ON COMPUTING INNER-OUTER FACTORIZATIONS OF RATIONAL MATRICES

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Abstract

In this paper we propose a new numerically reliable computational approach to determine the inner-outer factorization of a rational matrix G. The proposed approach is completely general being applicable whenever G is proper or not, or of full column/row rank or not. In contrast to existing "one-shot" methods which require the solution of Riccati or generalized Riccati equations, the new approach is recursive and avoids such computationally involved steps by using instead a recursive zeros dislocation state-space approach. The resulting factors have always minimal order descriptor representations.

1 Introduction

Let $G(\lambda)$ be a given $p \times m$ stable rational *transfer-function matrix* (TFM) of a linear time-invariant continuous-time or discrete-time descriptor system and let $G = (E, A, B, C, D)$ denote an equivalent nth order *regular* ($\det(\lambda E - A) \not\equiv 0$) irreducible descriptor representation satisfying $G(\lambda) = C(\lambda E - A)^{-1}B + D$, where λ is either s or z, depending on the type of the system. We assume that G has no zeros on the imaginary axis for a continuous-time system or on the unit circle circle for a discrete-time system. In this paper we address the problem to compute an *inner-outer factorization* of G, namely $G = G_i G_o$, where G_i is a *square* inner factor and G_o is an outer factor. Recall that G_i is *inner* means G_i is stable and $G_i^* G_i = I$, where $G_i^*(s) = G_i^T(-s)$ in continuous-time and $G_i^*(z) = G_i^T(1/z)$ in discrete-time. In this paper, G_o is *outer* means that it has only stable poles and zeros. This definition of outer matrices extends, by a slight abuse of language, the standard definition [6] (applicable to full row rank matrices), to rational matrices of arbitrary rank.

Two classes of methods have been proposed to compute inner-outer factorizations for several particular cases. The *first* class of methods relies on computing stabilizing solutions of certain Riccati or generalized Riccati equations. For *continuous-time systems*, the proper case with full rank $G(s)$ on the extended imaginary axis can be solved by employing the algorithms proposed by Tsai and Chen [14]. Here a particular Riccati equation is solved in order to compute the inner denominator of a right coprime factorization of a left or right inverse of G. The algorithm for surjective G represents a significant improvement over an earlier algorithm proposed by Chen and Francis [3] which led to a constrained Riccati equation. In contrast with the approach in [14], here the outer factor G_o is determined first by solving a spectral factorization problem. A similar approach for solving the more general factorization problem for a constant rank proper G (possibly of not maximal rank) was proposed by Weiss [23] and is based on linear pencil techniques. The strictly proper case for square systems has been considered by Hara and Sugie [8] by using a descriptor system formulation which involves the solution of a descriptor Riccati equation. The case of injective strictly proper G can be solved by the infinite zeros cancellation technique proposed by Copeland and Safonov [4]. For *discrete-time systems* Gu *et al.* [7] derived explicit formulas for the proper case with $G(z)$ injective on the unit circle. Their method, based on computing the outer factor by solving a discrete spectral factorization problem, is restricted to the case of invertible A as well as on other simplifying assumptions. Katayama [11] extended the results of [7] by using a descriptor system formulation, removing all the above constraining assumptions. Recently a method which parallels that of [23], has been proposed for constant rank proper discrete-time systems by Ionescu and Oara [9]. In both continuous- and discrete-time cases no methods are available for strictly proper non full rank or improper matrices. For all methods in this class the computational burden caused by the need to solve Ric-

cati equations represents the principal disadvantage from numerical point of view. Moreover, in all above methods, the resulting inner factors are usually non-minimal.

The *second* class of methods avoids the solution of Riccati equations and is based on recursive dislocation of unstable zeros of G by premultiplying it with suitable elementary all-pass factors. The general technique of zeros dislocation is described by Van Dooren [16]. Apparently the first algorithm to compute recursively the inner-outer factorization of a square invertible continuous-time system has been proposed by Kimura [12]. His method relies on using recursively a pole conjugation technique to reflect the unstable poles of the inverse system to symmetric positions with respect to the imaginary axis. Algorithms based on a similar idea, but without explicitly forming the inverse system, have been proposed for both continuous-time and discrete-time systems by Zhang and Freudenberg [24, 25]. These algorithms compute elementary inner factors which are used to reflect the unstable zeros of G to stable positions symmetrical with respect to the imaginary axis in continuous-time case or the unit circle in discrete-time case. The computation of the elementary first or second order inner factors which reflect one or two unstable zeros at each step is based on computing state and input zero directions of the system matrix pencil. The approach can be straightforwardly extended to handle descriptor representations too. The need to compute zero directions of the system matrix pencil at each step makes however these algorithms of complexity $0(n^4)$ in the worst case of n unstable zeros.

In this paper we propose a new numerically reliable computational approach to compute the inner-outer factorization of G. The proposed approach can be viewed as a recursive zeros dislocation technique. Generalized Lyapunov equations of order at most two are solved repeatedly to compute suitable elementary inner factors to reflect the unstable zeros into the stable region of the complex plane. The proposed approach is completely general being applicable whenever G is proper or not, or of full column/row rank or not. The method can be even employed when G has zeros on the extended imaginary axis in continuous-time case or on the unit circle in the discrete-time case. The resulting inner and outer factors have always minimal order descriptor representations.

The procedure to compute inner-outer factorizations is conceptually similar to that of [14] and has the following main steps:

1. Compute a generalized inverse G^+ of G such that the unstable poles of G^+ are *exactly* the unstable zeros of G.
2. Compute a *right coprime factorization with inner denominator* (RCFID) of G^+ as $G^+ = NG_i^{-1}$, where N and G_i are stable TFMs with G_i inner and of *least* order.
3. Compute $G_o = G_i^{-1}G$.

If the generalized inverse G^+ computed at step 1 is an (1,2)-generalized inverse satisfying $GG^+G = G$ and $G^+GG^+ = G^+$ [2], then it is easy to see that the computed G_o is outer. This follows by observing that N is a stable (1,2)-generalized inverse of G_o, satisfying $G_oNG_o = G_o$ and $NG_oN = N$. The requirements at steps 1 and 2 that G^+ has the only unstable poles, the unstable zeros of G, and that G_i has *least* order, ensure that the order of the resulting inner factor G_i equals the number of unstable zeros of G. The requirement for the least order of G_i is also a necessary condition to compute the corresponding G_o because the need for unstable poles-zeros cancellation in computing $G_i^{-1}G$ at step 3.

In the rest of the paper we discuss in detail the main computational problems appearing in the proposed computational approach.

2 Computation of generalized inverses

In this section we discuss a numerically reliable approach to compute a descriptor representation of a particular (1,2)-generalized inverse G^+ fulfilling the requirement to have as unstable poles the unstable zeros of G. It is known that for a non-square full rank TFM G the zeros of G (finite and infinite) are always poles of any left or right inverse G^+ [10, page 467]. However besides these fixed poles G^+ generally possesses also spurious poles resulting from the particular way the inverse was determined. In order to simplify the procedure at step 2, it is convenient to devise an inversion procedure by which these spurious poles result always stable. This allows an easy computation of a least order inner denominator at step 2.

Consider the *system matrix* pencil

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right]. \tag{1}$$

associated to the descriptor representation $G = (E, A, B, C, D)$. The computation of an (1,2)-generalized inverse of G relies on the following straightforward formula [21]

$$G^+(\lambda) = \left[\begin{array}{cc} 0 & I_m \end{array}\right] \mathcal{S}(\lambda)^+ \left[\begin{array}{c} 0 \\ I_p \end{array}\right], \tag{2}$$

where $\mathcal{S}(\lambda)^+$ is an (1,2)-generalized inverse of $\mathcal{S}(\lambda)$. It can be readily verified that if $\mathcal{S}(\lambda)^+$ is an (1,2)-generalized inverse then the corresponding $G^+(\lambda)$ in (2) is also an (1,2)-generalized inverse [21].

With the help of two orthogonal matrices Q and Z it is possible to reduce the system pencil (1) to the following Kronecker-like form which exhibits the complete Kronecker structure of $\mathcal{S}(\lambda)$:

$$\bar{\mathcal{S}}(\lambda) := Q\mathcal{S}(\lambda)Z = \left[\begin{array}{c|ccc} B_r & A_r-\lambda E_r & * & * \\ 0 & 0 & A_{reg}-\lambda E_{reg} & * \\ 0 & 0 & 0 & A_l-\lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array}\right], \tag{3}$$

where: 1) the pencil $[\ B_r\ \ A_r - \lambda E_r\ ]$, with E_r non-singular, contains the right Kronecker structure of $\mathcal{S}(\lambda)$ and the pair $(B_r, A_r - \lambda E_r)$ is controllable; 2) the *regular* pencil $A_{reg} - \lambda E_{reg}$ has the form

$$A_{reg} - \lambda E_{reg} = \begin{bmatrix} A_\infty - \lambda E_\infty & * \\ 0 & A_f - \lambda E_f \end{bmatrix}, \qquad (4)$$

where the pencil $A_\infty - \lambda E_\infty$, with A_∞ non-singular and E_∞ nilpotent, contains the infinity Kronecker structure of $\mathcal{S}(\lambda)$, while the pencil $A_f - \lambda E_f$, with E_f non-singular, contains the finite Kronecker structure of $\mathcal{S}(\lambda)$; the generalized eigenvalues of the pair (A_f, E_f) are the *finite zeros* of the G; 4) the pencil $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$, with E_l non-singular, contains the left Kronecker structure of $\mathcal{S}(\lambda)$ and the pair $(C_l, A_l - \lambda E_l)$ is observable.

By using additional left and right transformation matrices U and V of the special forms

$$U = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & K \\ 0 & 0 & 0 & I \end{bmatrix}, \quad V = \begin{bmatrix} I & F & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \qquad (5)$$

we obtain

$$\widetilde{\mathcal{S}}(\lambda) = \left[\begin{array}{c|c} \widetilde{\mathcal{S}}_{11}(\lambda) & \widetilde{\mathcal{S}}_{12}(\lambda) \\ \hline 0 & \widetilde{\mathcal{S}}_{22}(\lambda) \end{array}\right] := U\mathcal{S}(\lambda)V =$$

$$\left[\begin{array}{c|ccc} B_r & A_r + B_rF - \lambda E_r & * & * \\ 0 & 0 & A_{reg} - \lambda E_{reg} & * \\ 0 & 0 & 0 & A_l + KC_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array}\right] \qquad (6)$$

Because the pair $(B_r, A_r - \lambda E_r)$ is controllable and the pair $(C_l, A_l - \lambda E_l)$ is observable, we can arbitrarily assign stable spectrums for the matrix pairs $(A_r + B_rF, E_r)$ and $(A_l + KC_l, E_l)$ by choosing suitable state-feedback and output-injection matrices F and K, respectively.

With the partitioning of $\widetilde{\mathcal{S}}(\lambda)$ as in (6) it follows that for almost all λ, $\mathrm{rank}\,\mathcal{S}(\lambda) = \mathrm{rank}\,\widetilde{\mathcal{S}}_{12}(\lambda)$, and thus a generalized (1,2)-inverse of $\mathcal{S}(\lambda)$ can be computed as [2]

$$\mathcal{S}(\lambda)^+ = ZV \begin{bmatrix} 0 & 0 \\ \widetilde{\mathcal{S}}_{12}(\lambda)^{-1} & 0 \end{bmatrix} UQ. \qquad (7)$$

To compute a descriptor representation of the generalized inverse $G(\lambda)^+$ it is not necessary to explicitly evaluate $\widetilde{\mathcal{S}}_{12}(\lambda)^{-1}$. If we denote

$$\widetilde{A}_{12} - \lambda\widetilde{E}_{12} = \widetilde{\mathcal{S}}_{12}(\lambda), \quad \widetilde{B} = UQ \begin{bmatrix} 0 \\ I_p \end{bmatrix} = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{bmatrix}$$

$$\widetilde{C} = [\ 0\ I_m\ ]ZV = [\ \widetilde{C}_1\ \widetilde{C}_2\ ],$$

where $\widetilde{B}$ and $\widetilde{C}$ are partitioned analogously with the column and row partition of $\widetilde{\mathcal{S}}(\lambda)$ in (6), respectively, then the corresponding $G(\lambda)^+$ is given by

$$G(\lambda)^+ = -\widetilde{C}_2(\lambda\widetilde{E}_{12} - \widetilde{A}_{12})^{-1}\widetilde{B}_1, \qquad (8)$$

and thus $(\widetilde{E}_{12}, \widetilde{A}_{12}, \widetilde{B}_1, -\widetilde{C}_2, 0)$ is a descriptor representation of $G(\lambda)^+$.

From the above construction it is clear that the poles of G^+ include always the zeros of G (the generalized eigenvalues of the regular part (4)) as the fixed poles of G^+. The spurious poles are the union of generalized eigenvalues of the pairs $(A_r + B_rF, E_r)$ and $(A_l + KC_l, E_l)$ and as mentioned before, can be arbitrarily assigned. If G has only stable zeros, then this approach can be used to compute stable generalized inverses of G by appropriately determining stabilizing matrices F and K. Generally with F and K stabilizing, the only unstable poles of the generalized inverse (8) are the unstable zeros of G. Thus $G(\lambda)^+$ satisfies the condition imposed at step 1 of the procedure proposed in the previous section.

To compute at step 3 the outer factor G_o only the inner denominator G_i in the RCFID $G^+ = NG_i^{-1}$ is necessary. As it will be apparent in the next section, in this case the output matrix $-\widetilde{C}_2$ of G^+ plays no role in the computations. In fact, to compute G_i it is sufficient to have $\widehat{A} - \lambda\widehat{E}$, the lower right corner of $\widetilde{A}_{12} - \lambda\widetilde{E}_{12}$ defined as

$$\widehat{A} - \lambda\widehat{E} = \begin{bmatrix} A_f - \lambda E_f & * \\ 0 & A_l + KC_l - \lambda E_l \end{bmatrix}, \qquad (9)$$

and $\widehat{B}$, the corresponding rows of the input matrix $\widetilde{B}_1$. Thus the computation of the right transformation matrices Z and V is not necessary. Moreover, instead of the more complex Kronecker-like form (3), the following simpler Kronecker-like form can be used

$$\widehat{\mathcal{S}}(\lambda) := Q\mathcal{S}(\lambda)Z = \left[\begin{array}{ccc} A_r - \lambda E_r & * & * \\ 0 & A_f - \lambda E_f & * \\ \hline 0 & 0 & A_l - \lambda E_l \\ 0 & 0 & C_l \end{array}\right], \qquad (10)$$

where this time the pencil $A_r - \lambda E_r$ contains the right and the infinity Kronecker structures of the system pencil.

The reduction techniques to compute (3) or (10) is based on structure preserving algorithms similar to those described in [13] to compute the system zeros. A complete algorithm to compute (3) is described in detail in [19] together with the corresponding computational programs. The simpler form (10) is an intermediary step in computing (3). An algorithm to compute this form is given in the Appendix. The stabilizing matrix K in (9) can be efficiently computed by using either direct stabilization methods or pole assignment techniques for descriptor systems as those proposed in [20]. The computational complexity of computing a generalized inverse of G is $0(n^3)$ and all computations can be done with numerically stable or numerically reliable algorithms.

3 Computation of the inner factor

Let $G = (E, A, B, C, D)$ be a stabilizable descriptor representation of a rational TFM G. In this section we

present an algorithm to compute a least order inner denominator G_i of a RCFID of G, $G = NG_i^{-1}$. This algorithm can be used at step 2 of the proposed approach to determine the least order inner denominator factor of the RCFID of G^+. The algorithm to compute G_i relies on several simple facts (see also [18] for more details).

Fact 1. [22] *For any F of appropriate dimensions and for any invertible W, the descriptor representations*

$$\begin{aligned} N &= (E,\ A+BF,\ BW,\ C+DF,\ DW) \\ M &= (E,\ A+BF,\ BW,\ F,\ W) \end{aligned}$$

give a fractional representation of G as $G = NM^{-1}$. If F is chosen such that the pair $(E, A+BF)$ is stable, then the fractional representation is a right coprime factorization.

It is apparent that in computing the denominator factor M the output matrices C and D play no role. This fact leads to a sensible simplification of the procedure presented in section 1.

The algorithm to compute inner denominators uses recursively the following explicit updating technique of fractional representations.

Fact 2. *If $G = N_1M_1^{-1}$ and $N_1 = N_2M_2^{-1}$ are fractional representations with inner denominators, then G has the fractional representation $G = NM^{-1}$, where $N = N_2$ and $M = M_1M_2$ is inner.*

This simple fact allows us to obtain explicit formulas to update partial factorizations by using simple state space formulas. Let N_1 and M_1 be the factors computed as

$$\begin{aligned} N_1 &= (E,\ A+BF_1,\ BW_1,\ C+DF_1,\ DW_1) \\ M_1 &= (E,\ A+BF_1,\ BW_1,\ F_1,\ W_1) \end{aligned}$$

and let N_2 and M_2 be the factors of N_1 computed as

$$\begin{aligned} N_2 &= (E,\ A+BF,\ BW,\ C+DF,\ DW) \\ M_2 &= (E,\ A+BF,\ BW,\ F_2,\ W_2) \end{aligned}$$

where

$$\begin{aligned} F &= F_1 + W_1F_2 \\ W &= W_1W_2 \end{aligned} \qquad (11)$$

It easy to verify that the product M_1M_2 is given by

$$M_1M_2 = (E,\ A+BF,\ BW,\ F,\ W) \qquad (12)$$

and thus the equations (11) and (12) serve as explicit updating formulas of fractional representations.

Elementary first or second order inner factors can be used to reflect the unstable poles of G to symmetric positions with respect to the imaginary axis in case of continuous-time systems or with respect of the unit circle in case of discrete-time systems. This pole dislocation technique is used in our algorithm and is conceptually similar to the conjugation technique proposed in [12]. Formulas to compute elementary inner denominators are established below.

Fact 3. *Let $G = (E, A, B, *, *)$ a controllable descriptor representation with E non-singular and $\Lambda(E, A) \in \mathbf{C}^+$ (the appropriate unstable region of the complex plane $\mathbf{C}$). Then the denominator factor $G_i = (E, A+BF,\ BW,\ F,\ W)$ is* inner *by choosing F and W as:*

$$\begin{aligned} &AYE^T + EYA^T - BB^T = 0 \\ &F = -B^T(YE^T)^{-1}, \quad W = I, \end{aligned}$$

for a continuous-time system, and

$$\begin{aligned} &AYA^T - BB^T = EYE^T \\ &F = -B^T(EYE^T + BB^T)^{-1}A \\ &W = (I + B^T(EYE^T)^{-1}B)^{-\frac{1}{2}}, \end{aligned}$$

for a discrete-time system.

The following algorithm, based on the above facts, can be used to compute the inner denominator G_i of a RCFID of G.

RCFID Algorithm.

1. Find orthogonal matrices Q and Z to reduce the pair (E, A) to the ordered *generalized real Schur form* (GRSF) (QAZ, QEZ) such that

   $$QEZ = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix},\ QAZ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},\ QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

   where $E_{22}, A_{22} \in \mathbf{R}^{q\times q}$, $\Lambda(A_{11}, E_{11}) \cap \mathbf{C}^+ = \phi$ and $\Lambda(A_{22}, E_{22}) \subset \mathbf{C}^+$.
   Set $(\widehat{A}, \widehat{E}, \widehat{B}, \widehat{F}, \widehat{W}) = (A_{22}, E_{22}, B_2, 0, I)$ and $i = 1$.

2. If $q = 0$, go to 6.

3. Let (δ, α) be the last diagonal blocks of $(\widehat{E}, \widehat{A})$ of order k and let β be the $k \times m$ matrix formed from the last k rows of $\widehat{B}$. For the system $(\delta, \alpha, \beta, *, *)$ compute φ and V such that $(\delta, \alpha + \beta\varphi,\ \beta V, \varphi, V)$ is inner. Set $K = [\,0\ \varphi\,]$.

4. Compute $\widehat{A} \leftarrow \widehat{A} + \widehat{B}K$, $\widehat{F} \leftarrow \widehat{F} + \widehat{W}K$, $\widehat{W} \leftarrow \widehat{W}V$.

5. Compute the orthogonal $\widehat{Q}$ and $\widehat{Z}$ to move the last blocks of $(\widehat{E}, \widehat{A})$ to positions $(i+1, i+1)$ by interchanging the diagonal blocks of the GRSF. Compute $\widehat{E} \leftarrow \widehat{Q}\widehat{E}\widehat{Z}$, $\widehat{A} \leftarrow \widehat{Q}\widehat{A}\widehat{Z}$, $\widehat{B} \leftarrow \widehat{Q}\widehat{B}$, $\widehat{F} \leftarrow \widehat{F}\widehat{Z}$. Put $q \leftarrow q - k$, $i \leftarrow i + k$ and go to 2.

6. Set $G_i = (\widehat{E}, \widehat{A}, \widehat{B}\widehat{W}, \widehat{F}, \widehat{W})$.

If the RCFID Algorithm is applied to the TFM G^+ computed in the previous section, then the order of the inner factor is precisely the number of unstable zeros of G. The resulting pair $(\widehat{E}, \widehat{A})$ is in a generalized real Schur form having as stable eigenvalues the reflected unstable zeros of G.

The RCFID algorithm relies on reliable numerical techniques. It can be viewed as a pole assignment algorithm (similar to that of [20]) which assigns the unstable poles in symmetrical positions with respect to the imaginary axis in the continuous-time case or the unit circle in the discrete-time case. Because practically there is no freedom in assigning the poles, it is to be expected that the algorithm perform in a numerically stable way only if the norms of the elementary feedback matrices K computed at step 3 are not too high.

4 Computation of outer factor

The outer factor G_o can be computed by removing the uncontrollable unstable eigenvalues from the descriptor representation of $G_i^{-1}G$:

$$\left(\begin{bmatrix} \widehat{E} & 0 \\ 0 & E \end{bmatrix}, \begin{bmatrix} \widehat{A} - \widehat{B}\widehat{F} & \widehat{B}C \\ 0 & A \end{bmatrix}, \begin{bmatrix} \widehat{B}D \\ B \end{bmatrix}, [\,-\widehat{W}^{-1}\widehat{F}\;\; C\,], \widehat{W}^{-1}D\right)$$

These eigenvalues can be removed in a numerically sound way by using the following approach. First reduce the pair

$$\left(\begin{bmatrix} \widehat{A} - \widehat{B}\widehat{F} & \widehat{B}C \\ 0 & A \end{bmatrix}, \begin{bmatrix} \widehat{E} & 0 \\ 0 & E \end{bmatrix}\right)$$

by using an orthogonal similarity transformation, to an ordered GRSF where the unstable diagonal blocks are situated in the bottom right corner of the resulting pair. Because the unstable eigenvalues are uncontrollable, the corresponding rows in the transformed input matrix should be zero. Thus, after applying the transformations to the input and output matrices of $G_i^{-1}G$, the nth order outer factor results by simply retaining the subsystem corresponding to the first n stable eigenvalues.

5 Conclusions

A completely general implementable procedure to compute inner-outer factorizations of rational matrices has been proposed. All computational steps of this procedure can be performed by using exclusively numerically reliable algorithms. The procedure is well suited for robust and modular software implementation. It is worth to mention that the proposed procedure is applicable even when the given rational matrix has zeros on the imaginary axis for a continuous-time system or on the unit circle for a discrete-time system.

References

[1] T. Beelen and P. Van Dooren. An improved algorithm for the computation of Kronecker's canonical form of a singular pencil. *Lin. Alg. & Appl.*, 105:9–65, 1988.

[2] A. Ben Israel and T. N. E. Greville. Some topics in generalized inverses of matrices. In M. Z. Nashed, editor, *Generalized Inverses and Applications*, pp. 125–147. Academic Press, New York, 1976.

[3] T. Chen and B. A. Francis. Spectral and inner-outer factorizations of rational matrices. *SIAM J. Matrix Anal. Appl.*, 10:1–17, 1989.

[4] B. R. Copeland and M. G. Safonov. Zero cancelling compensators for singular control problems and their application to inner-outer factorization problem. *Int. J. Robust and Nonlinear Control*, 2:139–164, 1992.

[5] J. Demmel and B. Kågström. Stably computing the Kronecker structure and reducing subspaces of singular pencils $A-\lambda B$ for uncertain data. In J. Cullum and R. A. Willoughby, eds., *Large Scale Eigenvalue Problems*, vol. 127 of *Mathematics Studies Series*, pp. 283–323. North-Holland, Amsterdam, 1986.

[6] B. A. Francis. *A Course in H^∞ Theory.* Springer-Verlag, New York, 1991.

[7] D. W. Gu, M. C. Tsai, S. D. O'Young, and I. Postletwaite. State-space formulae for discrete-time H_∞ optimization. *Int. J. Control*, 49:1683–1723, 1989.

[8] S. Hara and T. Sugie. Inner-outer factorization for strictly proper functions with $j\omega$-axis zeros. *Systems & Control Lett.*, 16:179–185, 1991.

[9] V. Ionescu and C. Oara. Spectral and inner-outer factorizations for discrete-time systems: the general case. Submitted to *IEEE Trans. Autom. Control*, 1995.

[10] T. Kailath. *Linear Systems.* Prentice Hall, Englewood Cliffs, N.J., 1980.

[11] T. Katayama. A spectral factorization algorithm for discrete-time descriptor systems via generalized eigenproblems. *Lin. Alg. & Appl.*, 205-206:869–891, 1994.

[12] H. Kimura. Conjugation, interpolation and model-matching in H^∞. *Int. J. Control*, 49:269–307, 1989.

[13] P. Misra, P. Van Dooren, and A. Varga. Computation of structural invariants of generalized state-space systems. *Automatica*, 30:1921–1936, 1994.

[14] M. C. Tsai and L. W. Chen. A numerical algorithm for inner-outer factorizations for real-rational matrices. *Systems & Control Lett.*, 20:209–217, 1993.

[15] P. Van Dooren. The computation of Kronecker's canonical form of a singular pencil. *Lin. Alg. & Appl.*, 27:103–141, 1979.

[16] P. Van Dooren. Rational and polynomial matrix factorizations via recursive pole-zero cancellation. *Lin. Alg. & Appl.*, 137/138:663–697, 1990.

[17] A. Varga. Computation of irreducible generalized state-space realizations. *Kybernetika*, 26:89–106, 1989.

[18] A. Varga. Generalized Schur methods to compute coprime factorizations of rational matrices. In *Proc. 1st Asian Control Conference, Tokyo, Japan*, vol. 3, pp. 89–92, 1994.

[19] A. Varga. Computation of Kronecker-like forms of a system pencil: Applications, algorithms and software. TR R181-95, DLR-Oberpfaffenhofen, Inst. Robotics and System Dynamics, January 1995.

[20] A. Varga. On stabilization of descriptor systems. *Systems & Control Letters*, 24:133–138, 1995.

[21] A. Varga and P. Van Dooren. On computing generalized inverses of rational matrices with applications, in preparation. In *Proc. 1995 ICIAM, Hamburg*, 1995.

[22] F. Y. Wang and M. J. Balas. Doubly coprime fractional representations of generalized dynamical systems. *IEEE Trans. Autom. Control*, AC-34:733–744, 1989.

[23] M. Weiss. Spectral and inner-outer factorizations through the constrained Riccati equation. *IEEE Trans. Autom. Control*, AC-39:677–681, 1994.

[24] Z. Zhang and J. S. Freudenberg. Loop transfer recovery for non-minimum phase plants. *IEEE Trans. Autom. Control*, AC-35:547–553, 1990.

[25] Z. Zhang and J. S. Freudenberg. Formulas for minimum-phase/all-pass factorization of discrete-time transfer functions. *Control-Theory and Advanced Technology*, 8:743–754, 1992.

Appendix

The following procedure is a numerically stable $0(n^3)$ computational complexity approach to compute the Kronecker-like form (10) of Section 2.

Kronecker-like Form Reduction Procedure.

1. By using any appropriate rank revealing decomposition (for instance the SVD), determine orthogonal U_1 and V_1 to compute a complete orthogonal decomposition of E in the form

$$U_1 E V_1 = \begin{bmatrix} 0 & \widehat{E} \\ 0 & 0 \end{bmatrix}$$

where $\widehat{E}$ is upper-triangular and non-singular. Compute $\mathcal{S}_1(\lambda) = \operatorname{diag}\{U_1, I_p\}\,\mathcal{S}(\lambda)\operatorname{diag}\{I_m, V_1\}$ and partition $\mathcal{S}_1(\lambda)$ as

$$\mathcal{S}_1(\lambda) := \left[\begin{array}{c|c} \widehat{B} & \widehat{A}-\lambda\widehat{E} \\ \hline \widehat{D} & \widehat{C} \end{array}\right].$$

Set $Q = \operatorname{diag}\{U_1, I_p\}$, $Z = \operatorname{diag}\{I_m, V_1\}$.

2. By using the dual S-REDUCE algorithm of [13], determine orthogonal U_2 and V_2 to reduce the system pencil $\mathcal{S}_1(\lambda)$ to the form

$$\mathcal{S}_2(\lambda) = U_2\mathcal{S}_1(\lambda)V_2 = \left[\begin{array}{c|c:c} A_1-\lambda E_1 & * & \\ \hline 0 & \begin{array}{c:c} B_c & A_c-\lambda E_c \\ \hdashline D_c & C_c \end{array} \end{array}\right]$$

where D_c is upper-trapezoidal and has full column rank, E_c is upper-triangular and non-singular, and $A_1-\lambda E_1$ has full row rank. Compute $Q \leftarrow U_2 Q$, $Z \leftarrow Z V_2$.

3. By using the reduction technique of [1, pages 33-34], determine orthogonal U_3 to compress the rows of the matrix $\begin{bmatrix} B_c \\ D_c \end{bmatrix}$ such that

$$\mathcal{S}_3(\lambda) = \operatorname{diag}\{I, U_3\}\mathcal{S}_2(\lambda) = \begin{bmatrix} A_1-\lambda E_1 & * & * \\ 0 & D_i & * \\ 0 & 0 & A_2-\lambda E_2 \\ 0 & 0 & C_2 \end{bmatrix},$$

where D_i is upper-triangular and non-singular, C_2 is the part of C_c corresponding to the linearly dependent rows of D_c, and E_2 is upper-triangular and non-singular. Compute $Q \leftarrow \operatorname{diag}\{I, U_3\}Q$.

4. By using the dual of the controllability staircase algorithm of [17], determine orthogonal U_4 and V_4 to reduce the subpencil $\begin{bmatrix} A_2-\lambda E_2 \\ C_2 \end{bmatrix}$ to the observability staircase form

$$U_4 \begin{bmatrix} A_2-\lambda E_2 \\ C_2 \end{bmatrix} V_4 = \begin{bmatrix} A_f-\lambda E_f & * \\ 0 & A_l-\lambda E_l \\ 0 & C_l \end{bmatrix},$$

where the pair $(C_l, A_l-\lambda E_l)$ is observable, and both E_f and E_l are upper-triangular and non-singular matrices. Compute

$$\widehat{\mathcal{S}}(\lambda) = \operatorname{diag}\{I, U_4\}\mathcal{S}_3(\lambda)\operatorname{diag}\{I, V_4\}$$

and $Q \leftarrow \operatorname{diag}\{I, U_4\}Q$, $Z \leftarrow Z\operatorname{diag}\{I, V_4\}$.

The resulting system pencil $\widehat{\mathcal{S}}(\lambda)$ is in form (10) with

$$A_r - \lambda E_r := \begin{bmatrix} A_1-\lambda E_1 & * \\ 0 & D_i \end{bmatrix}$$

Note that in contrast with alternative algorithms to compute Kronecker-like forms [1, 5, 15], only a single rank determination is performed involving E (at step 1).

The reductions performed at steps 2 and 4 are based on a reduction technique similar to that introduced in [17] to compute controllability staircase forms of descriptor systems. This technique is also described in detail in [13]. The rank determinations are based on QR-decompositions with column pivoting. The main feature of these algorithms is the preservation, during computations of QR-decompositions, of the full rank and of the upper-triangular form of the intervening "E" matrices. This feature leads to two important advantages over existing methods. The first advantage is the computational complexity $0(n^3)$. In contrast, the algorithms of [5, 15] have computational complexity $0(n^4)$, because singular value decompositions are used instead of QR-decompositions, and thus the explicit accumulation of left and right transformation matrices is necessary. The second advantage arises in comparing the reduction algorithm S-REDUCE of [13] and the improved $0(n^3)$ complexity Algorithm 3.2.1 of [1]. The main weakness of this latter algorithm is the need to update during each QR-like reduction step the rank information on "E". This rank updating is in fact equivalent with rank decisions based on QR-decompositions without pivoting and thus it is potentially unreliable. In the S-REDUCE algorithm of [13], "E" having always full rank, no such updating is necessary. Instead, two QR-decompositions with column pivoting are necessary to be performed at each step.