Square-Root Balancing and Computation of Minimal Realizations of Periodic Systems

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Abstract

We propose a numerically reliable approach for balancing and minimal realization of linear periodic systems with time-varying dimensions. The proposed approach to balancing belongs to the family of *square-root* methods with guaranteed enhanced computational accuracy and can be also used to compute balanced minimal order realizations from non-minimal ones. An alternative *balancing-free square-root* method for minimal realization has the advantage of a potentially better numerical accuracy in case of poorly balanced original systems. The key numerical computation in both methods is the solution of nonnegative periodic Lyapunov equations directly for the Cholesky factors of the solutions. For this purpose, a numerically reliable computational algorithm is proposed to solve nonnegative periodic Lyapunov equations with time-varying dimensions.

1 Introduction

In the last few years there has been a constantly increasing interest to develope numerical algorithms for the analysis and design of linear periodic discrete-time control systems with constant state-, input- and output-vector dimensions [1, 6, 8, 11]. Areas where significant theoretical results have been achieved for periodic systems with time-varying dimensions are the solution of the minimal realization problem [4, 3] and robust pole assignment [7].

In this paper we develop a numerical approach for the balancing and minimal realization of linear periodic systems with time-varying dimensions. The proposed approach to balancing belongs to the family of *square-root* methods with guaranteed enhanced computational accuracy, where the balancing transformations are determined exclusively using the Cholesky factors of the periodic reachability and observability Gramians.

The minimal realization problem is solved by computing suitable projections to determine the matrices of the minimal order realization from the matrices of a given non-minimal periodic system. For this purpose, appropriate truncation matrices are computed directly from the Cholesky factors of the periodic Gramians. It is shown that the resulting low order periodic system is minimal, balanced and achieves the same input-output map as the original system. Analogously to the standard systems case [10], a *balancing-free square-root* approach is also proposed, with a potentially better numerical accuracy in case of poorly balanced original systems.

The key computation in the proposed computational approaches is the solution of nonnegative periodic Lyapunov equations directly for the Cholesky factors of the Gramians. For this purpose, a numerically reliable computational algorithm is proposed to solve nonnegative periodic Lyapunov equations with time-varying dimensions. The proposed algorithm is an extension of a method proposed by the author for constant dimensions [11].

2 Preliminaries

Consider the linear discrete-time K-periodic system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned} \tag{1}$$

where the matrices $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$, $B_k \in \mathrm{R}^{n_{k+1}\times m_k}$, $C_k \in \mathrm{R}^{p_k\times n_k}$, $D_k \in \mathrm{R}^{p_k\times m_k}$ and the integers n_k, m_k, p_k are periodic with period $K \geq 1$. The $n_j \times n_i$ transition matrix of the system (1) is defined by $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$, where $\Phi_A(i,i) := I_{n_i}$. The $n_j \times n_j$ matrix $\Phi_A(j+K,j)$ is the *monodromy matrix* of system (1) at time j and its eigenvalues $\Lambda(\Phi_A(j+K,j))$ are called *characteristic multipliers* at time j. There are always at least $n_j - \underline{n}$ zero eigenvalues, where $\underline{n} := \min_k\{n_k\}$. The rest of $\underline{n}$ eigenvalues are independent of time j and form the *core characteristic multipliers*. The periodic system (1) is *asymptotically stable* if all characteristic multipliers belong to the open unit disk.

For the definitions of reachability, observability and minimality of periodic systems we use the corresponding notions from [4] for general time-varying systems.

Definition 1. The periodic system (1) is *reachable at time* k if

$$\operatorname{rank} G_k = n_k, \tag{2}$$

where G_k is the infinite columns matrix

$$G_k = [\, B_{k-1} \;\; A_{k-1}B_{k-2} \;\cdots\; \Phi_A(k,i+1)B_i \;\cdots]. \tag{3}$$

The periodic system (1) is *completely reachable* if (2) holds for all k.

Definition 2. The periodic system (1) is *observable at time* k if

$$\operatorname{rank} F_k = n_k, \tag{4}$$

where F_k is the infinite rows matrix

$$F_k = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ \vdots \\ C_i\Phi_A(i,k) \\ \vdots \end{bmatrix}. \tag{5}$$

The periodic system (1) is *completely observable* if (4) holds for all k.

Definition 3. The periodic system (1) is *minimal* if it is completely reachable and completely observable.

For an asymptotically stable periodic system, the $n_k \times n_k$ reachability Gramian at time k is defined as

$$P_k := \sum_{i=-\infty}^{k-1} \Phi_A(k,i+1)B_iB_i^T\Phi_A(k,i+1)^T = G_kG_k^T \geq 0,$$

where G_k is defined in (3). Similarly, the $n_k \times n_k$ observability Gramian at time k is defined as

$$Q_k = \sum_{i=k}^{\infty} \Phi_A(i,k)^TC_i^TC_i\Phi_A(i,k) = F_k^TF_k \geq 0.$$

Note that both Gramians are K-periodic matrices. Using the definitions of reachability and observability we have the following results.

Proposition 1 *The periodic system* (1) *is reachable at time* k *iff* $P_k > 0$ *and is completely reachable iff* $P_k > 0$ *for* $k = 0,\ldots,K-1$.

Proposition 2 *The periodic system* (1) *is observable at time* k *iff* $Q_k > 0$ *and is completely observable iff* $Q_k > 0$ *for* $k = 0,\ldots,K-1$.

Notation and notational conventions. For a K-periodic matrix X_k we use alternatively the *script* notation $\mathcal{X} := \operatorname{diag}(X_0, X_1, \ldots, X_{K-1})$, which associates the block-diagonal matrix $\mathcal{X}$ to the cyclic matrix sequence X_k, $k = 0,\ldots,K-1$. This notation is consistent with the standard matrix operations as for instance addition, multiplication, inversion as well as with several standard matrix decompositions (Cholesky, SVD). We denote with $\sigma\mathcal{X}$ the K-cyclic shift $\sigma\mathcal{X} = \operatorname{diag}(X_1,\ldots,X_{K-1},X_0)$ of the cyclic sequence X_k, $k = 0,\ldots,K-1$. By using the script notation, the periodic system (1) will be alternatively denoted by the quadruple $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$.

3 Square-root Balancing

For an asymptotically stable periodic system the two Gramians are nonnegative definite and satisfy *nonnegative (or positive) discrete periodic Lyapunov equations* (PDPLEs): the reachability Gramian $\mathcal{P}$ satisfies the *forward-time* PDPLE

$$\sigma\mathcal{P} = \mathcal{A}\mathcal{P}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T, \tag{6}$$

while the observability Gramian $\mathcal{Q}$ satisfies the *reverse-time* PDPLE

$$\mathcal{Q} = \mathcal{A}^T\sigma\mathcal{Q}\mathcal{A} + \mathcal{C}^T\mathcal{C} \tag{7}$$

Let $T_k \in \mathrm{R}^{n_k\times n_k}$ be a K-periodic invertible matrix. Two periodic systems $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ and $(\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}},\widetilde{\mathcal{D}})$ related by the transformation

$$(\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}},\widetilde{\mathcal{D}}) = (\sigma\mathcal{T}^{-1}\mathcal{A}\mathcal{T}, \sigma\mathcal{T}^{-1}\mathcal{B}, \mathcal{C}\mathcal{T}, \mathcal{D}) \tag{8}$$

are called *Lyapunov-similar* and (8) is called a *Lyapunov similarity transformation*. The Gramians $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$ of the transformed system $(\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}},\widetilde{\mathcal{D}})$ satisfy

$$\widetilde{\mathcal{P}} = \mathcal{T}^{-1}\mathcal{P}\mathcal{T}^{-T}, \quad \widetilde{\mathcal{Q}} = \mathcal{T}^T\mathcal{Q}\mathcal{T}.$$

For a completely reachable and completely observable (i.e., minimal) periodic system, we can determine $\mathcal{T}$ such that the transformed Gramians are equal and diagonal

$$\widetilde{\mathcal{P}} = \widetilde{\mathcal{Q}} = \Sigma = \operatorname{diag}(\Sigma_0,\Sigma_1,\ldots,\Sigma_{K-1}),$$

where $\Sigma_k = \operatorname{diag}(\sigma_{k,1},\sigma_{k,2},\cdots,\sigma_{k,n_k})$. The quantities $\sigma_{k,i}, i = 1,\ldots,n_k$, are the positive square-roots of the eigenvalues of the product P_kQ_k and are called the *Hankel-singular values*.

Let $\mathcal{P} = \mathcal{S}^T\mathcal{S}$ and $\mathcal{Q} = \mathcal{R}^T\mathcal{R}$ be in Cholesky factorized forms. In analogy with the standard case [9], we can use the singular value decomposition

$$\mathcal{R}\mathcal{S}^T = \mathcal{U}\Sigma\mathcal{V}^T, \tag{9}$$

to compute the balancing transformation matrix $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ as

$$\mathcal{T} = \mathcal{S}^T\mathcal{V}\Sigma^{-1/2}, \qquad \mathcal{T}^{-1} = \Sigma^{-1/2}\mathcal{U}^T\mathcal{R}.$$

Note that the computation of the balancing transformation relies exclusively on *square-root* information (i.e. the Cholesky factors of Gramians) and this leads to a guaranteed enhancement of the overall numerical accuracy of computations. The key computation in determining $\mathcal{T}$ and $\mathcal{T}^{-1}$ is the solution of the two PDPLEs (6) and (7) with time-varying dimensions directly for the Cholesky factors of the Gramians. A numerically reliable procedure for this purpose is discussed Section 5.

4 Minimal Realization

Given an asymptotically stable non-minimal periodic system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, the balancing approach of the previous section is not applicable since the system is not completely reachable and/or not completely observable. Thus, from Proposition 1 or 2 follows that Σ in (9) is not invertible. We show in this section how to determine a realization of the given periodic system which is minimal, that is, completely reachable and completely observable, and moreover balanced. The main result (Theorem 1) of this section can be seen as an extension to the periodic case of a similar result in [10] for standard systems.

Let us write the singular value decomposition (9) for each time instant k in the partitioned form

$$R_k S_k^T = [\, U_{k,1} \; U_{k,2} \,] \begin{bmatrix} \widetilde{\Sigma}_k & 0 \\ 0 & 0 \end{bmatrix} [\, V_{k,1} \; V_{k,2} \,]^T, \qquad (10)$$

where $\widetilde{\Sigma}_k \in \mathrm{R}^{r_k \times r_k}$, $U_{k,1} \in \mathrm{R}^{n_k \times r_k}$, $V_{k,1} \in \mathrm{R}^{n_k \times r_k}$ and $\widetilde{\Sigma}_k > 0$. From the above decomposition define, with $\widetilde{\Sigma} = \mathrm{diag}\,(\widetilde{\Sigma}_0, \ldots, \widetilde{\Sigma}_{K-1})$, the truncation matrices

$$\mathcal{L} = \widetilde{\Sigma}^{-\frac{1}{2}} \mathcal{U}_1^T \mathcal{R}, \qquad \mathcal{T} = \mathcal{S}^T \mathcal{V}_1 \widetilde{\Sigma}^{-\frac{1}{2}},$$

which are used to compute the reduced system matrices as

$$\widehat{\mathcal{A}} = \sigma \mathcal{L} \mathcal{A} \mathcal{T}, \quad \widehat{\mathcal{B}} = \sigma \mathcal{L} \mathcal{B}, \quad \widehat{\mathcal{C}} = \mathcal{C} \mathcal{T}. \qquad (11)$$

The following is the main theoretical result of the paper.

Theorem 1 *The periodic system $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}, \mathcal{D})$ defined in (11) is a balanced minimal realization of the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$.*

Proof. The proof is lengthy, having essentially two steps. First we prove that $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}, \mathcal{D})$ is a minimal balanced realization and then we prove that it realizes the same input-output operator as $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. Because of lack of space the details of the proof are omitted. □

The computation of the balanced minimal realization is based on *square-root* information only, that is, the truncation matrices $\mathcal{L}$ and $\mathcal{T}$ are computed exclusively on the basis of the Cholesky factors of the Gramians. To obtain a minimal realization from a non-minimal one we do not actually need to obtain a balanced minimal realization since this could involve ill-conditioned $\mathcal{L}$ and $\mathcal{T}$ matrices, if the original system is poorly balanced. To avoid potential accuracy losses, an alternative is to use a *balancing-free* approach to compute the two truncation matrices. A *square-root balancing-free* approach can be easily devised analogously as in case of standard systems [10]. Consider the QR-decompositions

$$\mathcal{S}^T \mathcal{V}_1 = \widetilde{\mathcal{T}} \mathcal{X}, \quad \mathcal{R}^T \mathcal{U}_1 = \widetilde{\mathcal{Z}} \mathcal{Y},$$

where $\mathcal{X}$ and $\mathcal{Y}$ are nonsingular matrices and $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{Z}}$ are matrices with orthonormal columns. With the already computed $\widetilde{\mathcal{T}}$ we define the corresponding $\widetilde{\mathcal{L}}$ as

$$\widetilde{\mathcal{L}} = (\widetilde{\mathcal{Z}}^T \widetilde{\mathcal{T}})^{-1} \widetilde{\mathcal{Z}}^T.$$

The resulted system $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \mathcal{D}) := (\sigma \widetilde{\mathcal{L}} \mathcal{A} \widetilde{\mathcal{T}}, \sigma \widetilde{\mathcal{L}} \mathcal{B}, \mathcal{C} \widetilde{\mathcal{T}}, \mathcal{D})$ is clearly not balanced, but represents a minimal realization of the original system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ which is Lyapunov-similar to the balanced realization (11). For poorly balanced systems, the balancing-free square-root approach certainly leads to an improvement of the computational accuracy.

Remark. The proposed minimal realization approaches for periodic systems are not restricted to asymptotically stable periodic systems. For an unstable system, a simple scaling can be used to enforce the stability of the starting representation. For instance, it is possible to replace only A_0 by αA_0, where $0 < \alpha < 1$ is chosen such that $\alpha \Phi_A(K, 0)$ has eigenvalues in the open unit disc. For the modified system, we can apply either the square-root or balancing-free square-root approach to determine a minimal system. Then, the computed $\widehat{A}_0$ or $\widetilde{A}_0$ will be rescaled appropriately to $\widehat{A}_0/\alpha$ or $\widetilde{A}_0/\alpha$, respectively.

5 Solution of PDPLEs

The main computational problem to compute a balanced minimal realization of an asymptotically stable periodic system is the solution of a PDPLE of the form

$$\mathcal{U}^T \mathcal{U} = \mathcal{A}^T \sigma \mathcal{U}^T \sigma \mathcal{U} \mathcal{A} + \mathcal{R}^T \mathcal{R} \qquad (12)$$

directly for the Cholesky factor $\mathcal{U}$, where $U_k \in \mathrm{R}^{n_k \times n_k}$, $A_k \in \mathrm{R}^{n_{k+1} \times n_k}$, $R_k \in \mathrm{R}^{n_k \times n_k}$, and the dimension n_k are periodic with period $K \geq 1$. To solve PDPLEs with constant dimensions, numerically reliable algorithms have been recently proposed in [11], representing extensions of a method for standard systems [5]. In this section we describe an extension of the method of [11] to solve PDPLEs with time-varying dimensions.

A straightforward embedding of the problem with time-varying dimensions into a larger order problem with constant dimension allows to solve the PDPLE (12) by using algorithms for constant dimensions [11]. Let $\overline{n} = \max_k \{n_k\}$ and consider the extended $\overline{n} \times \overline{n}$ matrices

$$A_{k,e} = \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix}, \; R_{k,e} = \begin{bmatrix} R_k & 0 \\ 0 & 0 \end{bmatrix}, \; U_{k,e} = \begin{bmatrix} U_k & 0 \\ 0 & 0 \end{bmatrix}, \quad (13)$$

where the zeros matrices have appropriate dimensions. Then it is easy to see that $\mathcal{U}_e$ is the solution of the PDPLE

$$\mathcal{U}_e^T \mathcal{U}_e = \mathcal{A}_e^T \sigma \mathcal{U}_e^T \sigma \mathcal{U}_e \mathcal{A}_e + \mathcal{R}_e^T \mathcal{R}_e \qquad (14)$$

with constant dimensions. The main drawback of this approach is that working with extended matrices with many zero elements leads to an unnecessary loss of computational efficiency. Alternatively, an efficient approach can be devised which fully exploits the underlying problem structure.

In the approach which we propose, the key role plays a generalization of the *periodic Schur decomposition* (PSD) of a cyclic product of square matrices and of the corresponding algorithms for its computation [2, 6].

Proposition 3 *Let* $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$, $k = 0, 1, \ldots, K-1$, *with* $n_K = n_0$ *be arbitrary matrices and let* $\underline{n} = \min_k\{n_k\}$. *Then there exist orthogonal matrices* $Z_k \in \mathrm{R}^{n_k\times n_k}$ *such that the matrices* $\widetilde{A}_k = Z_{k+1}^T A_k Z_k$ *for* $k = 0, \ldots, K-1$ *are block upper triangular*

$$\widetilde{A}_k = \left[\begin{array}{cc} \widetilde{A}_{k,11} & \widetilde{A}_{k,12} \\ 0 & \widetilde{A}_{k,22} \end{array}\right], \qquad (15)$$

where $\widetilde{A}_{k,11} \in \mathrm{R}^{\underline{n}\times\underline{n}}$, $\widetilde{A}_{k,22} \in \mathrm{R}^{(n_{k+1}-\underline{n})\times(n_k-\underline{n})}$ *for* $k = 0, 1, \ldots, K-1$. *Moreover,* $\widetilde{A}_{K-1,11}$ *is in a real Schur form,* $\widetilde{A}_{k,11}$ *for* $k = 0, \ldots, K-2$ *are upper triangular and* $\widetilde{A}_{k,22}$ *for* $k = 0, \ldots, K-1$ *are upper trapezoidal.*

By using the above decomposition instead of the PSD of the product of extended matrices $A_{k,e}$, a notable reduction of computational costs arises if the difference $\overline{n} - \underline{n}$ is significant. Let $\mathcal{Z}$ be an orthogonal Lyapunov transformation to compute the generalization of the PSD of the monodromy matrix $\Phi_A(K, 0)$ in the Proposition 3 and define $\widetilde{\mathcal{A}} = \sigma\mathcal{Z}^T\mathcal{A}\mathcal{Z}$ and the upper triangular $\widetilde{\mathcal{R}}$ such that $\widetilde{\mathcal{R}}^T\widetilde{\mathcal{R}} = \mathcal{Z}^T\mathcal{R}^T\mathcal{R}\mathcal{Z}$. The equation (12) becomes after premultiplication with $\mathcal{Z}^T$ and postmultiplication with $\mathcal{Z}$

$$\widetilde{\mathcal{U}}^T\widetilde{\mathcal{U}} = \widetilde{\mathcal{A}}^T\sigma\widetilde{\mathcal{U}}^T\sigma\widetilde{\mathcal{U}}\widetilde{\mathcal{A}} + \widetilde{\mathcal{R}}^T\widetilde{\mathcal{R}} \qquad (16)$$

where $\widetilde{\mathcal{U}} = \mathcal{U}\mathcal{Z}$. After solving this reduced equation for $\widetilde{\mathcal{U}}$, the solution of (12) results as $\mathcal{U} = \widetilde{\mathcal{U}}\mathcal{Z}^T$.

To solve the reduced PDPLE (12) a procedure very similar to that of [11] can be used. In [11] the reduced PDPLE with constant dimensions is solved using the partitioning resulting from the structure of the PSD. At each main step a low order (1×1 or 2×2) PDPLE is solved followed by the solution of several low order periodic Sylvester equations. The original problem is then replaced by a lower order one by suitably updating the problem data. The only difference which arises in case of the procedure for varying dimension is that, after several steps some dimensions of submatrices become zero. In such cases, the computations can still continue since, we can freely assume that the missing blocks in all matrices are zero matrices as in (13). The rest of algorithmic details are almost the same as in case of the procedure for constant dimensions, although the efficient implementation for time-varying dimensions certainly requires an increased bookkeeping effort.

6 Conclusion

We proposed a numerically sound approach to perform the balancing and minimal realization of linear periodic systems with time-varying dimensions. The proposed approach relies on algorithms using exclusively square-root information in form of Cholesky factors of the Gramians and therefore they have guaranteed enhanced computational accuracy. A square-root balancing-free variant has been derived, which we believe to be a completely satisfactory numerical approach to solve periodic minimal realization problems. The key computation in the proposed approach is the numerical solution of PDPLEs directly for the Cholesky factors of the solutions. A numerically reliable computational algorithm has been proposed to solve PDPLEs with varying dimension.

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