

Correlations in homogeneous stratified shear turbulence

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Summary. Based on the budget of kinetic energy and simple estimates to relate dissipation and temperature or concentration fluctuations to shear, stratification, and the vertical velocity fluctuations, a consistent set of equations is deduced to estimate vertical fluxes of momentum and heat or mass. The estimates are designed for strongly sheared, neutral and stratified flows at high Reynolds numbers under approximately homogeneous conditions. The set is closed by using basically two empirical coefficients together with the turbulent Prandtl number and the growth rate of kinetic energy as a function of the gradient Richardson number. The correlations are tested using data from previous laboratory experiments and numerical simulations.

1 Introduction

For many applications, one needs estimates of the rate of turbulent mixing in neutrally and stably stratified shear flows. This is a particularly difficult topic because turbulence tends to degenerate to wavy motions under strongly stable stratification, and many models have been proposed in the past, some based on extensive second- or higher-order closure models with many model parameters. In view of the difficulties to determine the turbulence scales and even the mean profiles in such flows, simple relationships are required to estimate the magnitude of the mixing properties. Such relationships have been deduced, mainly for strongly stratified atmospheric and oceanic flows, on the basis of the energy budgets using simple closure assumptions for stationary flows [1–3].

Here, a new set of equations is deduced which takes into account the deviation from stationarity and applies to both pure shear flows and to moderately stratified shear flows. The paper is formulated for thermal stratification but the results can also be applied to density variations due to variable salt concentration. Hence, we consider the turbulence properties of a flow with given vertical velocity shear S and positive vertical potential temperature gradient s ,

$$S = dU/dz, \quad s = d\Theta/dz, \quad (1)$$

which define the Brunt-Väisälä frequency N and the gradient Richardson number Ri ,

$$N = (\beta g s)^{1/2}, \quad Ri = N^2/S^2. \quad (2)$$

Here, β is the thermal volumetric expansion coefficient, and g is the acceleration of gravity. We consider flows in between $Ri = 0$, and Ri of order one.

It is assumed that the density variations affect the buoyancy only, i.e. we employ the Boussinesq approximation. The analysis is restricted to flows at high Reynolds numbers, so that the molecular diffusivities are small in comparison to turbulent diffusivities. The discussion concentrates on approximately homogeneous but time-dependent flows in which the divergence

of energy fluxes is small in comparison to its local dissipation rate. Moreover, we assume approximately uniform vertical shear and stratification, remote from boundaries, so that the turbulence lengthscales are smaller than the scales of any variations in the mean profiles. The turbulence is assumed to be strongly sheared so that the timescale S^{-1} of shear is smaller than the time scale N^{-1} of stratification and both should be smaller than the turbulence timescales. Finally we assume that the exchange of energy between its kinetic and potential form has approached a local equilibrium so that the ensemble averaged fields are free of gravity wave oscillations.

Reliable data for homogeneous stratified shear flow have been measured by Rohr et al. [4], who used salt to produce the density stratification. The data are taken from appendix 2 of Rohr's thesis, as cited in [4], at shear times $tdU/dz > 6$ when the flow has approached structural equilibrium. Otherwise, data for homogeneous shear flows are available only for neutral stratification in wind tunnels [5–7]. These data will be used to calibrate and verify the model equations. Note the rather large molecular Prandtl or Schmidt number in salt (about 500) while that of air is about 0.7.

2 General consequences of the energy budget

In homogeneous turbulences the ensemble averaged kinetic energy $E_{\text{kin}} = (\overline{u^2} + \overline{v^2} + \overline{w^2})/2$ of the turbulent velocity fluctuations is a pure function of time t , and satisfies the budget

$$\frac{dE_{\text{kin}}}{dt} = P - B - \varepsilon. \quad (3)$$

It states that the local rate of change in kinetic energy equals the sum of shear production P , bouyancy destruction B , and viscous dissipation ε . If vertical shear and stratification dominate, the production terms are functions of the vertical turbulent fluxes of momentum and heat and of the related turbulent diffusivities,

$$P = -\overline{uw}S = K_m S^2, \quad B = -\overline{w\theta}N^2/s = K_h N^2. \quad (4)$$

Their ratio defines the flux Richardson number Ri_f and the turbulent Prandtl number Pr_t ,

$$Ri_f = \frac{B}{P} = \frac{Ri}{Pr_t}, \quad Pr_t = \frac{K_m}{K_h}. \quad (5)$$

Now the first essential assumption is introduced, namely that the parameter G ,

$$G = P/(\varepsilon + B), \quad (6)$$

is of similar universal importance for viscous flows as is the flux Richardson number for inviscid flows and controls the growth rate of kinetic energy,

$$\frac{dE}{dt} = (G - 1)(\varepsilon + B), \quad (7)$$

i.e. $G = 1$ for stationary flows, $G = 0$ for decaying flows without shear production, and $G = P/\varepsilon$ in neutral shear flows. As a consequence of the budget of kinetic energy and the above definitions of Ri_f and G , the rates of shear forcing and bouyancy destruction are related to the rate of

dissipation by

$$B = \frac{\text{Ri}_f G}{1 - \text{Ri}_f G} \varepsilon, \quad P = \frac{G}{1 - \text{Ri}_f G} \varepsilon. \quad (8)$$

Together with Eq. (4), these relationships determine the turbulent diffusivities,

$$K_m = \frac{G}{1 - \text{Ri}_f G} \frac{\varepsilon}{S^2}, \quad (9)$$

$$K_h = \frac{\text{Ri}_f G}{1 - \text{Ri}_f G} \frac{\varepsilon}{N^2}. \quad (10)$$

We also obtain estimates for the “structure parameter” of the momentum flux and for the correlation coefficient of the heat flux,

$$\alpha_{uw} \equiv -\frac{\overline{uw}}{w'^2} = \frac{G}{1 - \text{Ri}_f G} \frac{\varepsilon}{w'^2 S}, \quad (11)$$

$$\alpha_{w\theta} \equiv -\frac{\overline{w'\theta'}}{w'\theta'} = \frac{\text{Ri}_f G}{1 - \text{Ri}_f G} \frac{\varepsilon S}{N^2 w'\theta'} = \frac{G}{\text{Pr}_t(1 - \text{Ri}_f G)} \frac{\varepsilon S}{S^2 w'\theta'}, \quad (12)$$

where $w' = (\overline{w^2})^{1/2}$ and $\theta' = (\overline{\theta^2})^{1/2}$ are the root-mean square values of the turbulent fluctuations of vertical and temperature.

3 Approximations for strongly sheared turbulence

According to Hunt et al. [8], for strong shear but for moderate stratification, i.e. for $\text{Ri} \leq 1$, the dissipation due to small-scale mixing in turbulent flows (remote from boundaries) is controlled by shear S and the induced vertical motion velocity w' . Dimensional analysis and Prandtl's classical eddy mixing concept suggest

$$\varepsilon = A_S w'^2 S. \quad (13)$$

The temperature fluctuations are controlled by turbulent motions at the larger scales and are more sensitive to buoyancy. Therefore, the impact of buoyancy gets important at values of Ri considerably less than one. Hence, the mixing concept and dimensional analysis give

$$\theta' = \zeta_S w' s / S \quad \text{for } \text{Ri} \leq 0.25, \quad \theta' = \zeta_N w' s / N \quad \text{for } \text{Ri} > 0.25. \quad (14)$$

The limit $\text{Ri} = 0.25$ is certainly only approximately valid and is taken in correspondance with the linear stability criterion of inviscid flows. Here, A_S , ζ_S , and ζ_N are the yet undetermined model coefficients.

The two versions for the temperature fluctuations given in Eq. (14) are consistent with each other if

$$\zeta_S = \text{const}, \quad \zeta_N = \zeta_S \text{Ri}^{1/2}, \quad (15)$$

for $\text{Ri} \leq 0.25$, and

$$\zeta_S = \zeta_N \text{Ri}^{-1/2}, \quad \zeta_N = \text{const}, \quad (16)$$

for $Ri > 0.25$, with $\zeta_N = 0.5\zeta_S$ at the limit between the two ranges. Hence, only one of these coefficients is an independent model parameter.

Without any further assumption, these relations can be used to estimate

$$\alpha_{uw} = \frac{A_S G}{1 - Ri_f G}, \quad (17)$$

$$\alpha_{w\theta} = \frac{\alpha_{uw}}{\zeta_S Pr_t} = \frac{A_S G}{\zeta_S Pr_t (1 - Ri_f G)}, \quad (18)$$

$$K_h = c_S w'^2 / S = c_N w'^2 / N, \quad K_m = K_h Pr_t, \quad (19)$$

with

$$c_S = \alpha_{uw} / Pr_t, \quad c_N = \alpha_{uw} Ri^{1/2} / Pr_t. \quad (20)$$

4 Closure assumptions

In order to close the set of equations, one needs to specify A_S and ζ_S (for $Ri \leq 0.25$) as well as the growth-rate parameter $G(Ri)$, and the turbulent Prandtl number $Pr_t(Ri)$, which we assume to be pure functions of the Richardson number Ri . One expects that G and Pr_t are not unique functions of Ri , in particular for strong stratification, but we take this approach as a pragmatic procedure.

The function $G(Ri)$ is set up such that it equals unity at the stationary Richardson number Ri_S , for which the forcing by shear just balances dissipation and buoyancy destruction, and decays exponentially with Richardson number,

$$G = G_0^{(1 - Ri/Ri_S)}. \quad (21)$$

The value of Ri_S is less than the inviscid stability limit 0.25 because of finite dissipation in real flows.

The Prandtl number is specified to vary as

$$Pr_t = Pr_{t0} \exp \{ -Ri / (Pr_{t0} Ri_{f\infty}) \} + Ri / Ri_{f\infty}, \quad (22)$$

where Pr_{t0} is the Prandtl number for neutral stratification. The model is specified such that $Pr_t \geq Pr_{t0}$ with zero gradient at $Ri = 0$, and $Pr_t \rightarrow Ri / Ri_{f\infty}$ for $Ri \gg 1$. Also, it is assumed that $Ri_{f\infty} = 0.25$. The data do not allow to determine this parameter very precisely.

5 Determination of the model coefficients

The coefficients can only be fixed when suitable measurements are given. Table 1 collects the best available data. We found that the coefficients differ depending on the molecular Prandtl number (or Schmidt number). In salt-water, the damping of concentration fluctuations is much smaller than that of temperature fluctuations in air. Therefore, we have to give two sets of coefficients. For salt-water, based on the measurements of Rohr [4], one obtains for $Ri = 0$: $G_0 = P/\varepsilon = 1.8 \pm 0.36$, and $\zeta_S = 2.88 \pm 0.15$, so that $\zeta_N = 1.44$. The structure parameter was measured to be $\alpha_{uw} = 0.87 \pm 0.08$, and the scalar flux correlation coefficient

as $\alpha_{w\theta} = 0.42 \pm 0.03$. From Eq. (17) one obtains $A_S = \alpha_{uw}/G_0 = 0.5$, and from Eq. (18) $\text{Pr}_{t0} = \alpha_{uw}/(\alpha_{w\theta}\zeta_S) = 0.72$. For $\text{Ri} > 0$, the data suggest $G(0.36) = 0.5 \pm 0.3$, which defines $\text{Ri}_S \cong 0.16 \pm 0.06$.

From the wind-tunnel data given in [5–7] for $\text{Ri} = 0$, one obtains $G_0 = 1.47 \pm 0.13$, $\alpha_{uw} = 0.73 \pm 0.05$, $\alpha_{w\theta} = 0.45 \pm 0.03$, and $\zeta_S = 1.65 \pm 0.1$. Hence, $A_S = 0.48$, $\zeta_N = 0.825$, and $\text{Pr}_{t0} = 0.98$. In principle, there is no reason why G_0 should be different in air and salt-water flows. However, the differences are within the scatter of the data. No measurements exist for $\text{Ri} > 0$ in air, but from large-eddy simulations [9], which were performed for a Prandtl-number of one with respect to the subgrid-scale motions, we determine $\text{Ri}_S \cong 0.13$. The value is close to results from direct numerical simulations [10]. Compared to salt-water, a smaller value of Ri_S has to be expected in air because of the enhanced dissipation of total energy (kinetic and potential) by the larger thermal diffusion at the smaller Prandtl number.

6 Comparison to measurements and simulation results

The value of A_S is close to the value 0.45 deduced in [8] from the logarithmic law of the wall in the boundary layer. The value $\zeta_N = 0.825$ for air is very close to the values $\zeta_N = 0.8 \pm 0.25$ and $\zeta_N \cong 0.96$ found for the stable atmospheric boundary layer in [1] and [11].

Table 1 lists the data as obtained from the experiments together with the values which result from the above equations. We see that the model approximates the measurements mostly within the standard deviation of the measured data. Data for $\text{Ri} > 0$ are available only for the salt-water experiment. However, the comparison shows that A_S and ζ_S are indeed close to a constant, at least for $\text{Ri} \leq 0.25$. The growth-rate parameter G decreases with Ri , and the turbulent Prandtl number increases with Ri , as assumed above.

Table 1. Data and model results

Quantity	Air: $\text{Ri} = 0$	Salt: $\text{Ri} = 0$	0.018 ± 0.004	0.062 ± 0.008	0.186 ± 0.01	0.356 ± 0.005
G	1.47 ± 0.13	1.81 ± 0.36	1.47 ± 0.34	1.44 ± 0.13	0.84 ± 0.25	0.43 ± 0.38
model:	1.47	1.80	1.68	1.43	0.91	0.49
α_{uw}	0.73 ± 0.05	0.87 ± 0.07	0.70 ± 0.13	0.65 ± 0.04	0.43 ± 0.09	0.23 ± 0.18
	0.73	0.86	0.84	0.78	0.53	0.26
$\alpha_{w\theta}$	0.45 ± 0.03	—	0.38 ± 0.03	0.36 ± 0.03	0.156 ± 0.034	0.086 ± 0.026
	0.45	0.42	0.41	0.36	0.18	0.072
ζ_S	1.65 ± 0.1	—	2.63 ± 0.25	3.04 ± 0.09	3.29 ± 0.14	2.55 ± 0.14
	1.65	2.88	2.88	2.88	2.88	2.41
A_S	0.50 ± 0.08	0.50 ± 0.07	0.48 ± 0.07	0.46 ± 0.04	0.53 ± 0.07	0.53 ± 0.09
	0.50	0.48	0.48	0.48	0.48	0.48
Pr_t	1.1 ± 0.1	—	0.71 ± 0.14	0.60 ± 0.06	0.85 ± 0.18	1.16 ± 1.34
	0.98	0.72	0.72	0.76	1.00	1.52
c_S	0.75 ± 0.08	—	0.99 ± 0.06	1.09 ± 0.09	0.51 ± 0.11	0.22 ± 0.07
	0.75	1.20	1.17	1.03	0.52	0.17
c_N	0	—	0.13 ± 0.02	0.27 ± 0.04	0.22 ± 0.05	0.13 ± 0.04
	0	0	0.16	0.26	0.23	0.10

For each quantity, the first line gives the experimental mean value and its standard deviation, and the second line gives the model result. The first column of data refers to turbulence in air at neutral stratification [5–7], the following columns apply to neutrally and stably stratified shear flows in salt-water [4]

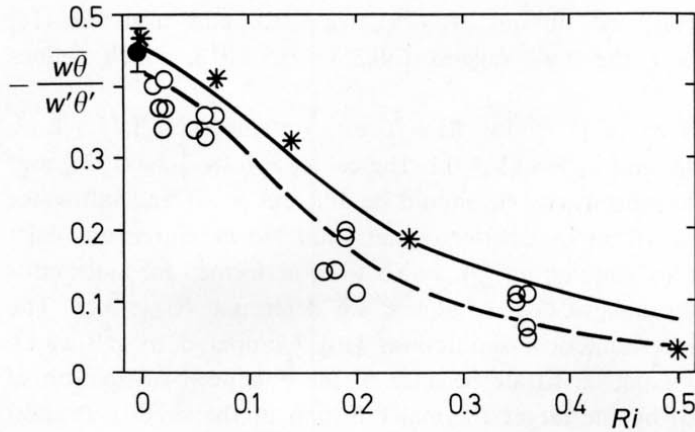


Fig. 1. Vertical scalar flux correlation coefficient $\alpha_{w\theta} = -\overline{w\theta}/(\overline{w'\theta'})$ versus gradient Richardson number Ri , based on the data of Rohr et al. [4] in salt-water (circles), the measurements in neutrally stratified wind tunnel shear flows of Tavoularis and Corrsin [6] (full circle with error bar), and the LES results of Kaltenbach [9] with a subgrid-Prandtl number of one (stars). The full curve corresponds to the present model, Eq. (18), for air, the dashed curve is the result for salt-water

In Fig. 1, the approximations are compared with the data for air and salt-water in terms of the correlation coefficient for vertical scalar fluxes. The curves are the consequences of the assumptions, and the data for $Ri > 0$ have not been used to calibrate the model parameters. Therefore, the comparison provides a check for the internal consistency of the present model. We see that the agreement is generally within the scatter of the data. The heat flux decays more quickly than the momentum flux, which is consistent with an increasing turbulent Prandtl number because of more efficient momentum than heat transport in wavy flows.

7 Conclusions

A consistent set of equations has been deduced for strongly sheared and stratified flow, based mainly on the local energy budget, and the basic assumptions of $\varepsilon = A_S w'^2 S$, and $\theta' = \zeta_S w' s / S$. If either ε or w' are given, the vertical diffusivities can be estimated from Eqs. (19) and (13). The coefficient c_N , see Eq. (20) and Table 1, is obviously a strong function of Ri . In [1], c_N was assumed to be a constant, but the data reported there show large scatter. It turns out that the results depend heavily on the variation of the growth rate G and the turbulent Prandtl number Pr_t with the gradient Richardson number Ri . Also, the coefficients are different for salt-water and air, in particular for the temperature fluctuations and the related coefficient ζ_S , because of different molecular mixing properties.

The given relationships let one conclude that $A_S = \text{const}$ implies a constant shear number Sw'^2/ε , and that the ratio of Ellison scale $L_\theta \equiv \theta'/s$ to the Ozmidov scale $L_0 \equiv (\varepsilon/N^3)^{1/2}$ equals $L_\theta/L_0 = \zeta_S A_S^{-1/2} Ri^{3/4}$ for $Ri \leq 0.25$, and the linear dependence of this scale ratio on $Ri^{3/4}$ is strongly supported by the data shown in Fig. 15 of Rohr et al. [4]. Hence, the present analysis supports in understanding and in predicting transport and dissipation properties of stratified and neutral shear flows. It shows, moreover, that a simple analysis using similarity properties and dimensional considerations, as I learned from Zierp [12], may sometimes be better suited to explain observations than complex closure models.

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