}\,[X_k^T X_k]\mathcal{V}_k + \mathrm{tr}\,[X_k^{-T} X_k^{-1}]\mathcal{W}_k)$$

which satisfies the PSE

$$\widetilde{\mathcal{A}}\mathcal{Z} - \sigma\mathcal{Z}\sigma\mathcal{A} - \sigma\mathcal{P} = 0, \quad (11)$$

where

$$\mathcal{P} = \sum_{k=0}^{K-1} (\mathrm{tr}\,[X_k^T X_k]\mathcal{R}_k + \mathrm{tr}\,[X_k^{-T} X_k^{-1}]\mathcal{Q}_k). \quad (12)$$

The corresponding expression of the gradient is

$$\nabla_{\mathcal{G}} J = (1-\alpha)(\mathcal{H}^T - \mathcal{B}^T\mathcal{U}^T) + \alpha\mathcal{B}^T\mathcal{Z}^T. \quad (13)$$

Further, we can even solve instead (8) and (11) a single PSE of the form

$$\widetilde{\mathcal{A}}\mathcal{U} - \sigma\mathcal{U}\sigma\mathcal{A} - \sigma\mathcal{Q} = 0 \quad (14)$$

where $\mathcal{Q} = (1-\alpha)\mathcal{H}\mathcal{F} + \alpha\mathcal{P}$, with $\mathcal{P}$ given by (12). The expression of the gradient is now

$$\nabla_{\mathcal{G}} J = (1-\alpha)\mathcal{H}^T + \mathcal{B}^T\mathcal{U}^T. \quad (15)$$

A somewhat simpler expression for the free term of the PSE (14) results by employing an alternative cost function J', which essentially leads to the same results. The proposed modified cost function is

$$J' = \alpha\frac{1}{2}\sum_{k=0}^{K-1} \mathrm{tr}\,([X_k^T X_k] + \mathrm{tr}\,[X_k^{-T} X_k^{-1}]) + (1-\alpha)\frac{1}{2}\sum_{k=0}^{K-1} \mathrm{tr}\,[F_k^T F_k] \quad (16)$$

and can be expressed by using the script notation as

$$J' = \alpha(\frac{1}{2}\mathrm{tr}\,[\mathcal{X}^T\mathcal{X}] + \frac{1}{2}\mathrm{tr}\,[\mathcal{X}^{-T}\mathcal{X}^{-1}]) + (1-\alpha)\frac{1}{2}\mathrm{tr}\,[\mathcal{F}^T\mathcal{F}]. \quad (17)$$

We have the following result for the expression of the gradient.

Proposition 2 *Let $\mathcal{F}$ be the periodic feedback computed as in (5), assigning the desired characteristic values Γ for given $\widetilde{\mathcal{A}}$ and $\mathcal{G}$. Then, the gradient of J' with respect to $\mathcal{G}$ is given by*

$$\nabla_{\mathcal{G}} J' = (1-\alpha)\mathcal{H}^T + \mathcal{B}^T\mathcal{U}^T, \quad (18)$$

where $\mathcal{H} = \mathcal{X}^{-1}\mathcal{F}^T$, and $\mathcal{U}$ satisfies the PSE (14) with

$$\mathcal{Q} = (1-\alpha)\mathcal{H}\mathcal{F} + \alpha(\mathcal{X} + \mathcal{X}^{-1}\mathcal{X}^{-T}\mathcal{X}^{-1}).$$

Proof. The proof is very similar to that of Proposition 1 and relies on the above employed technique to linearly combine the solutions of several Sylvester equations differing only in their free terms. □

Having explicit analytical expressions for the function and its gradient it is easy to employ any gradient based technique to minimize J or J'. However, since the dimension of the minimization problem Knm is potentially large, a particularly well suited class of methods to solve our problem is the class of unconstrained descent methods, as for instance, the limited memory BFGS method [9] used in conjunction with a line search procedure with guaranteed decrease as that described in [10]. Both methods are implemented within the MINPACK-2 project (the successor of MINPACK-1 [11]) offering a convenient reverse communication interface which allows an easy implementation of function and gradient computations. The guaranteed decrease feature of these methods ensures that for $\alpha > 0$ the condition numbers $\kappa_F(X_k)$ decrease and thus the solution $\mathcal{X}$ of (6) remains invertible at each iteration once an invertible solution has been determined at the first iteration.

3 Numerical Aspects

In this section we address the efficient calculation of the functions and gradients necessary in using gradient

based methods. For each function/gradient evaluation the main computation is the solution of two PSEs, both essentially of the form

$$\mathcal{A}\mathcal{X} - \sigma\mathcal{X}\widetilde{\mathcal{A}} + \mathcal{Q} = 0. \tag{19}$$

To ensure a satisfactory performance, these computations must be done efficiently by using numerically reliable algorithms. The key observation which allows a drastic reduction of the computational cost is that both cost functions J and J' are invariant to an orthogonal similarity transformation [5]. Thus, if $\mathcal{F}$ is the optimal feedback for the original pair $(\mathcal{A}, \mathcal{B})$, then $\widehat{\mathcal{F}} = \mathcal{F}\mathcal{Z}^T$ is the optimal feedback for the transformed pair $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}) = (\sigma\mathcal{Z}^T\mathcal{A}\mathcal{Z}, \sigma\mathcal{Z}^T\mathcal{B})$, where $\mathcal{Z}$ is an orthogonal matrix. By using such a transformation we can tremendously reduce the computational effort necessary to solve the PSEs.

Recall that by assumption $\widetilde{\mathcal{A}}$ in (19) is in a real Jordan form, having only 1×1 or 2×2 possibly nonzero diagonal blocks. To simplify further the solution of (19) we can determine an orthogonal $\mathcal{Z}$ such that $\widehat{\mathcal{A}} = \sigma\mathcal{Z}^T\mathcal{A}\mathcal{Z}$ is in a *periodic Schur form* (PSF) [12], where $\widehat{A}_0$ is in a real Schur form and the matrices $\widehat{A}_k$ for $k = 1, \ldots, K$–1 are upper triangular. By multiplying the equation (19) with $\sigma\mathcal{Z}^T$ from left, one obtains a reduced PSE

$$\widehat{\mathcal{A}}\widehat{\mathcal{X}} - \sigma\widehat{\mathcal{X}}\widetilde{\mathcal{A}} + \widehat{\mathcal{Q}} = 0, \tag{20}$$

where $\widehat{\mathcal{X}} = \mathcal{Z}^T\mathcal{X}$ and $\widehat{\mathcal{Q}} = \sigma\mathcal{Z}^T\mathcal{Q}$. Notice that by this transformation the resulted transformed PSE (20) has exactly the same form as the original one in (19). After solving equation (20) for $\widehat{\mathcal{X}}$, the solution of (19) results as $\mathcal{X} = \mathcal{Z}\widehat{\mathcal{X}}$. For the solution of the reduced PSE (20) an efficient method has been derived along the lines of a more general procedure proposed in [7] (see [6, 13] for details).

The procedure of [6] can exploit the block-diagonal structure of $\widetilde{\mathcal{A}}$ and performs about $0.5Kn^3$ *floating-point operations* (flops) to solve one reduced PSE of the form (20). By assuming $\mathcal{A}$ already reduced to the PSF, the total cost to evaluate either J or J' and their gradients is about $4Kn^3$ *flops*, from which about $3Kn^3$ *flops* are necessary to form the free term of (20). Note that without the preliminary reduction to the PSF, each function and gradient evaluation would require at least $12Kn^3$ additional *flops*.

The preliminary orthogonal transformation is also useful to remove the restrictive condition $\Lambda(\Phi_A(K, 0)) \cap \Lambda(\Phi_{\widetilde{A}}(K, 0)) = \emptyset$ and thus to allow a partial eigenvalue assignment. By using an orthogonal similarity transformation

$$\sigma\mathcal{Z}^T\mathcal{A}\mathcal{Z} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{bmatrix}, \quad \sigma\mathcal{Z}^T\mathcal{B} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix},$$

we can reduce the matrix $\mathcal{A}$ to an *ordered* PSF (see [12]), where the characteristic values of $\mathcal{A}_{11}$ are those corresponding to satisfactory poles (which will be kept unmodified), while the characteristic values of $\mathcal{A}_{22}$ will be moved to desired values. Now we can compute by using gradient search techniques the solution $\mathcal{F}_2$ of a reduced order PEAP such that the characteristic values of $\mathcal{A}_{22} + \mathcal{B}_2\mathcal{F}_2$ are assigned to desired locations. The final feedback results as $\mathcal{F} = [\, 0 \;\; \mathcal{F}_2 \,]\mathcal{Z}^T$.

4 Numerical Examples

Example 1. This is a constant system example from [2] with

$$A_k = \begin{bmatrix} e & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_k = \begin{bmatrix} e-1 & 0 \\ 1 & 1-e^{-1} \\ 1 & 0 \end{bmatrix}$$

for $k = 0, 1$. We assigned the poles at $\Gamma = \{0.6, 0.7, -0.7\}$ and we computed for $\alpha = 0$, 0.5 and 1 the optimum periodic feedback by minimizing both J and J'. For each solution we computed the resulting feedback norm $\|\mathcal{F}\|_F$ as well as the 2-norm and Frobenius-norm condition numbers $\kappa_2(V_k)$ and $\kappa_F(V_k)$, where V_k is the eigenvector matrix of the monodromy matrix $\Phi_{(A+BF)}(k+2, k)$. We also computed the optimal robust feedback minimizing $J'' := \sum \kappa_2(V_k)$. The results are summarized in the following table, where for completeness, we also included results for the solution computed in [2]:

Criteria	$\|\mathcal{F}\|_F$	$\kappa_2(V_0)$	$\kappa_2(V_1)$	$\kappa_F(V_0)$	$\kappa_F(V_1)$
Ex. in [2]	4.59	1.67	11.17	3.27	12.26
$J(\alpha = 0)$	1.62	57.59	40.72	59.14	44.46
$J(\alpha = 0.5)$	2.84	2.89	2.77	4.32	4.21
$J'(\alpha = 0.5)$	2.55	3.54	3.22	5.00	4.65
$J(\alpha = 1)$	3.60	2.87	2.89	4.24	4.24
$J'(\alpha = 1)$	3.92	2.87	2.16	4.24	3.67
J''	7.20	1.00	1.00	3.00	3.00

It can be seen that, for this example, that the proposed approach computes better results in terms of $\sum \kappa_2(V_k)$ than the method of [2] for all considered nonzero values of α. Moreover, in the case of minimizing J'', we even succeeded to achieve $\kappa_2(V_0) = \kappa_2(V_1) = 1$, that is, to obtain orthogonal eigenvector matrices. The explanation for this lies in the parametric freedom of the problem. The PEAP can freely manipulate $N = n(Km-1)$ parameters of the total of nmK parameters contained in G_k. The orthogonality of eigenvectors imposes additionally $\widetilde{N} = Kn(n-1)/2$ algebraic equations. In our case $N = 9 > \widetilde{N} = 6$, thus orthogonal eigenvectors can be achieved by using our parametrization. For reference purposes we give the values of the periodic

feedback minimizing J'':

$$F_1 = \begin{bmatrix} -2.9255 & 0.0979 & 0.4220 \\ -3.9647 & 0.5417 & 0.6962 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -2.1550 & 0.1820 & 0.4041 \\ -4.1761 & 1.6204 & 1.3413 \end{bmatrix}.$$

Example 2. This is the multi-rate sampled-data system example of [2] with

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & e^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e-1 & 0 \\ 0 & 1-e^{-1} \\ 1 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 1-e^{-1} & 0 & e^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ e-1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

We assigned the poles at $\Gamma = \{0.5, 0.6, -0.6, 0.7, -0.7\}$ obtaining the following results:

Criteria	$\|\mathcal{F}\|_F$	$\kappa_2(V_0)$	$\kappa_2(V_1)$	$\kappa_F(V_0)$	$\kappa_F(V_1)$
Ex. in [2]	4.83	9.12	4.33	20.18	9.12
$J(\alpha = 0)$	1.57	371.7	492.1	485.8	576.1
$J(\alpha = 0.5)$	4.45	8.0	6.30	15.40	11.09
$J'(\alpha = 0.5)$	4.36	7.99	7.19	15.02	12.23
$J(\alpha = 1)$	5.13	8.06	6.23	15.44	11.01
$J'(\alpha = 1)$	4.58	8.09	6.22	15.45	11.00
J''	4.97	7.86	3.86	18.35	8.17

As can be observed from the table, all solutions computed for the nonzero values of α are qualitatively the same with the results of [2]. For reference purpose we give the computed periodic feedback minimizing J'':

$$F_1 = \begin{bmatrix} 0.2687 & -0.0243 & -2.9971 & 0.0256 & 0.3901 \\ 0.0450 & -2.2733 & -0.1224 & 2.3404 & 0.0314 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -0.1446 & 0.0127 & -1.7855 & -0.0045 & 0.0725 \\ -0.1762 & 0.8689 & -0.4829 & -0.4798 & -0.6458 \end{bmatrix}.$$

5 Conclusions

We focussed on developing a numerical approach to exploit the intrinsic non-uniqueness of the PEAP. One possibility to address the non-uniqueness is by formulating the PEAP as a minimum norm robust pole assignment problem. By using a convenient parameterization, that solution of the PEAP is sought which minimizes a special cost function defined as a weighted sum between the Frobenius-norm of the periodic state feedback matrix and the condition numbers of the closed-loop eigenvector matrices. The derived explicit expressions for the cost function gradient allow the use of powerful gradient search based minimization techniques. The efficient evaluation of the cost function and its gradient is of paramount importance for the usefulness of the proposed approach. The main numerical aspects of these computations have been thoroughly discussed, including the efficient solution of the involved periodic Sylvester equations. The presented numerical examples indicate that our method provides comparable or even better results than the recently proposed robust pole assignment method in [2]. The extension of the proposed approach to periodic systems with variable dimensions of the state and input vectors is apparently possible. This aspect is however still under investigation.

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A Proofs

To prove Proposition 1 we use the following result [6].

Lemma 1 *Let A_k, B_k, C_k, D_k be $n \times n$ periodic matrices of period $K \geq 1$. Let $\mathcal{X}$ be the solution of the PSE*

$$\mathcal{A}\mathcal{X} + \sigma\mathcal{X}\mathcal{B} = \mathcal{C}. \tag{21}$$

Then

$$\mathrm{tr}\left[\mathcal{D}\mathcal{X}\right] = \mathrm{tr}\left[\mathcal{U}\mathcal{C}\right],$$

where $\mathcal{U}$ satisfies the PSE

$$\mathcal{B}\mathcal{U} + \sigma\mathcal{U}\sigma\mathcal{A} = \sigma\mathcal{D}. \tag{22}$$

Proof of Proposition 1. We can write the cost function (4) in the form

$$J = (1-\alpha)J_1 + \alpha\sum_{k=0}^{K-1} J_2^{(k)} J_3^{(k)}, \tag{23}$$

where $J_1 = \frac{1}{2}\mathrm{tr}\left[\mathcal{F}^T\mathcal{F}\right]$, $J_2^{(k)} = \frac{1}{2}\mathrm{tr}\left[\mathcal{I}_k\mathcal{X}^T\mathcal{X}\right]$ and $J_3^{(k)} = \frac{1}{2}\mathrm{tr}\left[\mathcal{I}_k\mathcal{X}^{-T}\mathcal{X}^{-1}\right]$. It follows that

$$\nabla_{\mathcal{G}} J = (1-\alpha)\nabla_{\mathcal{G}} J_1 + \alpha\sum_{k=0}^{K-1}(\nabla_{\mathcal{G}} J_2^{(k)} J_3^{(k)} + J_2^{(k)}\nabla_{\mathcal{G}} J_3^{(k)}).$$

For $J_1 = \frac{1}{2}\mathrm{tr}\left[\mathcal{F}^T\mathcal{F}\right]$ we deduce the gradient $\nabla_{\mathcal{G}} J_1$ from the first order variation

$$\Delta J_1 := \mathrm{tr}\left[(\nabla_{\mathcal{G}} J_1)^T\Delta\mathcal{G}\right] = \mathrm{tr}\left[\mathcal{F}^T\Delta\mathcal{F}\right].$$

From (5) we get

$$\Delta\mathcal{F} = \Delta\mathcal{G}\mathcal{X}^{-1} - \mathcal{G}\mathcal{X}^{-1}\Delta\mathcal{X}\mathcal{X}^{-1}.$$

and we have successively

$$\begin{aligned}\Delta J_1 &= \mathrm{tr}\left[\mathcal{F}^T(\Delta\mathcal{G} - \mathcal{F}\Delta\mathcal{X})\mathcal{X}^{-1}\right] \\ &= \mathrm{tr}\left[\mathcal{X}^{-1}\mathcal{F}^T(\Delta\mathcal{G} - \mathcal{F}\Delta\mathcal{X})\right] \\ &= \mathrm{tr}\left[\mathcal{X}^{-1}\mathcal{F}^T\Delta\mathcal{G}\right] - \mathrm{tr}\left[\mathcal{X}^{-1}\mathcal{F}^T\mathcal{F}\Delta\mathcal{X})\right].\end{aligned}$$

From (6) follows that $\Delta\mathcal{X}$ satisfies the PSE

$$\mathcal{A}\Delta\mathcal{X} - \sigma\Delta\mathcal{X}\widetilde{\mathcal{A}} + \mathcal{B}\Delta\mathcal{G} = 0.$$

By using Lemma 1, we can write

$$\mathrm{tr}\left[\mathcal{X}^{-1}\mathcal{F}^T\mathcal{F}\Delta\mathcal{X})\right] = \mathrm{tr}\left[\mathcal{U}\mathcal{B}\Delta\mathcal{G}\right],$$

where $\mathcal{U}$ satisfies the PSE

$$\widetilde{\mathcal{A}}\mathcal{U} - \sigma\mathcal{U}\sigma\mathcal{A} - \sigma(\mathcal{X}^{-1}\mathcal{F}^T\mathcal{F}) = 0,$$

which, with $\mathcal{H} = \mathcal{X}^{-1}\mathcal{F}^T$, is in fact the equation (8). We further obtain

$$\Delta J_1 = \mathrm{tr}\left[(\mathcal{H} - \mathcal{U}\mathcal{B})\Delta\mathcal{G}\right]$$

from which the gradient of J_1 results as

$$\nabla_{\mathcal{G}} J_1 = \mathcal{H}^T - \mathcal{B}^T\mathcal{U}^T. \tag{24}$$

We compute next the gradient of $J_2^{(k)} = \frac{1}{2}\mathrm{tr}\left[\mathcal{I}_k\mathcal{X}^T\mathcal{X}\right]$. As before, by using Lemma 1 we have

$$\Delta J_2^{(k)} = \mathrm{tr}\left[\mathcal{I}_k\mathcal{X}^T\Delta\mathcal{X})\right] = \mathrm{tr}\left[\mathcal{V}_k\mathcal{B}\Delta\mathcal{G}\right],$$

where $\mathcal{V}_k$ satisfies the PSE (9) with $\mathcal{R}_k = \mathcal{I}_k\mathcal{X}$. From $\Delta J_2^{(k)} := \mathrm{tr}\left[(\nabla_{\mathcal{G}} J_2^{(k)})^T\Delta\mathcal{G}\right]$, we obtain the gradient as

$$\nabla_{\mathcal{G}} J_2^{(k)} = \mathcal{B}^T\mathcal{V}_k^T. \tag{25}$$

For the term $J_3^{(k)} = \frac{1}{2}\mathrm{tr}\left[X_k^{-T}X_k^{-1}\right]$ we use again Lemma 1 and we have successively

$$\begin{aligned}\Delta J_3^{(k)} &= \mathrm{tr}\left[X_k^{-T}\Delta(X_k^{-1})\right] = \mathrm{tr}\left[X_k^{-T}X_k^{-1}\Delta X_k X_k^{-1}\right] \\ &= \mathrm{tr}\left[X_k^{-1}X_k^{-T}X_k^{-1}\Delta X_k\right] \\ &= \mathrm{tr}\left[\mathcal{I}_k\mathcal{X}^{-1}\mathcal{X}^{-T}\mathcal{X}^{-1}\Delta\mathcal{X}\right] \\ &= \mathrm{tr}\left[\mathcal{W}_k\mathcal{B}\Delta\mathcal{G}\right],\end{aligned}$$

where $\mathcal{W}_k$ satisfies (10) with $\mathcal{Q}_k = \mathcal{I}_k\mathcal{X}^{-1}\mathcal{X}^{-T}\mathcal{X}^{-1}$. We obtain the gradient of $J_3^{(k)}$ as

$$\nabla_{\mathcal{G}} J_3^{(k)} = \mathcal{B}^T\mathcal{W}_k^T. \tag{26}$$

To obtain (7), we assemble $\nabla_{\mathcal{G}} J$ according to (23) by using (24), (25), and (26). □