Computation of Kronecker-like Forms of a System Pencil: Applications, Algorithms and Software

Technical Report TR R181-95

January 1995

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Abstract. The reduction of system pencils to Kronecker-like canonical forms by orthogonal similarity transformations has many important applications, as for example: in the poles-zeros analysis of a system; in the structural analysis of a system (controllability, stabilizability, observability, detectability); in computing inverse systems; in computing minimum-phase factorizations of transfer-function matrices; in solving constrained matrix Riccati equations; in computing the Kronecker's canonical form of a general pencil. The reduction of system pencils can be performed by specially tailored $0(n^3)$ complexity numerically stable algorithms. The reduction technique can be also applied without modification to the more general case of an arbitrary pencil. A modular collection of LAPACK compatible FORTRAN 77 subroutines to perform the reduction of system pencil to several Kronecker-like forms is presented. Several lower level subroutines are useful in solving efficiently the above mentioned applications. A complete set of test programs together with a collection of test data accompanies the basic computational software.

Keywords: Kronecker's canonical form; descriptor systems; controllability; observability; system inversion; numerical algorithms; numerical control software.

1 Introduction

The most general representation of a linear time-invariant system is the *generalized state space* or *descriptor* model

$$\begin{aligned} \lambda \boldsymbol{E}\boldsymbol{x}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the descriptor state vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the input vector and $\boldsymbol{y}(t) \in \mathbb{R}^p$ is the output vector, and where $\boldsymbol{A} \in \mathbb{R}^{\ell\times n}$, $\boldsymbol{E} \in \mathbb{R}^{\ell\times n}$, $\boldsymbol{B} \in \mathbb{R}^{\ell\times m}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $\boldsymbol{D} \in \mathbb{R}^{p\times m}$. Notice that generally $\boldsymbol{A}$ and $\boldsymbol{E}$ are non-square matrices and even if these matrices are square, $\boldsymbol{E}$ may be singular. In the case of *standard systems* $\boldsymbol{E}$ is an invertible matrix and in most of cases $\boldsymbol{E} = \boldsymbol{I}_n$, the n-th order identity matrix. The operator λ is either the differential operator $\lambda \boldsymbol{x}(t) = d\boldsymbol{x}(t)/dt$ or the advance operator $\lambda \boldsymbol{x}(t) = \boldsymbol{x}(t+1)$. We denote alternatively the system (1) by the quadruple $(\boldsymbol{A} - \lambda \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$.

For this representation one can define the associated *system matrix* of (1) as the linear pencil

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \boldsymbol{B} & \boldsymbol{A} - \lambda \boldsymbol{E} \\ \hline \boldsymbol{D} & \boldsymbol{C} \end{array}\right]. \tag{2}$$

The *Kronecker's canonical form* (KCF) of this pencil [7] can be obtained by applying to $\mathcal{S}(\lambda)$ suitable invertible left and right transformations $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively, to yield a block diagonal decomposition of the form

$$\boldsymbol{U}\mathcal{S}(\lambda)\boldsymbol{V} = \mathrm{diag}\,\{\boldsymbol{L}_\epsilon, \boldsymbol{I} - \lambda \boldsymbol{J}_\infty, \boldsymbol{J}_f - \lambda I, \boldsymbol{L}_\eta^T\} \tag{3}$$

where

(a) $\boldsymbol{L}_\epsilon = \mathrm{diag}\,\{\boldsymbol{L}_{\epsilon_1}, \ldots, \boldsymbol{L}_{\epsilon_q}\}$, $\boldsymbol{L}_\eta^T = \mathrm{diag}\,\{\boldsymbol{L}_{\eta_1}^T, \ldots, \boldsymbol{L}_{\eta_r}^T\}$, and $\boldsymbol{L}_i$ is the $i \times (i+1)$ bidiagonal pencil

$$\boldsymbol{L}_i = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix} \tag{4}$$

(b) $\boldsymbol{J}_\infty = \mathrm{diag}\,\{\boldsymbol{J}_{\nu_1}(0), \ldots, \boldsymbol{J}_{\nu_s}(0)\}$ and $\boldsymbol{J}_i(0)$ is the Jordan block of order i corresponding to the null eigenvalue. Notice that the matrix $\boldsymbol{J}_\infty$ is nilpotent.

(c) $\boldsymbol{J}_f$ is a matrix in Jordan canonical form.

The pencils $\boldsymbol{J}_f - \lambda I$ and $I - \lambda \boldsymbol{J}_\infty$ contain the finite and infinite eigenvalues, respectively, and represent together the *regular* part of $\mathcal{S}(\lambda)$. The finite eigenvalues are also called the *finite zeros* of $\mathcal{S}(\lambda)$. To each Jordan block $\boldsymbol{J}_{\nu_i}(0)$ corresponds an infinite elementary divisor of order $\nu_i - 1$ in the Schmidt form of the polynomial matrix $\mathcal{S}(\lambda)$, and thus the union of the sets of $\nu_i - 1$ infinite eigenvalues is also called the *infinite zeros* of $\mathcal{S}(\lambda)$. The blocks $\boldsymbol{L}_\epsilon$ and $\boldsymbol{L}_\eta^T$ contain the *singularity* of $\mathcal{S}(\lambda)$ and the index sets $\{\epsilon_i\}$ and $\{\eta_i\}$ are the *left* (or *column*) and *right* (or *row*) *Kronecker indices* of $\mathcal{S}(\lambda)$, respectively. Notice that zero row or column indices correspond to null rows or null columns in the KCF of the system pencil, respectively.

The KCF is very useful in the structural analysis of descriptor systems. Particular system matrices can be used to study the pole-zeros structure or the controllability-observability properties of a system by computing various type of zeros (poles, input or output decoupling zeros) [14]. Controllability and observability indices can be easily deduced from the row and column Kronecker indices of the particular system pencils $[\, \boldsymbol{B} \mid \boldsymbol{A} - \lambda \boldsymbol{E} \,]$ and $\left[\begin{array}{c} \boldsymbol{A} - \lambda \boldsymbol{E} \\ \hline \boldsymbol{C} \end{array}\right]$. The KCF also provides information on the left and right null-space structure of $\mathcal{S}(\lambda)$ and generalized inverses of the system pencil can be computed using this information, having as main application the inversion of rational matrices [23]. The need to determine generalized inverses of rational matrices arises in some of recently developed algorithms to compute minimum-phase rational coprime factorizations, as for example the *inner-outer* factorization [24] or the *J-inner-outer* factorization [26]. The reduction of special system pencils to KCF can also be used to solve special classes of constrained Riccati equations [9]. The solution of such non-standard Riccati equations is the basic computational step in determining inner-outer factorizations with non-square inner factors [28], [12].

In all above applications the computation of the KCF is in fact not necessary and certainly not recommendable from numerical point of view. Instead, with the help of orthogonal left and right transformations, several condensed Kronecker-like forms can be computed which exhibit either the complete Kronecker structure or only a part of the Kronecker structure of the system pencil. In the next sections we introduce several Kronecker-like forms which can be computed with the help of a collection of recently implemented FORTRAN 77 subroutines. For an easy reference, we shall associate each form with the name of the corresponding subroutine implemented to compute it. We also indicate the main applicability of each form in solving some of the above mentioned problems.

The algorithms to compute various Kronecker-like forms are combinations of several recently developed numerically stable procedures [2, 20, 21, 14]. All implemented algorithms have $0(n^3)$ computational complexity and compare favorably in many aspects with existing methods [18, 2, 6].

2 The SPRED Form

The SPRED subroutine determines, by applying left and right orthogonal transformations, the following Kronecker-like form of the system pencil

$$\widehat{S}(\lambda) = Q^T S(\lambda) Z = \left[\begin{array}{c|ccccc} B_r & A_r - \lambda E_r & * & * & * & * \\ O & O & A_\infty - \lambda E_\infty & * & * & * \\ O & O & O & D_i & * & * \\ O & O & O & O & A_f - \lambda E_f & * \\ O & O & O & O & O & A_l - \lambda E_l \\ \hline O & O & O & O & O & C_l \end{array}\right], \quad (5)$$

where

(a) $B_r \in \mathbb{R}^{n_r \times m_r}$, $A_r \in \mathbb{R}^{n_r \times n_r}$, and $E_r \in \mathbb{R}^{n_r \times n_r}$ is invertible and upper-triangular; the pencil $[\, B_r \; A_r - \lambda E_r \,]$ contains the right Kronecker structure of $S(\lambda)$; the pair $(B_r, A_r - \lambda E_r)$ is controllable and the pencil $[\, B_r \; A_r - \lambda E_r \,]$ is in the controllability staircase form

$$[\, B_r \mid A_r - \lambda E_r \,] = \left[\begin{array}{c|cccc} A^r_{1,1} & A^r_{1,2} - \lambda E^r_{1,2} & \cdots & A^r_{1,k-1} - \lambda E^r_{1,k-1} & A^r_{1,k} - \lambda E^r_{1,k} \\ O & A^r_{2,2} & \cdots & A^r_{2,k-1} - \lambda E^r_{2,k-1} & A^r_{2,k} - \lambda E^r_{2,k} \\ O & O & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & A^r_{k-1,k-1} & A^r_{k-1,k} - \lambda E^r_{k-1,k} \\ O & O & \cdots & O & A^r_{k,k} \end{array}\right], \quad (6)$$

with the diagonal matrices $A^r_{i,i} \in \mathbb{R}^{\overline{\tau}_i \times \overline{\tau}_{i-1}}$ having full row rank $\overline{\tau}_i$ (with $\overline{\tau}_0 = m_r$ and $\overline{\tau}_k = 0$), and the upper diagonal matrices $E^r_{i,i+1} \in \mathbb{R}^{\overline{\tau}_i \times \overline{\tau}_i}$ being invertible and upper-triangular; moreover, each $A^r_{i,i}$ has the form $A^r_{i,i} = [\, O \;\; \widetilde{A}^r_{i,i} \,]$ with $\widetilde{A}^r_{i,i}$ invertible and upper-triangular.

(b) $A_\infty \in \mathbb{R}^{n_\infty \times n_\infty}$ is invertible and upper-triangular, $E_\infty \in \mathbb{R}^{n_\infty \times n_\infty}$ is nilpotent and upper-triangular, and $D_i \in \mathbb{R}^{n_s \times n_s}$ is invertible and upper-triangular; the *regular* pencil $A_\infty - \lambda E_\infty$ together with D_i contain the infinity Kronecker structure of $S(\lambda)$; the pencil $A_\infty - \lambda E_\infty$ is in the special form

$$A_\infty - \lambda E_\infty = \left[\begin{array}{ccccc} A^\infty_{1,1} & A^\infty_{1,2} - \lambda E^\infty_{1,2} & \cdots & A^\infty_{1,h-1} - \lambda E^\infty_{1,h-1} & A^\infty_{1,h} - \lambda E^\infty_{1,h} \\ O & A^\infty_{2,2} & \cdots & A^\infty_{2,h-1} - \lambda E^\infty_{2,h-1} & A^\infty_{2,h} - \lambda E^\infty_{2,h} \\ O & O & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & A^\infty_{h-1,h-1} & A^\infty_{h-1,h} - \lambda E^\infty_{h-1,h} \\ O & O & \cdots & O & A^\infty_{h,h} \end{array}\right], \quad (7)$$

with the diagonal matrices $A^\infty_{i,i} \in \mathbb{R}^{\rho_i \times \rho_i}$ invertible and upper-triangular, and the upper diagonal matrices $E^\infty_{i,i+1} \in \mathbb{R}^{\rho_i \times \rho_{i+1}}$ having full column rank.

(c) $A_f \in \mathbb{R}^{n_f \times n_f}$, and $E_f \in \mathbb{R}^{n_f \times n_f}$ is invertible and upper-triangular; the *regular* pencil $A_f - \lambda E_f$ contains the finite Kronecker structure of $S(\lambda)$.

(d) $C_l \in \mathbb{R}^{p_l \times n_l}$, $A_l \in \mathbb{R}^{n_l \times n_l}$, and $E_l \in \mathbb{R}^{n_l \times n_l}$ is invertible and upper-triangular; the pencil $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$ contains the left Kronecker structure of $\mathcal{S}(\lambda)$; the pair $(C_l, A_l - \lambda E_l)$ is observable and the pencil $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$ is in the observability staircase form

$$\left[\begin{array}{c} A_l - \lambda E_l \\ \hline C_l \end{array}\right] = \left[\begin{array}{ccccc} A^l_{j,j} & A^l_{j,j-1} - \lambda E^l_{j,j-1} & \cdots & A^l_{j,2} - \lambda E^l_{j,2} & A^l_{j,1} - \lambda E^l_{j,1} \\ O & A^l_{j-1,j-1} & \cdots & A^l_{j-1,2} - \lambda E^l_{j-1,2} & A^l_{j-1,1} - \lambda E^l_{j-1,1} \\ O & O & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & A^l_{2,2} & A^l_{2,1} - \lambda E^l_{2,1} \\ \hline O & O & \cdots & O & A^l_{1,1} \end{array}\right], \quad (8)$$

with the diagonal matrices $A^l_{i,i} \in \mathbb{R}^{\mu_{i-1} \times \mu_i}$ having full column rank μ_i (with $\mu_0 = p_l$ and $\mu_j = 0$), and the upper diagonal matrices $E^l_{i+1,i} \in \mathbb{R}^{\mu_i \times \mu_i}$ being invertible and upper-triangular; moreover each $A^l_{i,i}$ has the form $A^l_{i,i} = \begin{bmatrix} \widetilde{A}^l_{i,i} \\ O \end{bmatrix}$ with $\widetilde{A}^l_{i,i}$ invertible and upper-triangular.

Excepting the finite eigenvalues structure, the SPRED form contains identical structural information as the KCF. The index sets $\{\overline{\tau}_i\}$, $\{\rho_i\}$, $\{\mu_i\}$ determine the minimal indices and the infinite structure of the system pencil $\mathcal{S}(\lambda)$ as follows [18]:

Lemma 1 *From the structure of the pencil* $[\ B_r\ \ A_r - \lambda E_r\ ]$ *in (6), there are* $c_i = \overline{\tau}_{i-1} - \overline{\tau}_i$ *Kronecker column indices of size* $(i-1)$, $(i = 1, \ldots, k)$, *where* $\overline{\tau}_0 = m_r$ *and* $\overline{\tau}_k = 0$.

Lemma 2 *From the structure of the pencil* $A_\infty - \lambda E_\infty$ *in (7), there are* $d_i = \rho_i - \rho_{i+1}$ *infinite elementary divisors of degree* i, $(i = 1, \ldots, h)$, *where* $\rho_{h+1} = 0$.

Lemma 3 *From the structure of the pencil* $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$ *in (8), there are* $r_i = \mu_{i-1} - \mu_i$ *Kronecker row indices of size* $(i-1)$, $(i = 1, \ldots, j)$, *where* $\mu_0 = p_l$ *and* $\mu_j = 0$.

In what follows we discuss two interesting applications of the SPRED form.

2.1 Inversion of Rational Matrices.

Let $G(\lambda)$ be an $p \times m$ rational matrix for which we want to compute a generalized inverse $G(\lambda)^+$ satisfying

$$\begin{aligned} GG^+G &= G \\ G^+GG^+ &= G^+ \end{aligned} \quad (9)$$

In the nomenclature of [3], $G(\lambda)^+$ is called an (1,2)-*generalized inverse* of $G(\lambda)$. Such computation is a necessary first step in the recently developed algorithms to compute the *inner-outer* factorization [24] or the *J-inner-outer* factorization [26] of an arbitrary rational matrix.

$G(\lambda)$ can be assimilated with the *transfer-function matrix* (TFM) of a *regular* descriptor system $(A - \lambda E, B, C, D)$ ($\det(A - \lambda E) \not\equiv 0$), satisfying

$$G(\lambda) = C(\lambda E - A)^{-1}B + D. \quad (10)$$

If the descriptor representation of $G(\lambda)$ is *irreducible* (controllable and observable), it was shown in [27], that much of the structure of $G(\lambda)$ can be retrieved in that of $\mathcal{S}(\lambda)$, as for example the zero structure of $G(\lambda)$, and also the left and right null-space structures of $G(\lambda)$ and $\mathcal{S}(\lambda)$ are the same.

The generalized inverse $G(\lambda)^+$ of $G(\lambda)$ can be computed by using the formula [23]

$$G(\lambda)^+ = [\, O \; I_m \,] \; \mathcal{S}(\lambda)^+ \left[\begin{array}{c} I_p \\ O \end{array} \right]. \tag{11}$$

With the partitioning of $\widehat{\mathcal{S}}(\lambda)$ in (5) as

$$\widehat{\mathcal{S}}(\lambda) = \left[\begin{array}{c|c} \widehat{\mathcal{S}}_{11}(\lambda) & \widehat{\mathcal{S}}_{12}(\lambda) \\ \hline O & \widehat{\mathcal{S}}_{22}(\lambda) \end{array} \right] \tag{12}$$

it follows that for almost all λ, $\operatorname{rank} \mathcal{S}(\lambda) = \operatorname{rank} \widehat{\mathcal{S}}_{12}(\lambda)$, and thus a generalized (1,2)-inverse of $\mathcal{S}(\lambda)$ can be computed as [3]

$$\mathcal{S}(\lambda)^+ = Z \left[\begin{array}{cc} O & O \\ \widehat{\mathcal{S}}_{12}(\lambda)^{-1} & O \end{array} \right] Q^T. \tag{13}$$

It is easy to verify that $G(\lambda)^+$ in (11) is indeed an (1,2)-generalized inverse of $G(\lambda)$.

To compute a descriptor representation of the generalized inverse $G(\lambda)^+$, it is not necessary to explicitly evaluate $\widehat{\mathcal{S}}_{12}(\lambda)^{-1}$. If we denote

$$\widehat{A}_{12} - \lambda \widehat{E}_{12} = \widehat{\mathcal{S}}_{12}(\lambda), \quad \widehat{B} = Q^T \left[\begin{array}{c} I_p \\ O \end{array} \right] = \left[\begin{array}{c} \widehat{B}_1 \\ \widehat{B}_2 \end{array} \right], \quad \widehat{C} = [\, O \; I_m \,] Z = [\, \widehat{C}_1 \; \widehat{C}_2 \,], \tag{14}$$

where $\widehat{B}$ and $\widehat{C}$ are partitioned analogously with the column and row partition of $\widehat{\mathcal{S}}(\lambda)$ in (12), respectively, then the corresponding $G(\lambda)^+$ is given by

$$G(\lambda)^+ = \widehat{C}_2 (\widehat{A}_{12} - \lambda \widehat{E}_{12})^{-1} \widehat{B}_1 \tag{15}$$

and thus $(\widehat{A}_{12} - \lambda \widehat{E}_{12}, \widehat{B}_1, -\widehat{C}_2, O)$ is a descriptor representation of $G(\lambda)^+$. The compnted generalized inverse has minimal order only if $G(\lambda)$ is invertible. The compntation of lower order inverses is addressed in [23].

2.2 Computation of Stable Generalized Inverses

The finite pole structure of the generalized inverse G^+ computed with (15) results from the SPRED form (5) of the system pencil $\mathcal{S}(\lambda)$ used to compute it. Thus, the finite poles of G^+ are the union of generalized eigenvalnes of the pair (A_f, E_f) called also the *zeros* of G and and of the generalized eigenvalues of the pairs (A_r, E_r) and (A_l, E_l). Notice that the zeros of G are always present among the poles of any of its generalized inverses. Even if the zeros are stable, that is the descriptor system is *minimum-phase*, it is still possible that the generalized inverse computed with (15) has unstable poles because of possible unstable eigenvalues appearing in the pairs (A_r, E_r) and (A_l, E_l). The spectrums of these pairs can be arbitrarily modified by applying suitable left and right non-orthogonal transformations U and V to the SPRED form (5). By choosing U and V of the special forms

$$U = \left[\begin{array}{ccc} I_{n_r+n_\infty+n_s+n_f} & O & O \\ O & I_{n_l} & K \\ O & O & I_{p_l} \end{array} \right], \quad V = \left[\begin{array}{ccc} I_{m_r} & F & O \\ O & I_{n_r} & O \\ O & O & I_{n_\infty+n_s+n_f+n_l} \end{array} \right] \tag{16}$$

we obtain for $\widetilde{\mathcal{S}}(\lambda) = U \widehat{\mathcal{S}}(\lambda) V$

$$\widetilde{\mathcal{S}}(\lambda) = \left[\begin{array}{c|ccccc} B_r & A_r + B_r F - \lambda E_r & * & * & * & * \\ O & O & A_\infty - \lambda E_\infty & * & * & * \\ O & O & O & D_i & * & * \\ O & O & O & O & A_f - \lambda E_f & * \\ O & O & O & O & O & A_l + K C_l - \lambda E_l \\ \hline O & O & O & O & O & C_l \end{array} \right] \tag{17}$$

which has the same structure as $\widehat{\mathcal{S}}(\lambda)$ in (12). Because the pair $(\boldsymbol{B}_r, \boldsymbol{A}_r - \lambda \boldsymbol{E}_r)$ is controllable and the pair $(\boldsymbol{C}_l, \boldsymbol{A}_l - \lambda \boldsymbol{E}_l)$ is observable, we can arbitrarily assign the spectrum of the pairs $(\boldsymbol{A}_r + \boldsymbol{B}_r\boldsymbol{F}, \boldsymbol{E}_r)$ and $(\boldsymbol{A}_l + \boldsymbol{K}\boldsymbol{C}_l, \boldsymbol{E}_l)$ by choosing suitable state-feedback and output-injection matrices $\boldsymbol{F}$ and $\boldsymbol{K}$, respectively. These matrices can be efficiently computed by nsing either direct stabilization methods or pole assignment techniques for descriptor systems as those proposed in [25].

The generalized inverse $\boldsymbol{G}^+$ can be compnted similarly as in (13)

$$\mathcal{S}(\lambda)^+ = \boldsymbol{Z}\boldsymbol{V} \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \widetilde{\mathcal{S}}_{12}(\lambda)^{-1} & \boldsymbol{O} \end{bmatrix} \boldsymbol{U}\boldsymbol{Q}^T, \tag{18}$$

where $\widetilde{\mathcal{S}}_{12}(\lambda)$ is the submatrix of $\widetilde{\mathcal{S}}(\lambda)$ corresponding to $\widehat{\mathcal{S}}_{12}(\lambda)$ in (12). Further

$$\boldsymbol{G}(\lambda)^+ = \widetilde{\boldsymbol{C}}_2(\widetilde{\boldsymbol{A}}_{12} - \lambda \widetilde{\boldsymbol{E}}_{12})^{-1}\widetilde{\boldsymbol{B}}_1, \tag{19}$$

where

$$\widetilde{\boldsymbol{A}}_{12} - \lambda \widetilde{\boldsymbol{E}}_{12} = \widetilde{\mathcal{S}}_{12}(\lambda), \quad \widetilde{\boldsymbol{B}} = \boldsymbol{U}\boldsymbol{Q}^T \begin{bmatrix} \boldsymbol{I}_p \\ \boldsymbol{O} \end{bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{B}}_1 \\ \widetilde{\boldsymbol{B}}_2 \end{bmatrix}, \quad \widetilde{\boldsymbol{C}} = [\, \boldsymbol{O} \ \boldsymbol{I}_m \,]\boldsymbol{Z}\boldsymbol{V} = [\, \widetilde{\boldsymbol{C}}_1 \ \widetilde{\boldsymbol{C}}_2 \,] \tag{20}$$

Thus $(\widetilde{\boldsymbol{A}}_{12} - \lambda \widetilde{\boldsymbol{E}}_{12}, \widetilde{\boldsymbol{B}}_1, -\widetilde{\boldsymbol{C}}_2, \boldsymbol{O})$ is a descriptor representation of $\boldsymbol{G}(\lambda)^+$. It is clear that in general, the only unstable poles of the generalized inverse (19) are the unstable zeros of $\boldsymbol{G}$. Thus $\boldsymbol{G}(\lambda)^+$ is stable if the the given $\boldsymbol{G}$ is minimum-phase.

3 The SLRRED Form

The SLRRED subroutine determines, by applying left and right orthogonal transformations, the following Kronecker-like form of the system pencil

$$\widehat{\mathcal{S}}(\lambda) = \boldsymbol{Q}^T\mathcal{S}(\lambda)\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{A}_r - \lambda \boldsymbol{E}_r & * & * & * \\ \boldsymbol{O} & \boldsymbol{D}_i & * & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_f - \lambda \boldsymbol{E}_f & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_l - \lambda \boldsymbol{E}_l \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C}_l \end{bmatrix}, \tag{21}$$

where

(a) $\boldsymbol{A}_r - \lambda \boldsymbol{E}_r$ has full row rank and contains the right and infinite Kronecker structure of $\mathcal{S}(\lambda)$; the pencil $\boldsymbol{A}_r - \lambda \boldsymbol{E}_r$ is in the staircase form

$$\boldsymbol{A}_r - \lambda \boldsymbol{E}_r = \begin{bmatrix} \boldsymbol{A}^r_{1,1} & \boldsymbol{A}^r_{1,2} - \lambda \boldsymbol{E}^r_{1,2} & \cdots & \boldsymbol{A}^r_{1,k-1} - \lambda \boldsymbol{E}^r_{1,k-1} & \boldsymbol{A}^r_{1,k} - \lambda \boldsymbol{E}^r_{1,k} \\ \boldsymbol{O} & \boldsymbol{A}^r_{2,2} & \cdots & \boldsymbol{A}^r_{2,k-1} - \lambda \boldsymbol{E}^r_{2,k-1} & \boldsymbol{A}^r_{2,k} - \lambda \boldsymbol{E}^r_{2,k} \\ \boldsymbol{O} & \boldsymbol{O} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \boldsymbol{A}^r_{k-1,k-1} & \boldsymbol{A}^r_{k-1,k} - \lambda \boldsymbol{E}^r_{k-1,k} \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{A}^r_{k,k} \end{bmatrix}, \tag{22}$$

with the diagonal matrices $\boldsymbol{A}^r_{i,i} \in \mathbb{R}^{\bar{\tau}_i \times \bar{\mu}_i}$ having full row rank $\bar{\tau}_i$ and the upper diagonal matrices $\boldsymbol{E}^r_{i,i+1} \in \mathbb{R}^{\bar{\tau}_i \times \bar{\mu}_{i+1}}$ having full column rank.

(b) $\boldsymbol{D}_i \in \mathbb{R}^{n_s \times n_s}$ is invertible and upper-triangular;

(c) $\boldsymbol{A}_f \in \mathbb{R}^{n_f \times n_f}$, and $\boldsymbol{E}_f \in \mathbb{R}^{n_f \times n_f}$ is invertible and upper-triangular; the *regular* pencil $\boldsymbol{A}_f - \lambda \boldsymbol{E}_f$ contains the finite Kronecker structure of $\mathcal{S}(\lambda)$.

(d) $C_l \in \mathbb{R}^{p_l \times n_l}$, $A_l \in \mathbb{R}^{n_l \times n_l}$, and $E_l \in \mathbb{R}^{n_l \times n_l}$ is invertible and upper-triangular; the pencil $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$ contains the left Kronecker structure of $\mathcal{S}(\lambda)$; the pair $(C_l, A_l - \lambda E_l)$ is observable and the pencil $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$ is in the observability staircase form (8) with the diagonal matrices $A^l_{i,i} \in \mathbb{R}^{\mu_{i-1} \times \mu_i}$ having full column rank μ_i (with $\mu_0 = p_l$ and $\mu_j = 0$), and the upper diagonal matrices $E^l_{i+1,i} \in \mathbb{R}^{\mu_i \times \mu_i}$ being invertible and upper-triangular; moreover each $A^l_{i,i}$ has the form $A^l_{i,i} = \begin{bmatrix} \widetilde{A}^l_{i,i} \\ O \end{bmatrix}$ with $\widetilde{A}^l_{i,i}$ invertible and upper-triangular.

Excepting the finite eigenvalues structure, the SLRRED form (21) contains identical structural information as the KCF. The index sets $\{\bar{\tau}_i\}$, $\{\bar{\mu}_i\}$, $\{\mu_i\}$ determine the minimal indices and the infinite structure of the system pencil $\mathcal{S}(\lambda)$ as follows [18]:

Lemma 4 *From the structure of the pencil $A_r - \lambda E_r$ in (22), there are $c_i = \bar{\mu}_i - \bar{\tau}_i$ Kronecker column indices of size $(i-1)$, $(i = 1, \ldots, k)$ and $d_i = \bar{\tau}_i - \bar{\mu}_{i+1}$ infinite elementary divisors of degree i, $(i = 1, \ldots, k)$, where $\bar{\mu}_{k+1} = 0$.*

Lemma 5 *From the structure of the pencil $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$ in (8), there are $r_i = \mu_{i-1} - \mu_i$ Kronecker row indices of size $(i-1)$, $(i = 1, \ldots, j)$, where $\mu_0 = p_l$ and $\mu_j = 0$.*

3.1 Inner-Outer Factorization of Rational Matrices.

Let $G(\lambda)$ be a $p \times m$ rational matrix and let $(A - \lambda E, B, C, D)$ be a corresponding regular descriptor system representation satisfying (10). We assume that G is *stable*, that is all its finite poles are in the stability region $\mathbb{C}^-$ of the complex plane $\mathbb{C}$. $\mathbb{C}^-$ is either the left open complex half-plane for a continuous-time system or the interior of the nnit circle for a discrete-time system. Then $G(\lambda)$ has an *inner-outer factorization* $G = G_i G_o$, where G_i is a *square* inner factor and G_o is an outer factor. Recall that G_i is *inner* means $G_i^* G_i = I$, where $G_i^*(s) = G_i^T(-s)$ in continuous-time and $G_i^*(z) = G_i^T(1/z)$ in discrete-time, and G_o is *outer* means G_o has a stable generalized inverse. The main computational steps to determine the inner and outer factors of G are [24]:

1. Compute a generalized inverse G^+ of G such that the unstable poles of G^+ are exactly the unstable zeros of G.

2. Compute the right coprime factorization with minimal order inner denominator of G^+ as $G^+ = N G_i^{-1}$, where N and G_i are stable TFMs with G_i inner (the order of G_i equals the number of unstable zeros of G).

3. Compute $G_o = G_i^{-1} G$.

At step 1 of the above procedure the generalized inverse G^+ with the required properties can be computed by using the method presented in the subsection 2.2. At step 2 we have to compute a right coprime factorization of G^+ with minimal order inner denominator. For this purpose the recursive algorithm proposed in [22] is best suited. The inner denominator can be determined by applying this algorithm to the descriptor representation of $G(\lambda)^+$ given by (19).

The algorithm of [22] determines the matrices of the descriptor representation of the factors N and G_i in the form

$$\begin{bmatrix} N \\ G_i \end{bmatrix} := \left(\begin{bmatrix} A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} \\ 0 & A_{22} - \lambda E_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \\ 0 & F_2 \end{bmatrix}, \begin{bmatrix} 0 \\ I_p \end{bmatrix} \right)$$

and a minimal realization for the inner factor G_i is given by

$$G_i := (A_{22} - \lambda E_{22}, B_2, F_2, I_p),$$

where E_{22} is invertible and upper-triangular and the resulting pair (A_{22}, E_{22}) is in a *generalized real Schur form* (GRSF) having as eigenvalues the reflected stable eigenvalues corresponding to the unstable zeros of the given system. Thus, the order of the inner factor is precisely the number of unstable zeros of G.

In computing G_i the output matrix $\widetilde{C}_2$ in (19) plays no role and thus the right transformation matrices Z and V used in (18) to determine $\mathcal{S}(\lambda)^+$ are not necessary to be computed. Thus instead of using the SPRED form to compute an appropriate $G(\lambda)^+$, we can use the simpler SLRRED form, which provides all necessary information to compute G_i. A detailed description of the resulting algorithm is presented in [24]. A similar technique can be used to compute J-inner-outer factorizations [26].

4 The SRLRED Form

The SRLRED subroutine determines, by applying left and right orthogonal transformations, the following Kronecker-like form of the system pencil

$$\widehat{\mathcal{S}}(\lambda) = Q^T \mathcal{S}(\lambda) Z = \begin{bmatrix} B_r & A_r - \lambda E_r & * & * & * \\ O & O & A_f - \lambda E_f & * & * \\ O & O & O & D_i & * \\ O & O & O & O & A_l - \lambda E_l \end{bmatrix}, \qquad (23)$$

where

(a) $B_r \in \mathbb{R}^{n_r \times m_r}$, $A_r \in \mathbb{R}^{n_r \times n_r}$, and $E_r \in \mathbb{R}^{n_r \times n_r}$ is invertible and upper-triangular; the pencil $[\ B_r\ A_r - \lambda E_r\ ]$ contains the right Kronecker structure of $\mathcal{S}(\lambda)$; the pair $(B_r, A_r - \lambda E_r)$ is controllable and the pencil $[\ B_r\ A_r - \lambda E_r\ ]$ is in the controllability staircase form (6) with the diagonal matrices $A^r_{i,i} \in \mathbb{R}^{\bar{\tau}_i \times \bar{\tau}_{i-1}}$ having full row rank $\bar{\tau}_i$ (with $\bar{\tau}_0 = m_r$ and $\bar{\tau}_k = 0$), and the upper diagonal matrices $E^r_{i,i+1} \in \mathbb{R}^{\bar{\tau}_i \times \bar{\tau}_i}$ being invertible and upper-triangular; moreover, each $A^r_{i,i}$ has the form $A^r_{i,i} = [\ O\ \ \widetilde{A}^r_{i,i}\ ]$ with $\widetilde{A}^r_{i,i}$ invertible and upper-triangular.

(b) $A_f \in \mathbb{R}^{n_f \times n_f}$, and $E_f \in \mathbb{R}^{n_f \times n_f}$ is invertible and upper-triangular; the *regular* pencil $A_f - \lambda E_f$ contains the finite Kronecker structure of $\mathcal{S}(\lambda)$.

(c) $D_i \in \mathbb{R}^{n_s \times n_s}$ is invertible and upper-triangular;

(d) $A_l - \lambda E_l$ has full column rank and contains the left and infinite Kronecker structure of $\mathcal{S}(\lambda)$; the pencil $A_l - \lambda E_l$ is in the staircase form

$$A_l - \lambda E_l = \begin{bmatrix} A^l_{j,j} & A^l_{j,j-1} - \lambda E^l_{j,j-1} & \cdots & A^l_{j,2} - \lambda E^l_{j,2} & A^l_{j,1} - \lambda E^l_{j,1} \\ O & A^l_{j-1,j-1} & \cdots & A^l_{j-1,2} - \lambda E^l_{j-1,2} & A^l_{j-1,1} - \lambda E^l_{j-1,1} \\ O & O & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & A^l_{2,2} & A^l_{2,1} - \lambda E^l_{2,1} \\ O & O & \cdots & O & A^l_{1,1} \end{bmatrix}, \qquad (24)$$

with the diagonal matrices $A^l_{i,i} \in \mathbb{R}^{\tau_i \times \mu_i}$ having full column rank μ_i and the upper diagonal matrices $E^l_{i+1,i} \in \mathbb{R}^{\mu_i \times \tau_{i+1}}$ having full row rank.

Excepting the finite eigenvalues structure, the SRLRED form contains identical structural information as the KCF. The index sets $\{\bar{\tau}_i\}$, $\{\tau_i\}$, $\{\mu_i\}$ determine the minimal indices and the infinite structure of the system pencil $\mathcal{S}(\lambda)$ as follows [18]:

Lemma 6 *From the structure of the pencil* $[\ B_r\ A_r - \lambda E_r\ ]$ *in (6), there are* $c_i = \bar{\tau}_{i-1} - \bar{\tau}_i$ *Kronecker column indices of size* $(i-1)$, $(i = 1, \ldots, k)$, *where* $\bar{\tau}_0 = m_r$ *and* $\bar{\tau}_k = 0$.

Lemma 7 *From the structure of the pencil* $A_l - \lambda E_l$ *in (24), there are* $r_i = \tau_i - \mu_i$ *Kronecker row indices of size* $(i-1)$, $(i = 1, \ldots, j)$ *and* $d_i = \mu_i - \tau_{i+1}$ *infinite elementary divisors of degree* i, $(i = 1, \ldots, j)$, *where* $\tau_{j+1} = 0$.

4.1 Computation of Maximal Proper Stable Deflating Subspaces

Let $M - \lambda N$ be an arbitrary pencil with $M, N \in \mathbb{R}^{r \times q}$, let $\mathbb{C}^-$ be the stability region of $\mathbb{C}$ and let $\mathbb{C}^+$ be the complement of $\mathbb{C}^-$ in $\mathbb{C}$. The following particular reducing subspace introduced in [10] has important applications in solving various nonstandard Riccati equations.

Definition 1 *A subspace* $\mathcal{V} \subset \mathbb{R}^q$ *of dimension* ρ *is called a proper stable deflating subspace of* $M - \lambda N$ *to the right if* $NV = MVS$ *and* MV *is monic, where* $V \in \mathbb{R}^{q \times \rho}$ *is any basis matrix for* $\mathcal{V}$ *and* $S \in \mathbb{R}^{\rho \times \rho}$ *is an adequate matrix having all its eigenvalues in* $\mathbb{C}^-$.

Here the term *proper* restricts the definition of reducing subspaces introduced in [19] to the finite eigenvalue structure of the pencil $M - \lambda N$, ruling out basically the structure corresponding to infinite eigenvalues. Similar definition can be given for proper stable deflating subspaces to the left.

Let n_f^- be the number of stable generalized eigenvalues of the pair (A_f, E_f) in the SRLRED form of the pencil $M - \lambda N$ (viewed as a particular system pencil) and let n_r the dimension of the $A_r - \lambda E_r$ block. The following result [10] characterizes the existence of a stable proper deflating subspace of maximal dimension.

Theorem 1 *The pencil* $M - \lambda N$ *has a stable proper deflating subspace to the right if and only if* $n_r + n_f^- > 0$. *Moreover, the maximal dimension of a stable proper deflating subspace to the right is* $n_r + n_f^-$.

For the computation of a stable proper deflating subspace of maximal dimension the SRLRED form of the pencil $M - \lambda N$ can be used as the starting form for further reductions. Notice that generally the accumulation of left transformations is not necessary in this case. The procedure to compute the basis matrix V for a proper deflating subspace of maximal dimension of $M - \lambda N$ has the following main steps [16]:

1. Compute the orthogonal matrices Q and Z to reduce the pencil $M - \lambda N$ to the SRLRED form (23).

2. Apply the pole assignment algorithm of [25] to determine the orthogonal matrices Q_1 and Z_1 and the feedback matrix F such that $\Lambda(Q_1^T(A_r + B_r F)Z_1, Q_1^T E_r Z_1) \subset \mathbb{C}^-$ and the pair $(Q_1^T(A_r + B_r F)Z_1, Q_1^T E_r Z_1)$ is in a GRSF.

3. Compute the orthogonal matrices Q_2 and Z_2 to reduce the pair (A_f, E_f) to the ordered GRSF
$$Q_2^T A_f Z_2 = \begin{bmatrix} A_{11}^f & A_{12}^f \\ 0 & A_{22}^f \end{bmatrix}, \quad Q_2^T E_f Z_2 = \begin{bmatrix} E_{11}^f & E_{12}^f \\ 0 & E_{22}^f \end{bmatrix}, \tag{25}$$
where $A_{11}^f, E_{11}^f \in \mathbb{R}^{n_f^- \times n_f^-}$, $\Lambda(A_{11}^f, E_{11}^f) \subset \mathbb{C}^-$ and $\Lambda(A_{22}^f, E_{22}^f) \subset \mathbb{C}^+$.

4. Compute V as
$$V = Z \begin{bmatrix} F & O \\ Z_1 & O \\ O & Z_2 \\ O & O \end{bmatrix} \tag{26}$$

Notice that in the above procedure, the left transformations should be not accumulated. It is easy to verify that $NV = MVS$ holds with MV monic,

$$S := \begin{bmatrix} Z_1^T E_r^{-1}(A_r + B_r F) Z_1 & * \\ O & (E_{11}^f)^{-1} A_{11}^f \end{bmatrix}, \tag{27}$$

and $\Lambda(S) \subset \mathbb{C}^-$.

The reduction to the ordered GRSF at step 3 can be performed by using the well-know QZ algorithm of [15] followed by the recently developed numerically stable algorithms to reorder the GRSF [13].

4.2 Solving Nonstandard Continuous-Time Riccati Equations

We discuss in this subsection the computation of the symmetric solution X of the following nonstandard so-called *constrained continuous-time algebraic Riccati equation* (CCTARE)

$$\begin{aligned} A^T X + XA - (XB + L)R^+(B^T X + L^T) + Q = O \\ \ker R \subset \ker (XB + L) \end{aligned} \tag{28}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$ with $Q = Q^T$, $R \in \mathbb{R}^{m\times m}$ with $R = R^T$, and $L \in \mathbb{R}^{n\times m}$. A solution to CCTARE is called *stabilizing* if the pair $(A+BF_0, B_0)$ is stabilizable, where $F_0 := -R^+(B^T X + L^T)$ and where B_0 is any basis matrix for $\operatorname{Im} B \cap \ker R$. Notice that no other assumptions than symmetry are made on matrices Q and R.

If R is invertible, then the second equation in (28) is trivially satisfied and the first equation becomes

$$A^T X + XA - (XB + L)R^{-1}(B^T X + L^T) + Q = O. \tag{29}$$

This equation is called the *the continuous-time algebraic Riccati equation* (CTARE).

In his most general form, the CCTARE appears in computing inner-outer factorizations with non-square inner factors [5, 28]. The CTARE appears in solving various H_∞ synthesis problems and in computing spectral and J-spectral factorizations.

For the solution of CTARE and CCTARE consider the *extended Hamiltonian pencil* (EHP)

$$M - \lambda N := \begin{bmatrix} B & A - \lambda I & O \\ -L & -Q & -A^T - \lambda I \\ R & L^T & B^T \end{bmatrix} \tag{30}$$

and let $V \in \mathbb{R}^{(2n+m)\times p}$ a basis matrix for the maximal proper stable deflating subspace of the pencil (30) partitioned compatibly

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{31}$$

The following theorems [10] establish necessary and sufficient conditions for the existence of stabilizing solutions of CTARE and CCTARE.

Theorem 2 *The CTARE has a stabilizing solution X if and only if the EHP (30) is regular and has a maximal n dimensional stable proper deflating subspace to the right $\mathcal{V} = \operatorname{Im} V$ with V_2 in (31) invertible. Moreover, if such solution exists it is unique and $X = V_3 V_2^{-1}$ and $F = V_1 V_2^{-1}$ makes $A + BF$ stable.*

Theorem 3 *The CCTARE has a stabilizing solution X if and only if the EHP (30) has a maximal n dimensional stable proper deflating subspace to the right $\mathcal{V} = \operatorname{Im} V$ with V_2 in (31) invertible. Moreover, if such solution exists it is unique and $X = V_3 V_2^{-1}$ and $F = V_1 V_2^{-1}$ makes $A + BF$ stable.*

The EHP (30) has the form of a system matrix of a $2n$ standard system. Thus the computation of the solution of either CTARE or CCTARE can be accomplished by computing the maximal proper stable deflatiug subspace of this pencil using the method of previous subsection and verifying if this subspace has dimension n with $\boldsymbol{V}_2$ invertible.

4.3 Solving Nonstandard Discrete-Time Riccati Equations

We discuss in this subsection the computation of the symmetric solution $\boldsymbol{X}$ of the following nonstandard so-called *constrained discrete-time algebraic Riccati equation* (CDTARE)

$$\begin{array}{r} \boldsymbol{A}^T\boldsymbol{X}\boldsymbol{A} - \boldsymbol{X} - (\boldsymbol{A}^T\boldsymbol{X}\boldsymbol{B} + \boldsymbol{L})(\boldsymbol{R} + \boldsymbol{B}^T\boldsymbol{X}\boldsymbol{B})^{+}(\boldsymbol{B}^T\boldsymbol{X}\boldsymbol{A} + \boldsymbol{L}^T) + \boldsymbol{Q} = \boldsymbol{O} \\ \ker(\boldsymbol{R} + \boldsymbol{B}^T\boldsymbol{X}\boldsymbol{B}) \subset \ker(\boldsymbol{A}^T\boldsymbol{X}\boldsymbol{B} + \boldsymbol{L}) \end{array} \qquad (32)$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{L}$ are as in the previous subsection. A solution to CDTARE is called *stabilizing* if the pair $(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F}_0, \boldsymbol{B}_0)$ is stabilizable, where $\boldsymbol{F}_0 := -(\boldsymbol{R}+\boldsymbol{B}^T\boldsymbol{X}\boldsymbol{B})^{+}(\boldsymbol{B}^T\boldsymbol{X}\boldsymbol{A}+\boldsymbol{L}^T)$ and where $\boldsymbol{B}_0$ is any basis matrix for $\operatorname{Im}\boldsymbol{B} \cap \ker(\boldsymbol{R} + \boldsymbol{B}^T\boldsymbol{X}\boldsymbol{B})$.

If $\boldsymbol{R} + \boldsymbol{B}^T\boldsymbol{X}\boldsymbol{B}$ is invertible, then the second equation in (32) is trivially satisfied and the first equation becomes

$$\boldsymbol{A}^T\boldsymbol{X}\boldsymbol{A} - \boldsymbol{X} - (\boldsymbol{A}^T\boldsymbol{X}\boldsymbol{B} + \boldsymbol{L})(\boldsymbol{R} + \boldsymbol{B}^T\boldsymbol{X}\boldsymbol{B})^{-1}(\boldsymbol{B}^T\boldsymbol{X}\boldsymbol{A} + \boldsymbol{L}^T) + \boldsymbol{Q} = \boldsymbol{O}. \qquad (33)$$

This equation is called the *the discrete-time algebraic Riccati equation* (DTARE).

For the solution of DTARE and CDTARE consider the *extended simplectic pencil* (ESP)

$$\boldsymbol{M} - \lambda\boldsymbol{N} := \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{L} & -\boldsymbol{Q} & -\boldsymbol{I} \\ \boldsymbol{R} & \boldsymbol{L}^T & \boldsymbol{O} \end{bmatrix} - \lambda \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & -\boldsymbol{A}^T \\ \boldsymbol{O} & \boldsymbol{O} & -\boldsymbol{B}^T \end{bmatrix} \qquad (34)$$

and let $\boldsymbol{V} \in \mathbb{R}^{(2n+m)\times p}$ a basis matrix for the maximal proper stable deflating subspace of the pencil (34) partitioned compatibly as in (31). The following theorems [11] establish necessary and sufficient conditions for the existence of stabilizing solutions of DTARE and CDTARE.

Theorem 4 *The DTARE has a stabilizing solution $\boldsymbol{X}$ if and only if the ESP (34) is regular and has a maximal n dimensional stable proper deflating subspace to the right $\mathcal{V} = \operatorname{Im}\boldsymbol{V}$ with $\boldsymbol{V}_2$ in (31) invertible. Moreover, if such solution exists it is unique and $\boldsymbol{X} = \boldsymbol{V}_3\boldsymbol{V}_2^{-1}$ and $\boldsymbol{F} = \boldsymbol{V}_1\boldsymbol{V}_2^{-1}$ makes $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$ stable.*

Theorem 5 *The CDTARE has a stabilizing solution $\boldsymbol{X}$ if and only if the ESP (34) has a maximal n dimensional stable proper deflating subspace to the right $\mathcal{V} = \operatorname{Im}\boldsymbol{V}$ with $\boldsymbol{V}_2$ in (31) invertible. Moreover, if such solution exists it is unique and $\boldsymbol{X} = \boldsymbol{V}_3\boldsymbol{V}_2^{-1}$ and $\boldsymbol{F} = \boldsymbol{V}_1\boldsymbol{V}_2^{-1}$ makes $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$ stable.*

The ESP (34) has no longer the form of a standard system matrix as in the continuous-time case. Nevertheless, we can assimilate this pencil with a particular system pencil of a system without inputs and without outputs. We can compute the solution of either DTARE or CDTARE by computing the maximal proper stable deflating subspace of this particular system pencil using the method of subsection 4.1 and verifying if this subspace has dimension n with $\boldsymbol{V}_2$ invertible. Other non-standard discrete-time Riccati equations are discussed in [9].

5 Other Condensed Forms

We present in this section several condensed form which are computed during the reduction of the system pencil to the more complex forms already discussed. These forms are useful in determining alternative Kronecker-like forms or simply by using in revealing partial structural information on the system pencil.

5.1 The SRSET form

This form is the starting form to compute all Kronecker-like forms presented in this paper. By using suitable left and right transformation matrices $\boldsymbol{U}_1$ and $\boldsymbol{V}_1$ respectively, the system matrix $\mathcal{S}(\lambda)$ is reduced to the following equivalent form

$$\widehat{\mathcal{S}}(\lambda) = \boldsymbol{U}_1^T \mathcal{S}(\lambda) \boldsymbol{V}_1 = \left[\begin{array}{c|c} \widehat{\boldsymbol{B}} & \widehat{\boldsymbol{A}} - \lambda \widehat{\boldsymbol{E}} \\ \hline \widehat{\boldsymbol{D}} & \widehat{\boldsymbol{C}} \end{array}\right] \tag{35}$$

where $\widehat{\boldsymbol{E}}$ is upper-triangular non-singular matrix of order rank $(\boldsymbol{E})$. Notice that in the case of a standard system $\boldsymbol{U}$ and $\boldsymbol{V}$ can be chosen identity matrices.

5.2 The SRISEP form

The SRISEP form is useful for determining the left and infinity Kronecker structure of the system pencil. It is obtained by further reducing the SRSET form, by applying left and right orthogonal transformations, to the following particular structure

$$\boldsymbol{U}_2^T \widehat{\mathcal{S}}(\lambda) \boldsymbol{V}_2 = \left[\begin{array}{cc|c} \boldsymbol{B}_r & \boldsymbol{A}_r - \lambda \boldsymbol{E}_r & * \\ \boldsymbol{D}_r & \boldsymbol{C}_r & \\ \hline \multicolumn{2}{c|}{\boldsymbol{O}} & \boldsymbol{A}_l - \lambda \boldsymbol{E}_l \end{array}\right] \tag{36}$$

where

(a) $\boldsymbol{D}_r$ has full row rank p_r and is in an upper-trapezoidal form.

(b) $\boldsymbol{A}_r, \boldsymbol{E}_r \in \mathbb{R}^{n_r \times n_r}$ and $\boldsymbol{E}_r$ is upper-triangular and non-singular.

(c) $\boldsymbol{A}_l - \lambda \boldsymbol{E}_l$ has full column rank and contains the left and infinite Kronecker structure of $\mathcal{S}(\lambda)$; the pencil $\boldsymbol{A}_l - \lambda \boldsymbol{E}_l$ is in the staircase form (24) with the diagonal matrices $\boldsymbol{A}_{i,i}^l \in \mathbb{R}^{\tau_i \times \mu_i}$ having full column rank μ_i and the upper diagonal matrices $\boldsymbol{E}_{i+1,i}^l \in \mathbb{R}^{\mu_i \times \tau_{i+1}}$ having full row rank.

The index sets $\{\tau_i\}$ and $\{\mu_i\}$ determine the minimal row indices and the infinite structure of the system pencil $\mathcal{S}(\lambda)$ according to lemma 7.

Remark. For a standard system with $\boldsymbol{E} = \boldsymbol{I}_n$, the reduction to the SRISEP form can be accomplished such that the resulting $\boldsymbol{E}_r$ is an identity matrix.

5.3 The SLISEP form

The SLISEP form is useful for determining the right and infinity Kronecker structure of the system pencil. It is obtained by further reducing the SRSET form, by applying left and right orthogonal transformations, to the following particular structure

$$\boldsymbol{U}_3^T \widehat{\mathcal{S}}(\lambda) \boldsymbol{V}_3 = \left[\begin{array}{c|cc} \boldsymbol{A}_r - \lambda \boldsymbol{E}_r & \multicolumn{2}{c}{*} \\ \hline \boldsymbol{O} & \boldsymbol{B}_c & \boldsymbol{A}_c - \lambda \boldsymbol{E}_c \\ & \boldsymbol{D}_c & \boldsymbol{C}_c \end{array}\right] \tag{37}$$

where

(a) $\boldsymbol{A}_r - \lambda \boldsymbol{E}_r$ has full row rank and contains the right and infinite Kronecker structure of $\mathcal{S}(\lambda)$; the pencil $\boldsymbol{A}_r - \lambda \boldsymbol{E}_r$ is in the staircase form (22) with the diagonal matrices $\boldsymbol{A}_{i,i}^r \in \mathbb{R}^{\bar{\tau}_i \times \bar{\mu}_i}$ having full row rank $\bar{\tau}_i$ and the upper diagonal matrices $\boldsymbol{E}_{i,i+1}^r \in \mathbb{R}^{\bar{\tau}_i \times \bar{\mu}_{i+1}}$ having full column rank.

(b) $\boldsymbol{D}_c$ has full column rank m_r and is in an upper-trapezoidal form.

(c) $\boldsymbol{A}_r, \boldsymbol{E}_r \in \mathbb{R}^{n_r \times n_r}$ and $\boldsymbol{E}_r$ is upper-triangular and non-singular.

The index sets $\{\bar{\tau}_i\}$ and $\{\bar{\mu}_i\}$ determine the minimal row indices and the infinite structure of the system pencil $\mathcal{S}(\lambda)$ according to lemma 4.

Remark. For a standard system with $E = I_n$, the reduction to the SLISEP form can be accomplished such that the resulting E_c is an identity matrix.

5.4 The SRISEP-SLISEP and SLISEP-SRISEP forms

These forms are useful for determining the complete Kronecker structure of the system pencil. They can be obtained by calling successively the SRISEP-SLISEP or the SLISEP-SRISEP subroutines to reduce the SRSET form of the system pencil to the following particular structure

$$U_4^T \widehat{S}(\lambda) V_4 = \left[\begin{array}{c|cc|c} A_r - \lambda E_r & \multicolumn{2}{c|}{*} & * \\ \hline O & B_{rc} & A_{rc} - \lambda E_{rc} & * \\ & D_{rc} & C_{rc} & \\ \hline O & \multicolumn{2}{c|}{O} & A_l - \lambda E_l \end{array}\right] \qquad (38)$$

where

(a) $A_r - \lambda E_r$ has full row rank and contains the right Kronecker structure of $S(\lambda)$, and in the case of SLISEP-SRISEP form also the infinite Kronecker structure; the pencil $A_r - \lambda E_r$ is in the staircase form (22) with the diagonal matrices $A^r_{i,i} \in \mathbb{R}^{\bar{\tau}_i \times \bar{\mu}_i}$ having full row rank $\bar{\tau}_i$ and the upper diagonal matrices $E^r_{i,i+1} \in \mathbb{R}^{\bar{\tau}_i \times \bar{\mu}_{i+1}}$ having full column rank. In the case of SRISEP-SLISEP form $\bar{\mu}_i = \bar{\tau}_{i-1}$ for $i = 2, \ldots, k$ and $\bar{\tau}_k = 0$.

(b) $D_{rc} \in \mathbb{R}^{p_r \times p_r}$ is upper-triangular and non-singular.

(c) $A_{rc}, E_{rc} \in \mathbb{R}^{n_r \times n_r}$ and E_r is upper-triangular and non-singular.

(d) $A_l - \lambda E_l$ has full column rank and contains the left Kronecker structure of $S(\lambda)$, and in the case of SRISEP-SLISEP form also the infinite Kronecker structure; the pencil $A_l - \lambda E_l$ is in the staircase form (24) with the diagonal matrices $A^l_{i,i} \in \mathbb{R}^{\tau_i \times \mu_i}$ having full column rank μ_i and the upper diagonal matrices $E^l_{i+1,i} \in \mathbb{R}^{\mu_i \times \tau_{i+1}}$ having full row rank. In the case of SLISEP-SRISEP form, $\tau_i = \mu_{i-1}$ for $i = 2, \ldots, j$ and $\mu_j = 0$.

The index sets $\{\bar{\tau}_i\}$, $\{\bar{\mu}_i\}$, $\{\tau_i\}$ and $\{\mu_i\}$ determine the minimal row indices the system pencil $S(\lambda)$ according to lemmas 4 and 7. The finite eigenvalue structure of $S(\lambda)$ is determined by the eigenstructure of the matrix pair $(A_{rc} - B_{rc} D_{rc}^{-1} C_{rc}, E_{rc})$.

Remark. For a standard system with $E = I_n$, the reduction to the SRISEP-SLISEP or SLISEP-SRISEP forms can be accomplished such that the resulting E_{rc} is an identity matrix.

6 Algorithms

In this section we discuss the computational approaches implemented in the subroutines to compute the Kronecker-like forms introduced in previous sections. A common characteristics of all these procedures is that they consist of combinations of several highly specialized structure revealing subprocedures, as those to separate the left and infinity structures, the right and infinity structures, the controllable and uncontrollable or the observable and unobservable parts of particular system pencils. As an example, we present the main computational steps of the most complex procedure to compute the SPRED form. We also discuss the computational ingredients which ensure the $O(n^3)$ computational complexity of this procedure. The procedures to compute other Kronecker-like forms are either parts of the SPRED Procedure or rely on similar dual algorithms applied to implicitly pertransposed pencils (transposed with respect to the main antidiagonal). Notice however that all implemented procedures avoid explicit pertransposing.

SPRED **Procedure.**

1. Determine orthogonal U_1 and V_1 to compute a complete orthogonal decomposition of E in the form
$$U_1^T E V_1 = \begin{bmatrix} O & \widehat{E} \\ O & O \end{bmatrix} \tag{39}$$
where $\widehat{E}$ is upper-triangular and non-singular. Compute the SRSET form of $\mathcal{S}(\lambda)$ as
$$\mathcal{S}_1(\lambda) = \operatorname{diag}\{U_1^T, I_p\}\,\mathcal{S}(\lambda)\operatorname{diag}\{I_m, V_1\} := \left[\begin{array}{c|c} \widehat{B} & \widehat{A} - \lambda\widehat{E} \\ \hline \widehat{D} & \widehat{C} \end{array}\right], \tag{40}$$
and set $Q = \operatorname{diag}\{U_1, I_p\}$, $Z = \operatorname{diag}\{I_m, V_1\}$.

2. By using the dual S-REDUCE algorithm of [14], determine orthogonal U_2 and V_2 to reduce the system pencil $\mathcal{S}_1(\lambda)$ to the SLISEP form
$$\mathcal{S}_2(\lambda) = U_2^T \mathcal{S}_1(\lambda) V_2 = \left[\begin{array}{c|cc} A_1 - \lambda E_1 & \multicolumn{2}{c}{*} \\ \hline O & B_c & A_c - \lambda E_c \\ & D_c & C_c \end{array}\right] \tag{41}$$
where D_c is upper-trapezoidal and has full column rank, E_c is upper-triangular and non-singular, and $A_1 - \lambda E_1$ has full row rank. Compute $Q \leftarrow QU_2$, $Z \leftarrow ZV_2$.

3. By using the reduction technique of [2, pages 33-34], determine orthogonal U_3 to compress the rows of the matrix $\begin{bmatrix} B_c \\ D_c \end{bmatrix}$ such that
$$\mathcal{S}_3(\lambda) = \operatorname{diag}\{I, U_3^T\}\mathcal{S}_2(\lambda) = \begin{bmatrix} A_1 - \lambda E_1 & * & * \\ O & D_i & * \\ O & O & A_2 - \lambda E_2 \\ O & O & C_2 \end{bmatrix}, \tag{42}$$
where D_i is upper-triangular and non-singular, C_2 is the part of C_c corresponding to the linearly dependent rows of D_c, and E_2 is upper-triangular and non-singular. Compute $Q \leftarrow Q\operatorname{diag}\{I, U_3\}$.

4. By using the dual of the controllability staircase algorithm of [20], determine orthogonal U_4 and V_4 to reduce the sub-pencil $\begin{bmatrix} A_2 - \lambda E_2 \\ C_2 \end{bmatrix}$ to the observability staircase form
$$U_4^T \begin{bmatrix} A_2 - \lambda E_2 \\ C_2 \end{bmatrix} V_4 = \begin{bmatrix} A_f - \lambda E_f & * \\ O & A_l - \lambda E_l \\ O & C_l \end{bmatrix}, \tag{43}$$
where the pair $(C_l, A_l - \lambda E_l)$ is observable, and both E_f and E_l are upper-triangular and non-singular matrices. Compute
$$\mathcal{S}_4(\lambda) = \operatorname{diag}\{I, U_4^T\}\mathcal{S}_3(\lambda)\operatorname{diag}\{I, V_4\} \tag{44}$$
and $Q \leftarrow Q\operatorname{diag}\{I, U_4\}$, $Z \leftarrow Z\operatorname{diag}\{I, V_4\}$.

5. By using Algorithms 3.3.1 and 3.3.3 in [2], determine orthogonal U_5 and V_5 to reduce the full row rank sub-pencil $A_1 - \lambda E_1$ to the following form
$$U_5^T(A_1 - \lambda E_1)V_5 = \begin{bmatrix} B_r & A_r - \lambda E_r & * \\ O & O & A_\infty - \lambda E_\infty \end{bmatrix}, \tag{45}$$

where $A_\infty - \lambda E_\infty$ contains the infinity structure of the system pencil $\mathcal{S}(\lambda)$ and the pair $(B_r, A_r - \lambda E_r)$ is controllable with E_r upper-triangular and non-singular. Compute the final SPRED form

$$\widehat{\mathcal{S}}(\lambda) = \operatorname{diag}\{U_5^T, I\}\mathcal{S}_3(\lambda)\operatorname{diag}\{V_5, I\} \qquad (46)$$

and $Q \leftarrow Q\operatorname{diag}\{U_5, I\}$, $Z \leftarrow Z\operatorname{diag}\{V_5, I\}$.

The SRSET, SLISEP and SLRRED forms of the system pencil $\mathcal{S}(\lambda)$ are computed at intermediary steps 1, 2 and 4 as the pencils $\mathcal{S}_1(\lambda)$, $\mathcal{S}_2(\lambda)$, and $\mathcal{S}_4(\lambda)$, respectively. For computing the SRISEP and SRLRED forms the same procedure applied to the pertransposed system pencil $\mathcal{S}(\lambda)^P$ can be used. However the implemented algorithms to compute these forms avoid explicit pertransposing by working directly on the original matrices.

For computing the complete orthogonal decomposition at step 1, any rank revealing decomposition of E can be used. The most reliable approach is to compute the singular value decomposition of E and to determine the rank of E on the basis of computed singular values. A less expensive approach is to use the QR-decomposition with column pivoting of E. Excepting very special examples (for instance the so-called *Kahan*-matrices), this decomposition has almost the same reliability in determining the rank of a matrix as the singular value decomposition [17]. Thus we decided to use it in implementing the SRSET subroutine in combination with the incremental rank estimation technique proposed in [4]. Reliable software for both decompositions as well as auxiliary routines for the incremental rank estimation, are provided in LAPACK [1]. Notice that in contrast with alternative algorithms [2, 6, 18], a single rank determination is performed involving E. In all subsequent computations the preservation of the triangular form and of the full rank structure of intervening "E" matrices are crucial for performing the various pencil reductions and for ensuring the $0(n^3)$ computational complexity.

The reductions performed at steps 2 and 4 are based on a reduction technique similar to that introduced in [20] to compute controllability staircase forms of descriptor systems. This technique was used in conjunction with computing system zeros [21] and is described in detail in [14]. The rank determinations are based on QR-decompositions with column pivoting. The main feature of these algorithms is the preservation, during computations of QR-decompositions, of the full rank and of the upper-triangular form of the intervening "E" matrices. This feature leads to two important advantages over existing methods. The first advantage is the computational complexity $0(n^3)$. In contrast, the algorithms of [6, 18] have computational complexity $0(n^4)$, because singular value decompositions are used instead of QR-decompositions, and thus the explicit accumulation of left and right transformation matrices is necessary. Notice that the $0(n^4)$ computational complexity is a generic feature of these algorithms and always occurs for example for a randomly generated single-input single-output system. The second advantage arises in comparing the reduction algorithm S-REDUCE of [14] and the improved $0(n^3)$ complexity Algorithm 3.2.1 of [2]. The main weekness of this latter algorithm is the need to update during each QR-like reduction step the rank information on "E". This rank updating is in fact equivalent with rank decisions based on QR-decompositions without pivoting and thus it is potentially unreliable. In the S-REDUCE algorithm of [14], "E" having always full rank, no such updating is necessary. Instead, two QR-decompositions with column pivoting are necessary to be performed at each step.

The row compression of $\begin{bmatrix} B_c \\ D_c \end{bmatrix}$ at step 3 is performed in two steps. First the rows of D_c are compressed to an invertible matrix D_1 such that

$$W_1^T[\,D_c\,C_c\,] = \begin{bmatrix} D_1 & C_1 \\ O & C_2 \end{bmatrix}. \qquad (47)$$

Then the rows of $\begin{bmatrix} B_c \\ D_1 \end{bmatrix}$ are compressed with an appropriate orthogonal matrix W_2 such that

$$W_2^T \begin{bmatrix} B_c & A_c - \lambda E_c \\ D_1 & C_1 \end{bmatrix} = \begin{bmatrix} D_i & * \\ O & A_2 - \lambda E_2 \end{bmatrix}, \tag{48}$$

where the resulting E_2 is upper-triangular and non-singular. The compression method efficiently combines Given rotations and row permutations and is described in detail in [2, pages 33-34].

To perform the reductions at step 5 the S-REDUCE algorithm can be used after compressing E_1 to a full rank invertible matrix. A more efficient approach is to use the Algorithms 3.3.1 and 3.3.3 proposed in [2] which requires no rank determinations. In computing the SPRED form we implemented these two algorithms.

7 Software Outline

The following higher level user callable FORTRAN 77 subroutines have been implemented to compute the Kronecker-like forms described in the previous sections:

- SRSET - to compute the SRSET form of the system pencil $\mathcal{S}(\lambda)$ corresponding to a given 5-tuple (E, A, B, C, D). Particular system pencils corresponding to: D, (E, A), (E, A, B), (E, A, C), or $E = I$ can be also handled.
- SLISEP - to reduce a system pencil in SRSET form to the SLISEP form.
- SRISEP - to reduce a system pencil in SRSET form to the SRISEP form.
- SLRRED - to reduce a system pencil in SRSET form to the SLRRED form.
- SRLRED - to reduce a system pencil in SRSET form to the SRLRED form.
- SPRED - to reduce a general system pencil to the SPRED form.

Several lower level subroutines are called by the above subroutines:

- SCFRED - to reduce a subpencil $[\, B \;\; A - \lambda E \,]$ to the descriptor controlability staircase form
- SOFRED - to reduce a subpencil $\begin{bmatrix} A - \lambda E \\ C \end{bmatrix}$ to the descriptor observability staircase form
- SLIUTR - to reduce a full row rank subpencil $A - \lambda E$ to a standardized staircase form with all diagonal blocks in A and all upper diagonal blocks in E triangularized
- SLIRED - to a full row rank subpencil $A - \lambda E$ to a form with separated left and infinite Kronecker structure

All routines optionally accumulates the left and right orthogonal transformations made during the reductions.

A prerequisite for the reliable usage of all reduction routines is the assumption of a certain uniformity in the ranges of elements of the system matrices. This assumption is necessary because all rank decissions are taken using only two tolerance parameters, which serve in detecting negligible elements in the matrices $M = \begin{bmatrix} B & A \\ D & C \end{bmatrix}$ and E. If the ranges of nonzero elements in M or in E is too different, usually certain balancing of system data is necessary before calling the above routines in order to obtain meaningful results. A preconditioning algorithm of system data for computing the zeros of standard systems has been recently proposed in [8]. An extension of this algorithm is seemingly straightforward for the case of reducing the system pencils of descriptor systems.

The implementations of all routines rely on LAPACK [1] and BLAS calls. The user interface conforms with the implementation standards of the SLICOT library [29]. All routines are extensively commented and in line comments can be used for documentation purposes. As an example, the complete listing of the source code of the SPRED routines is presented in Appendix A. The inline comments for all implemented routines are listed in Appendices D-L.

Test programs with files containing test data and test results are available for all user callable routines. An example of a test program for the subroutine `SPRED` is presented in Appendix B and the obtained results are listed in Appendix C. Special routines to evaluate the incurred rounding errors are called by all test programs.

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