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On Computing Generalized Inverses of Rational Matrices with Applications

A numerically reliable computational approach based on descriptor systems techniques is proposed for the computation of the week generalized inverses of rational matrices. The main applications of the proposed method are in solving model matching problems and in computing the inner-outer or J-inner-outer factorizations of rational matrices.

1. Introduction

The computation of generalized inverses of rational matrices is strongly related to the inversion of linear time-invariant systems and has several important applications in areas such as control theory, filtering and coding theory. The recent interest to develop numerically reliable methods for the computation of generalized inverses is motivated by the need to compute such inverses in some new algorithms for computation of inner-outer or J-inner-outer factorizations of rational matrices [3], [5].

Consider a $p \times m$ rational matrix $G(\lambda)$ with normal rank k (rank over rationals). Let (E, A, B, C, D) be an equivalent n-th order regular descriptor representation of G such that $G(\lambda) = C(\lambda E - A)^{-1}B + D$. We assume that the descriptor representation of G is minimal, thus n is the least integer for which the above relation holds. Because of minimality, the poles of $G(\lambda)$ are the generalized eigenvalues (finite and infinite) of the pair (A, E) and the zeros of G are the those complex frequencies (counting multiplicities) where the system matrix

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]$$

looses its maximal rank $n + k$. The *week* or (1,2)-generalized inverse of G we denote with G^+ and satisfies the following two conditions [1]: (1) $GG^+G = G$, and (2) $G^+GG^+ = G^+$. Note that the (1,2)-generalized inverse is generally not unique. The computational method which we propose is based on the following result:

Lemma 1. *If $\mathcal{S}(\lambda)^+$ is an (1,2)-generalized inverse of $\mathcal{S}(\lambda)$ then*

$$G^+(\lambda) = \left[\begin{array}{cc} 0 & I_m \end{array}\right] \mathcal{S}(\lambda)^+ \left[\begin{array}{c} 0 \\ I_p \end{array}\right] \tag{1}$$

is an (1,2)-generalized inverse of $G(\lambda)$.

Proof. The expression of G^+ follows easily from the following straightforward formula

$$\left[\begin{array}{cc} A - \lambda E & 0 \\ 0 & G(\lambda) \end{array}\right] = \left[\begin{array}{cc} I_n & 0 \\ -C(A - \lambda E)^{-1} & I_p \end{array}\right] \mathcal{S}(\lambda) \left[\begin{array}{cc} I_n & -(A - \lambda E)^{-1}B \\ 0 & I_m \end{array}\right]$$

Provided $\mathcal{S}(\lambda)^+$ is an (1,2)-generalized inverse, the above expression can be also used to check that G^+ in (1) is also an (1,2)-generalized inverse of G. It can be also proven that the expression (1) can be used to generate any (1,2)-generalized inverse of G by appropriately choosing $\mathcal{S}(\lambda)^+$.

2. Computational Approach and Applications

By using the expression (1) the computation of the (1,2)-generalized inverse of any rational matrix can be accomplished by computing the (1,2)-generalized inverse of the associated system pencil. The computation of $\mathcal{S}(\lambda)^+$ will be done by reducing $\mathcal{S}(\lambda)^+$ to an appropriate Kronecker-like form from which a maximal rank regular sub-pencil can be easily separated. Then the generalized inverse is determined by using simple formulas as given in [1]. An important aspect from computational point of view of the proposed approach is that a descriptor representation of G^+ is determined without explicitly inverting $\mathcal{S}(\lambda)$. The computational procedure has the following three main steps:

1. Choose orthogonal Q and Z to reduce $\mathcal{S}(\lambda)$ to the Kronecker-like form

$$\bar{\mathcal{S}}(\lambda) := Q\mathcal{S}(\lambda)Z = \left[\begin{array}{c|cccc} B_r & A_r - \lambda E_r & * & * \\ 0 & 0 & A_{reg} - \lambda E_{reg} & * \\ 0 & 0 & 0 & A_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array}\right]$$

Note. In the reduced pencil $\bar{\mathcal{S}}(\lambda)$, the regular part $A_{reg} - \lambda E_{reg}$ contains the finite and infinite system zeros,, the pair $(A_r - \lambda E_r, B_r)$ is controllable with E_r nonsingular and the pair $(C_l, A_l - \lambda E_l)$ is observable with E_l nonsingular.

2. Choose F such that $\Lambda(A_r + B_rF, E_r) \subset \mathbb{C}_g$ and choose K such that $\Lambda(A_l + KC_l, E_l) \subset \mathbb{C}_g$. With

$$U = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & K \\ 0 & 0 & 0 & I \end{bmatrix}, \quad V = \begin{bmatrix} I & F & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

compute $\widetilde{\mathcal{S}}(\lambda) = U\bar{\mathcal{S}}(\lambda)V$ as

$$\widetilde{\mathcal{S}}(\lambda) = \left[\begin{array}{c|cccc} B_r & A_r + B_rF - \lambda E_r & * & * \\ 0 & 0 & A_{reg} - \lambda E_{reg} & * \\ 0 & 0 & 0 & A_l + KC_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array}\right] := \left[\begin{array}{c|c} \widetilde{\mathcal{S}}_{11}(\lambda) & \widetilde{\mathcal{S}}_{12}(\lambda) \\ \hline 0 & \widetilde{\mathcal{S}}_{22}(\lambda) \end{array}\right]$$

Note. $\operatorname{rank} \mathcal{S}(\lambda) = \operatorname{rank} \widetilde{\mathcal{S}}_{12}(\lambda)$.

3. Compute $G(\lambda)^+ := (\widetilde{E}_{12}, \widetilde{A}_{12}, \widetilde{B}_1, \widetilde{C}_2, 0)$, where

$$\widetilde{A}_{12} - \lambda\widetilde{E}_{12} = \widetilde{\mathcal{S}}_{12}(\lambda), \quad \widetilde{B} = UQ\begin{bmatrix} 0 \\ I_p \end{bmatrix} = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{bmatrix}, \quad \widetilde{C} = [\, 0 \ \ -I_m \,]ZV = [\, \widetilde{C}_1 \ \widetilde{C}_2 \,]$$

The computations at step 1 can be done by using a structure preserving numerically stable reduction algorithm similar to that proposed in [2]. At step 2 the eigenvalue placement problems can be solved by using stabilization or pole assignment techniques for descriptor systems as those proposed in [4]. This step is necessary only if the corresponding eigenvalues must lie in a specific region $\mathbb{C}_g$ of the complex plane, as for instance in the stability domain in [3], [5]. The proposed method has overall computational complexity $0(n^3)$ and provides flexibility to cope with various conditions on the poles of the computed generalized inverse. For instance, a stable generalized inverse can be computed whenever exists.

Two important applications of the proposed method are: (1) *the solution of model matching problem:* compute stable X (if exists) such that

$$T_1 = T_2XT_3$$

where T_1 is stable, and T_2 and T_3 are minimum phase; and (2) *the computation of inner-outer and J-inner-outer factorizations* (for details see [3], [5]). Interesting problems for further research are the computation of minimum order (stable) inverses.

3. References

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