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On Periodic Linear Matrix Equations: Applications, New Algorithms and Software

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Consider the linear discrete-time periodic system of the form

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned} \tag{1}$$

where the matrices $A_k \in R^{n\times n}$, $B_k \in R^{n\times m}$, $C_k \in R^{p\times n}$ and $D_k \in R^{p\times m}$ are periodic with period $K \geq 1$. In the last few years there has been a constantly increasing interest for the development of numerical algorithms for the analysis and design of linear periodic control systems [2, 8, 6]. Of particular interest in many applications is the efficient and numerically reliable solution of various types of periodic Lyapunov and Sylvester matrix equations. Several applications are mentioned in [8, 9].

The reachability gramian of an exponentially stable time-varying system of the form (1) is defined as $P_k = \sum_{i=-\infty}^{k-1} \Phi(k,i+1)B_iB_i^T\Phi^T(k,i+1)$, where $\Phi(j,i) := A_{j-1}A_{j-2}\cdots A_i$ for $j > i$ and $\Phi(i,i) := I$ is the so-called *monodromy matrix* of (1). It can be shown that for the periodic system (1) P_k satisfies the forward-time periodic Lyapunov equation

$$P_{k+1} = A_kP_kA_k^T + B_kB_k^T, \quad k = 1,\ldots,K; \quad P_{K+1} = P_1 \tag{2}$$

and the system (1) is uniformly controllable iff $P_k > 0$ for $k = 1,\ldots,K$ [4]. Similarly, the observability gramian defined as $Q_k = \sum_{i=k}^{\infty} \Phi(i,k)^T C_i^T C_i \Phi(i,k)$, satisfies the reverse-time periodic Lyapunov equation

$$Q_k = A_k^TQ_{k+1}A_k + C_k^TC_k \quad k = 1,\ldots,K; \quad Q_{K+1} = Q_1 \tag{3}$$

and the system (1) is uniformly observable iff $Q_k > 0$ for $k = 1,\ldots,K$ [4]. A particular feature of equations (2) and (3) for exponentially stable periodic systems is that the non-negative definite gramians can be determined directly in terms of their Cholesky factors, i.e. $P_k = S_k^TS_k$ and $Q_k = R_k^TR_k$. These factors are useful for instance in determining the Hankel-singular values and the Hankel-norm of the given periodic system [4]. They can be further used to compute balancing Lyapunov transformation matrices [9]. Forward- and reverse-time Lyapunov equations are also encountered in solving periodic Riccati equations by a Newton-type iterative technique [2], in stabilizing periodic systems or in evaluating gradients for optimal parametric output feedback control of periodic systems [10]. A closely related computational problem is the solution of various types of forward- or reverse-time Sylvester equations as for instance the forward-time Sylvester equation

$$F_k P_{k+1} + P_k G_k = H_k \quad k = 1, \dots, K; \quad P_{K+1} = P_1 \tag{4}$$

which arises in computing additive spectral decompositions of linear periodic systems by simultaneously block-diagonalizing all component matrices of the monodromy matrix [9]. A similar equation also appears in the periodic observer design [8].

One class of existing numerical methods to solve periodic Lyapunov and Sylvester equations [2, 8] is based on reducing these problems to a single Lyapunov or Sylvester equation to compute a periodic generator, say P_1. The rest of solution is computed by forward or reverse recursion. The main drawback of such methods is the need to form explicitly matrix products and sums of matrix products. An alternative approach discussed also in [8] is to solve the periodic Lyapunov equations as particular periodic Riccati equations. In this approach the construction of products is avoided but the method has a substantially increased computational complexity, much greater than usually necessary to solve such a problem.

In this paper we propose for the numerical solution of periodic Lyapunov and Sylvester equations a set of new methods, which essentially parallel the methods available for standard systems [1, 7, 5]. The key role in all these methods plays the recent discovery of the so-called *periodic Schur decomposition* (PSD) of a cyclic matrix product and of the corresponding algorithms for its computation [3, 6]. To illustrate the proposed solution techniques we discuss the main aspects of solving the forward-time periodic Lyapunov equation

$$P_{k+1} = A_k P_k A_k^T + M_k, \quad k = 1, \dots, K; \quad P_{K+1} = P_1 \tag{5}$$

By using the PSD algorithm, we can determine the orthogonal matrices Z_k, $k = 1, \dots, K$ to reduce the matrix product $A = A_1 A_2 \cdots A_K$ to the periodic Schur form $\widetilde{A} = \widetilde{A}_1 \widetilde{A}_2 \cdots \widetilde{A}_K$, where $\widetilde{A}_k = Z_k^T A_k Z_{k+1}$, $\widetilde{A}_1$ is in *real Schur form* (RSF) and $\widetilde{A}_k$ for $k > 1$ is upper triangular. Note that by definition $Z_{K+1} = Z_1$. By premultiplying the k-th equation in (5) with Z_k^T from left and with Z_k from right, one obtains

$$\widetilde{P}_{k+1} = \widetilde{A}_k \widetilde{P}_k \widetilde{A}_k^T + \widetilde{M}_k, \quad k = 1, \dots, K; \quad \widetilde{P}_{K+1} = \widetilde{P}_1 \tag{6}$$

where $\widetilde{P}_k = Z_{k-1}^T P_k Z_{k-1}$ and $\widetilde{M}_k = Z_k^T M_k Z_k$. Notice that by this transformation the resulted transformed equations (6) have exactly the same form as the original ones in (5). After solving the transformed equations for the matrices $\widetilde{P}_k$, the solution of (5) results as $P_k = Z_{k-1} \widetilde{P}_k Z_{k-1}^T$ for $k = 1, \dots, K$.

Thus, by using the PSD we reduced the original problem to an equivalent one with all coefficient matrices in upper triangular form excepting $\widetilde{A}_1$ which is in a RSF. Notice that for the computation of controllability and observability gramians satisfying the forward- and reverse-time Lyapunov equations (2) and (3), respectively, the computation of a single PSD is sufficient. To simplify the notations, in what follows we assume that the coefficient matrices of the original equations (5) are already in the reduced forms corresponding to the PSD.

Let us partition the matrix A_k and the symmetric matrices P_k and M_k according to the RSF structure of the matrix A_1

$$A_k = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} & \cdots & A_{1\bar{n}}^{(k)} \\ 0 & A_{22}^{(k)} & \cdots & A_{2\bar{n}}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\bar{n}\bar{n}}^{(k)} \end{bmatrix}, \; P_k = \begin{bmatrix} P_{11}^{(k)} & P_{12}^{(k)} & \cdots & P_{1\bar{n}}^{(k)} \\ P_{21}^{(k)} & P_{22}^{(k)} & \cdots & P_{2\bar{n}}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{\bar{n}1}^{(k)} & P_{\bar{n}2}^{(k)} & \cdots & P_{\bar{n}\bar{n}}^{(k)} \end{bmatrix}, \; M_k = \begin{bmatrix} M_{11}^{(k)} & M_{12}^{(k)} & \cdots & M_{1\bar{n}}^{(k)} \\ M_{21}^{(k)} & M_{22}^{(k)} & \cdots & M_{2\bar{n}}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ M_{\bar{n}1}^{(k)} & M_{\bar{n}2}^{(k)} & \cdots & M_{\bar{n}\bar{n}}^{(k)} \end{bmatrix}$$

The (r,l)-th blocks $P_{rl}^{(k)}$ of P_k, $k = 1,\ldots,K$ satisfy the simultaneous equations

$$A_{rr}^{(k)} P_{rl}^{(k)} A_{ll}^{(k)T} - P_{rl}^{(k+1)} = -M_{rl}^{(k)} - Y_{rl}^{(k)}, \quad k = 1,\ldots,K; \quad P_{rl}^{(K+1)} = P_{rl}^{(1)},$$

where

$$Y_{rl}^{(k)} = \sum_{i=r}^{\bar{n}} A_{ri}^{(k)} \left(\sum_{j=l+1}^{\bar{n}} P_{ij}^{(k)} A_{lj}^{(k)T} \right) + \left(\sum_{j=r+1}^{\bar{n}} A_{rj}^{(k)} P_{jl}^{(k)} \right) A_{ll}^{(k)T}$$

By starting from the bottom-right corner, we can compute P_k, $k = 1,\ldots,K$ column by column by solving repeatedly periodic discrete Sylvester equations of the form

$$E^{(k)} X^{(k)} F^{(k)T} - X^{(k+1)} = G^{(k)}, \quad k = 1,\ldots,K; \quad X^{(K+1)} = X^{(1)}$$

where $E^{(k)} \in \mathrm{R}^{n_1 \times n_1}$, $F^{(k)} \in \mathrm{R}^{n_2 \times n_2}$ and $G^{(k)} \in \mathrm{R}^{n_1 \times n_2}$ with n_1 and n_2 at most 2.

Several methods to solve the above low order equations can be devised. By rewriting these equations with the help of Kronecker products one obtains a set of $n_1 n_2 K$ simultaneous linear equations. Even for moderate values of K, say $K = 20$, this technique leads to rather expensive computations because for each second order block we have to solve a system of 80 linear equations. A solution method based on a specialized block LU decomposition algorithm which fully exploits the sparse cyclic structure of the coefficient matrix is described in [9]. Alternatively the method of periodic generators [2, 8] can be employed to solve the low order equations, but the construction of products can lead to severe accuracy losses. Thus this approach is effective only when used in conjunction with iterative refinement techniques. Such an approach is particularly helpful in solving non-negative definite periodic Lyapunov equations [9]. The two techniques can be combined in order to exploit the advantages of both methods.

The proposed technique can be readily adapted to solve any other type of Lyapunov equations, as for instance the reverse-time equation (3) or the transposed variants of equations (2) and (3). Similarly, the method can be employed to solve the non-negative definite periodic Lyapunov equations as well the periodic Sylvester equations. For details see [9].

A set of LAPACK based computational routines have been implemented to compute the PSD and to solve four types of periodic Lyapunov equations. The implemented software is available on request from the author for testing purposes. It was successfully used to evaluate gradients of linear-quadratic functionals for optimal periodic output feedback control [10].

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