Proposals for Control Data Objects in ANDECS 2.0

ANDECS Working Note: AWN-01

May, 1994

A. Varga

DLR - Oberpfaffenhofen
German Aerospace Research Establishment
Institute for Robotics and System Dynamics
P.O.B. 1116, D-82230 Wessling, Germany
E-mail: Andres.Varga@dlr.de

Abstract. In this working note we introduce the system theoretical definitions of the control data objects (CDOs) intended to be implemented in Version 2.0 of ANDECS. Besides the already supported CDOs: LS - Linear State Space Model, RS - Real Signals and FR - Frequency Responses, several new CDOs are defined to handle other systems descriptions: GS - Generalized Linear State Space (Descriptor) Model, PM - Polynomial Differential State Space Model, TM - Transfer Matrix Model. The proposed set of CDOs for systems representations is considerably more general than those employed presently, including information on dead-times structures as well on parametric uncertainty structures modeled by LFTs. CDOs subclasses are also defined for particular model types (as for example models without dead-times) and a standard nomenclature for these subclasses is introduced. In parallel with the description of the new CDOs, the necessary model transformations applicable to each model class or subclass are specified.
1 LS – Linear State Space Model

Definition:

\[
\begin{align*}
\lambda x(t) &= \sum_{i=0}^{k_A} A_i(p)x(t-i\tau_i) + \sum_{i=0}^{k_B} B_i(p)u(t-i\tau_i) \\
y(t) &= \sum_{i=0}^{k_C} C_i(p)x(t-i\tau_i) + \sum_{i=0}^{k_D} D_i(p)u(t-i\tau_i)
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^r, p \in \mathbb{R}^q, \tau_0 = 0\) and \(\tau_i > 0, i = 1, \ldots, \max\{k_A, k_B, k_C, k_D\}\).

Notations:

\[
\begin{align*}
\text{LS}_\tau &:= \text{Linear System with Delays:} \quad \max\{k_A, k_B, k_C, k_D\} > 0 \\
\text{LS}_0 &:= \text{Linear System without Delays:} \quad k_A = k_B = k_C = k_D = 0 \\
\text{LS}_{c0} \text{ or LS}_{c-} &:= \text{Continuous Linear System:} \quad \lambda x(t) = \dot{x}(t) \\
\text{LS}_{d0} \text{ or LS}_{d+} &:= \text{Discrete Linear System:} \quad \lambda x(t) = x(t+T), \tau_i = h_i T \quad \text{(def: } T = 1). \\
\end{align*}
\]

1.1 General Transformations

- **LS \rightarrow RS**: Simulation for \(\{u(t), t \in [t_0, t_f]\}, x(t_0) = x_0, p = p_0\)
- **LS \rightarrow FR**: Frequency response for \(\omega \in [\omega_{\min}, \omega_{\max}]\), \(p = p_0\)
  (a) **LS_{c-} \rightarrow FR**: Evaluation in s-Domain

\[
G(j\omega) = \left(\sum_{i=0}^{k_C} C_i e^{-j\omega \tau_i}\right) \left(j\omega I - \sum_{i=0}^{k_A} A_i e^{-j\omega \tau_i}\right)^{-1} \left(\sum_{i=0}^{k_B} B_i e^{-j\omega \tau_i}\right) + \sum_{i=0}^{k_D} D_i e^{-j\omega \tau_i}
\]

(b) **LS_{d+} \rightarrow FR**: Evaluation in z-Domain

\[
G(z) = \left(\sum_{i=0}^{k_C} C_i z^{-h_i}\right) \left(zI - \sum_{i=0}^{k_A} A_i z^{-h_i}\right)^{-1} \left(\sum_{i=0}^{k_B} B_i z^{-h_i}\right) + \sum_{i=0}^{k_D} D_i z^{-h_i}, \quad z = e^{j\omega T}
\]

- **LS \rightarrow GS**: Transformation to generalized state space (descriptor) form \((E = I)\)
- **LS_{d+} \rightarrow LS_{d0}**: Transformation of discrete systems to representations without delays

Simplified notation:

(a) \(k_1 = k_A = k_C, k_2 = k_B = k_D\);

(b) \(0 = h_0 \leq h_1 \leq \cdots \leq h_k\).

Resulting system (without delays):

\[
\begin{align*}
\tilde{x}(kT + T) &= \tilde{A}\tilde{x}(kT) + \tilde{B}u(kT) \\
y(kT) &= \tilde{C}\tilde{x}(kT) + \tilde{D}u(kT)
\end{align*}
\]

where \(\tilde{x}(t) = [x(t) x(t-T) \cdots x(t-h_k T) u(t-T) \cdots u(t-h_k T)]^T\). The matrices of the extended system are:
\[
\tilde{A} = \begin{bmatrix}
A_0 & 0 & \cdots & A_{k_1} & 0 & \cdots & B_1 & 0 & \cdots & B_{k_2} \\
0 & I_n & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & I_n & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}
B_0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}
\]

\[
\tilde{C} = \begin{bmatrix}
C_0 & 0 & \cdots & C_{k_1} & 0 & \cdots & D_1 & 0 & \cdots & D_{k_2}
\end{bmatrix}, \quad \tilde{D} = \begin{bmatrix}
D_0
\end{bmatrix}
\]

1.2 Transformations with Restrictions on Delays

\[
\lambda x(t) = A x(t) + \sum_{i=0}^{k_B} B_i u(t - \tau_i)
\]

\[
y(t) = \sum_{i=0}^{k_C} C_i x(t - \tau_i) + \sum_{i=0}^{k_D} D_i u(t - \tau_i)
\]

- \textbf{LS}_r \rightarrow \textbf{TM} : Evaluation of the transfer-function matrix

\[
\text{LS}_r : G(s) = \sum_{j=0}^{k_B} \sum_{i=0}^{k_C} C_{ij} (sI - A)^{-1} B_i e^{-s(\tau_1 + \tau_j)} + \sum_{i=0}^{k_D} D_i e^{-s\tau_i} := \sum_{i=0}^{k_b+k_D} \tilde{G}_i(s) e^{-s\tau_i}
\]

\[
\text{LS}_d : G(z) = \sum_{j=0}^{k_B} \sum_{i=0}^{k_C} C_{ij} (zI - A)^{-1} B_i z^{-(h_1 + h_j)} + \sum_{i=0}^{k_D} D_i z^{-h_1} := \sum_{i=0}^{k_B+k_D} \tilde{G}_i(z) z^{-h_1}
\]

- \textbf{LS}_e \rightarrow \textbf{LS}_d : Discretization with a sampling period \( T \)

Let \( h_i, i = 0, 1, \ldots, \max(k_B, k_C, k_D) \) be positive integers such that \( \tau_i = h_i T + \lambda_i, \ 0 \leq \lambda_i < T \) and \( \lambda_i = 0 \) for \( i = 0, \ldots, k_C \). The last conditions on \( \lambda_i \) should be satisfied by an appropriate choice of the sampling period \( T \). The resulting discretized system is

\[
x(kT + T) = F x(kT) + \sum_{i=0}^{k_B} H_i' u(kT - h_i T) + \sum_{i=1}^{k_B} H_i'' u(kT - h_i T - T)
\]

\[
y(kT) = \sum_{i=0}^{k_C} C_i x(kT - h_i T) + \sum_{i=0}^{k_D} D_i u(kT - h_i T) + \sum_{i=k_C+1}^{k_D} D_i u(kT - h_i T - T)
\]

where

\[
F = e^{AT}, \quad H_i' = \int_0^{T - \lambda_i} e^{At} B_i dt, \quad i = 0, \ldots, k_B
\]

\[
H_i'' = \int_0^{\lambda_i} e^{At} B_i dt, \quad i = 1, \ldots, k_B
\]
1.3 Input-Output Equivalence Transformations on Systems without Delays

• \( \text{LS}_0 \rightarrow \text{LS}_0 \): Coordinate transformations

Original system:
\[
\begin{align*}
\lambda x(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

Coordinate transformation: \( x(t) = T \tilde{x}(t), \quad u(t) = V \tilde{u}(t), \quad y(t) = W \tilde{y}(t) \)

Resulting system:
\[
\begin{align*}
\lambda \tilde{x}(t) &= \tilde{A} \tilde{x}(t) + \tilde{B} \tilde{u}(t) \\
\tilde{y}(t) &= \tilde{C} \tilde{x}(t) + \tilde{D} \tilde{u}(t)
\end{align*}
\]

where
\[(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) := (T^{-1}AT, T^{-1}BV, W^{-1}CT, W^{-1}DV)\]

- General coordinate transformation
  \( T, V \) and \( W \) general invertible matrices
- General orthogonal coordinate transformation
  \( T, V \) and \( W \) general orthogonal matrices
- Scaling of system matrices
  \( T, V \) and \( W \) diagonal matrices
- Balancing transformation
  \( T \) non-orthogonal, \( V = I, W = I \)
- Reduction of \( A \) to block-diagonal form
  \( T \) non-orthogonal but well conditioned, \( V = I, W = I \)
- Reduction of \( A \) to Hessenberg form
  \( T \) orthogonal, \( V = I, W = I \)
- Reduction of \( A \) to real Schur form or ordered real Schur form
  \( T \) orthogonal, \( V = I, W = I \)
- Reduction of system matrices to controllability or observability forms
  \( T \) orthogonal, \( V = I, W = I \)

• \( \text{LS}_0 \rightarrow \text{LS}_0 \): Minimal state space realization

Original system of order \( n \):
\[
\begin{align*}
\lambda x(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

Resulting system of minimal order \( n' \):
\[
\begin{align*}
\lambda \tilde{x}(t) &= \tilde{A} \tilde{x}(t) + \tilde{B} \tilde{u}(t) \\
\tilde{y}(t) &= \tilde{C} \tilde{x}(t) + \tilde{D} \tilde{u}(t)
\end{align*}
\]

such that
\[
\tilde{C}(\lambda I - \tilde{A})^{-1} \tilde{B} = C(\lambda I - A)^{-1} B
\]

• \( \text{LS}_{0}\rightarrow\text{LS}_{20} \): Continuous to discrete and discrete to continuous bilinear transformations
\[
\tilde{G}(z) = G(s)|_{s = \frac{\pi z + \pi}{\pi z + 2\pi}}
\]
1.4 Building an LFT Uncertainty System Model: LS → LFT

Consider a partitioned rational matrix

\[ M(s) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{R}(s)^{m_1 \times r_1} \times (q_2 + q_2) \]

and a rational matrix \( \Delta \in \mathbb{R}(s)^{q_1 \times m_1} \) and define the upper linear fractional transformation (LFT) \( F_u(M, \Delta) \) as

\[ F_u(M, \Delta) = M_{22} + M_{21}(I - \Delta M_{11}^{-1} \Delta M_{12}) \]

Any parametric uncertainty in the elements of matrices \( A, B, C \) or \( D \) of the form \( p \in [p_{min}, p_{max}] \) can be expressed as a local LFT uncertainty model with constant matrices

\[ p = F_u \left( \begin{bmatrix} 0 & s_0 \\ 1 & p_0 \end{bmatrix}, \delta \right), \]

where \( p_0 = (p_{min} + p_{max})/2 \) and \( s_0 = (p_{max} - p_{min})/2 \). It is easy to see that \( p = p_0 + s_0 \delta \) with \(|\delta| \leq 1\). By using elementary coupling operations with LFTs, for each of system matrices, LFT uncertainty models can be generated in the forms

\[ A(p) = F_u \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_0 \end{bmatrix}, \Delta_A \right), \quad B(p) = F_u \left( \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_0 \end{bmatrix}, \Delta_B \right), \]

\[ C(p) = F_u \left( \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_0 \end{bmatrix}, \Delta_C \right), \quad D(p) = F_u \left( \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_0 \end{bmatrix}, \Delta_D \right) \]

where \( \Delta_A, \Delta_B, \Delta_C \) and \( \Delta_D \) are diagonal matrices having on the diagonal the normalized uncertainty parameters \( \delta_1, \delta_2, \ldots \). Note that \( A_0, B_0, C_0 \) and \( D_0 \) can be viewed as nominal values for the respective matrices (for all \( \delta_i \) set to zero). The parametric uncertainties at component level can be transformed to structured uncertainties at the system level by using the properties of LFTs. The LFT uncertainty system model can be expressed as

\[ G_p(\lambda) = F_u(G(\lambda), \Delta) \]

where \( \Delta = \text{diag}(\Delta_A, \Delta_B, \Delta_C, \Delta_D) \) and \( G(\lambda) \) is the following partitioned transfer function matrix with the corresponding state space realization

\[ G(\lambda) = \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_{22}(\lambda) \end{bmatrix} \]

\[ := \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & B_{12} \\ 0 & 0 & C_{11} & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} \\ 0 & 0 & C_{21} & D_{21} & D_0 \end{bmatrix} \lambda (A_0 - \Lambda_0)^{-1} \begin{bmatrix} A_{12} \\ 0 \\ 0 \\ C_{12} \\ 0 \end{bmatrix} \]

\[ - \begin{bmatrix} A_{21} & B_{21} & 0 & 0 & B_0 \end{bmatrix} \begin{bmatrix} A_{12} \\ 0 \\ 0 \\ C_{12} \\ 0 \end{bmatrix} \lambda (A_0 - \Lambda_0)^{-1} \begin{bmatrix} A_{21} & B_{21} & 0 & 0 & B_0 \end{bmatrix} \]

Note that \( G_0(\lambda) = C_0(\lambda I - A_0)^{-1}B_0 + D_0 \) is the nominal transfer-function matrix.
2 GS – Generalized Linear State Space (Descriptor) Model

Definition:
\[
\sum_{i=0}^{k_R} E_i(p) \lambda x(t - \tau_i) = \sum_{i=0}^{k_A} A_i(p)x(t - \tau_i) + \sum_{i=0}^{k_B} B_i(p)u(t - \tau_i)
\]
\[
y(t) = \sum_{i=0}^{k_C} C_i(p)x(t - \tau_i) + \sum_{i=0}^{k_D} D_i(p)u(t - \tau_i)
\]

where \(x(t) \in \mathbb{R}^n\), \(u(t) \in \mathbb{R}^n\), \(y(t) \in \mathbb{R}^r\), \(p \in \mathbb{R}^5\), \(\tau_0 = 0\) and \(\tau_i > 0\), \(i = 1, \ldots, \max\{k_E, k_A, k_B, k_C, k_D\}\).

Notations:

- \(\text{GS}_r\) := Generalized Linear System with Delays: \(\max\{k_E, k_A, k_B, k_C, k_D\} > 0\)
- \(\text{GS}_0\) := Generalized Linear System without Delays: \(k_E = k_A = k_B = k_C = k_D = 0\)
- \(\text{GS}_{cd} \) or \(\text{GS}_{cr}\) := Continuous Generalized Linear System:
  \[\lambda x(t) = \dot{x}(t)\]
- \(\text{GS}_{dr} \) or \(\text{GS}_{dt}\) := Discrete Generalized Linear System:
  \[\lambda x(t) = x(t + T)\] , \(\tau_i = h_i T\)
  (def: \(T = 1\)).

2.1 General Transformations

- \(\text{GS}_r \rightarrow \text{RS}\): Simulation for \(\{u(t), t \in [t_0, t_f]\}, x(t_0) = x_0, p = p_0\)

- \(\text{GS}_r \rightarrow \text{FR}\): Frequency response for \(\{\omega \in [\omega_{\text{min}}, \omega_{\text{max}}]\}, p = p_0\)
  (a) \(\text{GS}_{cr} \rightarrow \text{FR}\): Evaluation in \(s\)-Domain

\[
G(j\omega) = \left( \sum_{i=0}^{k_C} C_i e^{-j\omega \tau_i} \right) \left( \sum_{i=0}^{k_X} E_i e^{-j\omega \tau_i} j\omega - \sum_{i=0}^{k_A} A_i e^{-j\omega \tau_i} \right)^{-1} \left( \sum_{i=0}^{k_B} B_i e^{-j\omega \tau_i} \right) + \sum_{i=0}^{k_D} D_i e^{-j\omega \tau_i}
\]

(b) \(\text{GS}_{dt} \rightarrow \text{FR}\): Evaluation in \(z\)-Domain

\[
G(z) = \left( \sum_{i=0}^{k_C} C_i z^{-h_i} \right) \left( \sum_{i=0}^{k_E} E_i z^{-h_i+1} - \sum_{i=0}^{k_A} A_i z^{-h_i} \right)^{-1} \left( \sum_{i=0}^{k_B} B_i z^{-h_i} \right) + \sum_{i=0}^{k_D} D_i z^{-h_i}, \quad z = e^{j\omega T}
\]

- \(\text{GS}_{dr} \rightarrow \text{GS}_{cd}\) : Transformation of discrete systems to representations without delays

Simplified notation:
(a) \(k_1 = k_A = k_E = k_C, k_2 = k_B = k_D\);
(b) \(0 = h_0 \leq h_1 \leq \cdots \leq h_k\).

Resulting system (without delays):
\[
\tilde{E}\tilde{x}(kT + T) = \tilde{A}\tilde{x}(kT) + \tilde{B}u(kT)
\]
\[
y(kT) = \tilde{C}\tilde{x}(kT) + \tilde{D}u(kT)
\]

where \(\tilde{x}(t) = [x(t) x(t - T) \cdots x(t - h_k T) u(t - T) \cdots u(t - h_k T)]^T\). The matrices of the extended system are:
\[
\tilde{E} = \begin{bmatrix}
E_0 & 0 & \cdots & E_1 & 0 & \cdots & E_{k_1} & 0 & \cdots & 0 \\
0 & I_n & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & I_n & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
\tilde{A} = \begin{bmatrix}
A_0 & 0 & \cdots & A_1 & 0 & \cdots & A_{k_1} & 0 & \cdots & B_1 & 0 & \cdots & B_{k_2} \\
0 & I_n & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & I_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}
B_0 \\
0 \\
\vdots \\
0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\tilde{C} = \begin{bmatrix}
C_0 & 0 & \cdots & C_1 & 0 & \cdots & C_{k_1} & 0 & \cdots & D_1 & 0 & \cdots & D_{k_2}
\end{bmatrix}, \quad \tilde{D} = \begin{bmatrix}
D_0
\end{bmatrix}
\]

### 2.2 Transformations with Restrictions on Delays

\[
E\lambda x(t) = A x(t) + \sum_{i=0}^{k_B} B_i u(t - \tau_i)
\]
\[
y(t) = \sum_{i=0}^{k_C} C_i x(t - \tau_i) + \sum_{i=0}^{k_D} D_i u(t - \tau_i)
\]

- **GSₜ→TM**: Evaluation of the transfer-function matrix

\[
\text{GS}_{xτ} : G(s) = \sum_{j=0}^{k_C} \sum_{i=0}^{k_B} C_j (sE - A)^{-1} B_i e^{-s(\tau_i + \tau_j)} + \sum_{i=0}^{k_D} D_i e^{-s\tau_i} \overset{\text{GS}_{xτ}}{=} \tilde{G}_i(s)e^{-s\tau_i}
\]

\[
\text{GS}_{zτ} : G(z) = \sum_{j=0}^{k_C} \sum_{i=0}^{k_B} C_j (zE - A)^{-1} B_i z^{-(h_i + i)} + \sum_{i=0}^{k_D} D_i z^{-h_i} \overset{\text{GS}_{zτ}}{=} \tilde{G}_i(z)z^{-h_i}
\]

### 2.3 Input-Output Equivalence Transformations on Systems without Delays

- **GS₀→GS₀**: Coordinate transformations

Original system:
\[
E\lambda x(t) = A x(t) + B u(t)
\]
\[
y(t) = C x(t) + D u(t)
\]

Coordinate transformation and left multiplication:
\[
x(t) = Z\tilde{x}(t), \quad u(t) = V\tilde{u}(t), \quad y(t) = W\tilde{y}(t), \quad \text{left multiplication matrix } Q
\]
Resulting system:

\[
\begin{align*}
\bar{E}\lambda\bar{x}(t) &= \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\
\bar{y}(t) &= \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t)
\end{align*}
\]

where

\[(\bar{E}, \bar{A}, \bar{B}, \bar{C}, \bar{D}) := (QEZ, QAZ, QBV, W^{-1}CZ, W^{-1}DV)\]

- General coordinate transformation
  \(Q, Z, V \text{ and } W\) general invertible matrices

- General orthogonal coordinate transformation
  \(Q, Z, V\) and \(W\) general orthogonal matrices

- Scalling of system matrices
  \(Q, Z, V\) and \(W\) diagonal matrices

- Reduction of pair \((E, A)\) to block-diagonal form (two blocks)
  \(Q, Z\) non-orthogonal but well conditioned, \(V = I, W = I\)

- Reduction of the pair \((E, A)\) to generalized Hessenberg form
  \(Q, Z\) orthogonal, \(V = I, W = I\)

- Reduction of pair \((E, A)\) to generalized real Schur form or ordered generalized real Schur form
  \(Q, Z\) orthogonal, \(V = I, W = I\)

- Reduction of system matrices to controllability or observability forms
  \(Q, Z\) orthogonal, \(V = I, W = I\)

- \textbf{GS}_0 → \textbf{GS}_0 : Minimal state space realization

  Original system of order \(n\):

  \[
  \begin{align*}
  E\lambda x(t) &= Ax(t) + Bu(t) \\
  y(t) &= Cx(t) + Du(t)
  \end{align*}
  \]

  Resulting system of minimal order \(n'\):

  \[
  \begin{align*}
  \bar{E}\lambda\bar{x}(t) &= \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\
  \bar{y}(t) &= \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t)
  \end{align*}
  \]

  such that

  \[
  \bar{C}(\lambda\bar{E} - \bar{A})^{-1}\bar{B} = C(\lambda E - A)^{-1}B
  \]

- \textbf{GS}_0 → \textbf{LS}_0 : Reduction to standard state-space representation

  Condition to be fulfilled: number of finite poles equals \(\text{rank}(E)\).
2.4 Building an LFT Uncertainty System Model: GS₀→LFT

Consider the LFT uncertainty models of the system matrices

\[
E(p) = \mathcal{F}_u\left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_0 \end{bmatrix}, \Delta E\right),
\]

\[
A(p) = \mathcal{F}_u\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_0 \end{bmatrix}, \Delta A\right), \quad B(p) = \mathcal{F}_u\left(\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_0 \end{bmatrix}, \Delta B\right),
\]

\[
C(p) = \mathcal{F}_u\left(\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_0 \end{bmatrix}, \Delta C\right), \quad D(p) = \mathcal{F}_u\left(\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_0 \end{bmatrix}, \Delta D\right)
\]

where \(\Delta E, \Delta A, \Delta B, \Delta C\) and \(\Delta D\) are diagonal matrices having on the diagonal the normalized uncertainty parameters \(\delta_1, \delta_2, \ldots\). Note that \(E_0, A_0, B_0, C_0\) and \(D_0\) can be viewed as nominal values for the respective matrices (for all \(\delta_i\) set to zero). The parametric uncertainties at component level can be transformed to structured uncertainties at the system level by using the properties of LFTs. The LFT uncertainty system model can be expressed as

\[
G_p(\lambda) = \mathcal{F}_u(G(\lambda), \Delta)
\]

where \(\Delta = \text{diag}(\Delta E, \Delta A, \Delta B, \Delta C, \Delta D)\) and \(G(\lambda)\) is the following partitioned transfer function matrix with the corresponding state space realization

\[
G(\lambda) = \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_{0}(\lambda) \end{bmatrix}
\]

\[
\begin{bmatrix}
E_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & A_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & B_{11} & 0 & 0 & B_{12} \\
0 & 0 & 0 & C_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{11} & D_{12} \\
0 & 0 & 0 & C_{21} & D_{21} & D_0
\end{bmatrix}
\begin{bmatrix}
\lambda E_{12} \\
A_{12} \\
0 \\
C_{12} \\
0 \\
0
\end{bmatrix} = (\lambda E_0 - A_0)^{-1} \begin{bmatrix} E_{21} & A_{21} & B_{21} & 0 & 0 & B_0 \\
0 & 0 & C_0 & 0 \end{bmatrix}
\]

Note that \(G_0(\lambda) = C_0(\lambda E_0 - A_0)^{-1}B_0 + D_0\) is the nominal transfer-function matrix.
3 PM – Polynomial Differential State Space Model

Definition:

\[
\sum_{i=0}^{k_p} P_i(\lambda; p)x(t - \tau_i) = \sum_{i=0}^{k_Q} Q_i(\lambda; p)u(t - \tau_i)
\]

\[
y(t) = \sum_{i=0}^{k_V} V_i(\lambda; p)x(t - \tau_i) + \sum_{i=0}^{k_W} W_i(\lambda; p)u(t - \tau_i)
\]

where \(x(t) \in \mathbb{R}^n\), \(u(t) \in \mathbb{R}^m\), \(y(t) \in \mathbb{R}^r\), \(p \in \mathbb{R}^j\), \(\tau_0 = 0\) and \(\tau_i > 0\), \(i = 1, \ldots, \max\{k_p, k_Q, k_V, k_W\}\); \(P(\lambda; p) \in \mathbb{R}^{n \times n}[\lambda]\), \(Q(\lambda; p) \in \mathbb{R}^{m \times m}[\lambda]\), \(V(\lambda; p) \in \mathbb{R}^{r \times n}[\lambda]\), \(W(\lambda; p) \in \mathbb{R}^{r \times m}[\lambda]\) are polynomial matrices.

Notations for polynomial model representation:

\[
\text{PM}^{(m)}(\lambda) := \text{Polynomial Model in Matrix Polynomial Representation in } \lambda
\]

\[
Z_k(\lambda) = \sum_{i=0}^{k_z} Z_{ij}(p)\lambda^i \text{ for } Z = P, Q, V, W.
\]

\[
\text{PM}^{(m)}(\lambda^{-1}) := \text{Polynomial Model in Matrix Polynomial Representation in } \lambda^{-1}
\]

\[
Z_k(\lambda^{-1}) = \sum_{i=0}^{k_z} Z_{ij}(p)\lambda^{-i} \text{ for } Z = P, Q, V, W.
\]

\[
\text{PM}^{(c)}(\lambda) := \text{Polynomial Model with Coefficient Representation of Polynomials in } \lambda
\]

\[
Z_k(\lambda) := [z_{ij}^k(\lambda)] \text{ for } Z = P, Q, V, W, \text{ where each } z_{ij}^k(\lambda) \text{ has the form } c_0(p) + c_1(p)\lambda + \ldots + c_l(p)\lambda^l.
\]

\[
\text{PM}^{(c)}(\lambda^{-1}) := \text{Polynomial Model with Coefficient Representation of Polynomials in } \lambda^{-1}
\]

\[
Z_k(\lambda^{-1}) := [z_{ij}^k(\lambda^{-1})] \text{ for } Z = P, Q, V, W, \text{ where each } z_{ij}^k(\lambda^{-1}) \text{ has the form } c_0(p)\lambda^{-l} + c_1(p)\lambda^{-(l-1)} + \ldots + c_l(p).
\]

\[
\text{PM}^{(r)}(\lambda) := \text{Polynomial Model with Factored Representation of Polynomials in } \lambda
\]

\[
Z_k(\lambda) := [z_{ij}^k(\lambda)] \text{ for } Z = P, Q, V, W, \text{ where each } z_{ij}^k(\lambda) \text{ has the form } g(\lambda - \tau_1(p))\ldots(\lambda - \tau_l(p)).
\]

\[
\text{PM}^{(r)}(\lambda^{-1}) := \text{Polynomial Model with Factored Representation of Polynomials in } \lambda^{-1}
\]

\[
Z_k(\lambda^{-1}) := [z_{ij}^k(\lambda^{-1})] \text{ for } Z = P, Q, V, W, \text{ where each } z_{ij}^k(\lambda^{-1}) \text{ has the form } g(\lambda^{-1} - \tau_1(p))\ldots(\lambda^{-1} - \tau_l(p)).
\]

Notations for polynomial model classes:

\[
\text{PM}_r := \text{Polynomial Model with Delays: } \max\{k_p, k_Q, k_V, k_W\} > 0
\]

\[
\text{PM}_0 := \text{Polynomial Model without Delays: } k_p = k_Q = k_V = k_W = 0
\]

\[
\text{PM}_{\text{cd}} \text{ or } \text{PM}_{\text{cr}} := \text{Continuous Polynomial Model: } \lambda z(t) = \dot{z}(t), \text{ for } z = x, u, y.
\]

\[
\text{PM}_{\text{dr}} \text{ or } \text{PM}_{\text{dr}} := \text{Discrete Polynomial Model: } \lambda z(t) = z(t + T), \text{ for } z = x, u, y; \tau_i = h_iT\text{ (def. } T = 1)\text{.}
\]

3.1 General Transformations

- \(\text{PM}^{(m)}_r \leftrightarrow \text{PM}^{(c)}_r \leftrightarrow \text{PM}^{(r)}_r \leftrightarrow \text{PM}^{(m)}_r\)

- \(\text{PM}^{(r)}_d(z)(\lambda) \leftrightarrow \text{PM}^{(r)}_d(z^{-1}), z = m, c, r,\)

- \(\text{PM}_r \rightarrow \text{FR}: \text{Frequency response for } \{\omega \in [\omega_{\text{min}}, \omega_{\text{max}}], p = p_0\}\)
(a) $\text{PM}_{cr} \rightarrow \text{FR}$: Evaluation in $z$-Domain

$$G(j\omega) = \left( \sum_{i=0}^{k_F} P_i(j\omega) e^{-j\omega T_i} \right)^{-1} \left( \sum_{j=0}^{k_Q} Q_j(j\omega) e^{-j\omega T_j} \right) + \sum_{i=0}^{k_W} W_i(j\omega) e^{-j\omega T_i}$$

(b) $\text{PM}_{dr} \rightarrow \text{FR}$: Evaluation in $z$-Domain

$$G(z) = \left( \sum_{i=0}^{k_F} P_i(z) z^{-h_i} \right)^{-1} \left( \sum_{j=0}^{k_Q} Q_j(z) z^{-h_j} \right) + \sum_{i=0}^{k_W} W_i(z) z^{-h_i}, \quad z = e^{j\omega T}$$

- **$\text{PM}_{dr} \rightarrow \text{PM}_{d0}$**: Transformation of discrete systems to representations without delays

Let $h = \max \{h_i\}$ and define

$$\tilde{P}(z) = \sum_{i=0}^{k_F} P_i(z) z^{-h_i}, \quad \tilde{Q}(z) = \sum_{i=0}^{k_Q} Q_i(z) z^{-h_i}, \quad \tilde{V}(z) = \sum_{i=0}^{k_F} V_i(z) z^{-h_i}, \quad \tilde{W}(z) = \sum_{i=0}^{k_W} W_i(z) z^{-h_i};$$

Resulting system:

$$\tilde{P}(z) \tilde{x}(kT) = \tilde{Q}(z) \tilde{u}(kT)$$

$$y(kT) = \tilde{V}(z) \tilde{x}(kT) + \tilde{W}(z) u(kT)$$

where $\tilde{x}(t) = [v(t) \ x(t)]^T$ and the matrices of the system without delays are:

$$\tilde{P}(z) = \begin{bmatrix} z^h I_r & -\tilde{V}(z) \\ 0 & \tilde{P}(z) \end{bmatrix}, \quad \tilde{Q}(z) = \begin{bmatrix} \tilde{W}(z) \\ \tilde{Q}(z) \end{bmatrix}, \quad \tilde{V}(z) = \begin{bmatrix} I_r & 0 \end{bmatrix}, \quad \tilde{W}(z) = 0.$$

### 3.2 Input-Output Equivalent Transformations on Systems without Delays

- **$\text{PM}_{d0} \rightarrow \text{GS}_{d0}$**: Minimal state space realization

Original polynomial system of order $n$:

$$P(\lambda)\bar{x}(t) = Q(\lambda)u(t)$$

$$y(t) = V(\lambda)\bar{x}(t) + W(\lambda)u(t)$$

Resulting generalized state space system of minimal order $n'$:

$$E\lambda \bar{x}(t) = A\bar{x}(t) + Bu(t)$$

$$y(t) = C\bar{x}(t) + Du(t)$$

such that

$$C(\lambda E - A)^{-1} B + D = V(\lambda)P(\lambda)^{-1}Q(\lambda) + W(\lambda)$$

- **$\text{PM}_{d0} \rightarrow \text{LS}_{d0}$**: Reduction to standard state-space representation

Condition to be fulfilled: number of finite poles equals rank($E$).
3.3 Building an LFT Uncertainty System Model: PM\(_{\text{d}}\)→LFT

Parametric uncertainties in the coefficients or roots of polynomials in the system matrices can be expressed as LFT uncertainty models of the forms

\[
P(\lambda, p) = \mathcal{F}_u \left( \begin{bmatrix} P_{11} & P_{12}(\lambda) \\ P_{21} & P_0(\lambda) \end{bmatrix}, \Delta_P \right), \quad Q(\lambda, p) = \mathcal{F}_u \left( \begin{bmatrix} Q_{11} & Q_{12}(\lambda) \\ Q_{21} & Q_0(\lambda) \end{bmatrix}, \Delta_Q \right),
\]

\[
V(\lambda, p) = \mathcal{F}_u \left( \begin{bmatrix} V_{11} & V_{12}(\lambda) \\ V_{21} & V_0(\lambda) \end{bmatrix}, \Delta_V \right), \quad W(\lambda, p) = \mathcal{F}_u \left( \begin{bmatrix} W_{11} & W_{12}(\lambda) \\ W_{21} & W_0(\lambda) \end{bmatrix}, \Delta_W \right)
\]

where \(\Delta_P, \Delta_Q, \Delta_V\) and \(\Delta_W\) are diagonal matrices having on the diagonal the normalized uncertainty parameters \(\delta_1, \delta_2, \ldots\). Note that \(P_0(\lambda), Q_0(\lambda), V_0(\lambda)\) and \(W_0(\lambda)\) can be viewed as nominal values for the respective matrices (for all \(\delta_i\) set to zero). The parametric uncertainties at component level can be transformed to \textit{structural uncertainties at the system level} by using the properties of LFTs. The LFT uncertainty system model can be expressed as

\[
G_p(\lambda) = \mathcal{F}_u(G(\lambda), \Delta)
\]

where \(\Delta = \text{diag}(\Delta_P, \Delta_Q, \Delta_V, \Delta_W)\) and \(G(\lambda)\) is the following partitioned transfer function matrix with the corresponding state space realization

\[
G(\lambda) = \begin{bmatrix}
G_{11}(\lambda) & G_{12}(\lambda) \\
G_{21}(\lambda) & G_0(\lambda)
\end{bmatrix}
\]

\[
:= \begin{bmatrix}
P_{11} & 0 & 0 & 0 & \alpha_0 \\
0 & Q_{11} & 0 & 0 & Q_{12}(\lambda) \\
0 & 0 & V_{11} & 0 & 0 \\
0 & 0 & 0 & W_{11} & W_{12}(\lambda) \\
0 & 0 & V_{21} & W_{21} & W_0(\lambda)
\end{bmatrix} - \begin{bmatrix}
P_{12}(\lambda) \\
0 \\
V_{12}(\lambda) \\
0 \\
V_0(\lambda)
\end{bmatrix} P_0(\lambda)^{-1} \begin{bmatrix}
P_{21} & Q_{21} & 0 & 0 & Q_0(\lambda)
\end{bmatrix}
\]

Note that \(G_0(\lambda) = V_0(\lambda)P_0(\lambda)^{-1}Q_0(\lambda) + W_0(\lambda)\) is the \textit{nominal} transfer-function matrix.
4 TM – Transfer Matrix Model

Definition of continuous-time transfer matrix model:

\[ Y(s) = G(s;p)U(s) = \sum_{i=0}^{k_G} G_i(s;p)e^{-s\tau_i}U(s) \]

where \( Y(s) \) and \( U(s) \) are the Laplace-transforms of \( y(t) \in \mathbb{R}^r \) and \( u(t) \in \mathbb{R}^m \), respectively, \( p \in \mathbb{R}^q \), \( \tau_0 = 0 \) and \( G_i(s;p) \in \mathbb{R}^{r \times m}(s) \), \( i = 0, \ldots, k_G \) are rational matrices.

Definition of discrete-time transfer matrix model:

\[ Y(z) = G(z;p)U(z) = \sum_{i=0}^{k_G} G_i(z;p)z^{-h_i}U(z) \]

where \( Y(z) \) and \( U(z) \) are the Z-transforms of \( y(t) \in \mathbb{R}^r \) and \( u(t) \in \mathbb{R}^m \), respectively, \( p \in \mathbb{R}^q \), \( h_0 = 0 \) and \( G_i(z;p) \in \mathbb{R}^{r \times m}(z) \), \( i = 0, \ldots, k_G \) are rational matrices.

Notations for transfer matrix model representation:

\[ \text{TM}^{(c)}(\lambda) := \text{Transfer Matrix Model with Coefficient Representation of Polynomials in } \lambda \]
\[ G_k(\lambda) := [q_{ij}^k(\lambda)/t_{ij}^k(\lambda)] \text{ where each } q_{ij}^k(\lambda) \text{ and } t_{ij}^k(\lambda) \text{ has the form} \]
\[ c_i(\lambda)\lambda^{r_i} + \ldots + c_0(\lambda). \]

\[ \text{TM}^{(c)}(\lambda^{-1}) := \text{Transfer Matrix Model with Coefficient Representation of Polynomials in } \lambda^{-1} \]
\[ G_k(\lambda^{-1}) := [q_{ij}^k(\lambda^{-1})/t_{ij}^k(\lambda^{-1})] \text{ where each } q_{ij}^k(\lambda^{-1}) \text{ and } t_{ij}^k(\lambda^{-1}) \text{ has the form} \]
\[ c_i(\lambda^{-1})\lambda^{-r_i} + \ldots + c_0(\lambda). \]

\[ \text{TM}^{(f)}(\lambda) := \text{Transfer Matrix Model with Factored Representation of Polynomials in } \lambda \]
\[ G_k(\lambda) := [q_{ij}^k(\lambda)/t_{ij}^k(\lambda)] \text{ where each } q_{ij}^k(\lambda) \text{ and } t_{ij}^k(\lambda) \text{ has the form} \]
\[ g(\lambda) = (\lambda - r_1(p))\ldots(\lambda - r_i(p))/(\lambda - z_1(p))\ldots(\lambda - z_i(p)). \]

\[ \text{TM}^{(f)}(\lambda^{-1}) := \text{Transfer Matrix Model with Factored Representation of Polynomials in } \lambda^{-1} \]
\[ G_k(\lambda^{-1}) := [q_{ij}^k(\lambda^{-1})/t_{ij}^k(\lambda^{-1})] \text{ where each } q_{ij}^k(\lambda^{-1}) \text{ and } t_{ij}^k(\lambda^{-1}) \text{ has the form} \]
\[ g(\lambda^{-1}) = (\lambda^{-1} - r_1(p))\ldots(\lambda^{-1} - r_i(p))/(\lambda^{-1} - z_1(p))\ldots(\lambda^{-1} - z_i(p)). \]

Notations for transfer matrix model classes:

\[ \text{TM}_r \] is Transfer Matrix Model with Delays: \( k_G > 0 \)
\[ \text{TM}_b \] is Transfer Matrix Model without Delays: \( k_G = 0 \)
\[ \text{TM}_{dr} \] or \[ \text{TM}_{d_r} \] is Continuous Transfer Matrix Model
\[ \text{TM}_{d0} \] or \[ \text{TM}_{d_r} \] is Discrete Transfer Matrix Model.

4.1 General Transformations

- \( \text{TM}_{dr}^{(c)}(\lambda^{-1}) \rightarrow \text{RS} \): Time response of discrete filters
- \( \text{TM}_{r}^{(c)} \leftrightarrow \text{TM}_{r}^{(r)} \leftrightarrow \text{TM}_{r}^{(m)} \)
- \( \text{TM}_{dr}^{(c)}(\lambda) \leftrightarrow \text{TM}_{dr}^{(c)}(\lambda^{-1}) \), \( z = c, r \).
- \( \text{TM}, \rightarrow \text{FR} \): Frequency response for \( \{\omega \in [\omega_{min}, \omega_{max}], \ p = p_0\} \)
  - (a) \( \text{TM}_{cr} \rightarrow \text{FR} \): Evaluation in s-Domain
    \[
    G(j\omega) = \sum_{i=0}^{k_G} G_i(j\omega)e^{-j\omega r_i}
    \]
(b) $\textbf{TM}_{d \rightarrow FR}$: Evaluation in $z$-Domain

$$G(z) = \sum_{i=0}^{k_G} G_i(z) e^{-h_i}, \quad z = e^{\omega T}$$

- **$\textbf{TM}_{cT} \rightarrow \textbf{TM}_{d0}$**: Discretization of continuous transfer matrix models
  Restriction: proper systems

- **$\textbf{TM}_{cT} \rightarrow \textbf{TM}_{d0}$**: Transformation of continuous-time systems to representations without delays
  Method: Approximating the irrational terms with Padé rational approximations, as for instance
  $$e^{-s \tau_i} \approx \frac{1 - s \tau_i/2}{1 - s \tau_i/2}$$

- **$\textbf{TM}_{d \rightarrow \textbf{TM}_{d0}}$**: Transformation of discrete-time systems to representations without delays

- **$\textbf{TM}_{r} \rightarrow \textbf{GS}_r$**: Minimal generalized state space realization

  Resulting generalized state space system of minimal order:

  $$\begin{align*}
  E \lambda x(t) &= A x(t) + \sum_{i=0}^{k_G} B_i u(t - \tau_i) \\
  y(t) &= C x(t) + \sum_{i=0}^{k_G} D_i u(t - \tau_i)
  \end{align*}$$

  such that

  $$G_i(\lambda) = C(\lambda E - A)^{-1} B_i + D_i, \quad i = 1, \ldots, k_G$$

- **$\textbf{TM}_{r} \rightarrow \textbf{LS}_r$**: Minimal standard state space realization

  Restriction: Proper systems

  Resulting state space system of minimal order:

  $$\begin{align*}
  \lambda x(t) &= A x(t) + \sum_{i=0}^{k_G} B_i u(t - \tau_i) \\
  y(t) &= C x(t) + \sum_{i=0}^{k_G} D_i u(t - \tau_i)
  \end{align*}$$

  such that

  $$G_i(\lambda) = C(\lambda I - A)^{-1} B_i + D_i, \quad i = 1, \ldots, k_G$$

- **$\textbf{TM}_{c0} \leftrightarrow \textbf{TM}_{d0}$**: Continuous to discrete and discrete to continuous bilinear transformations

  $$\tilde{G}(z) = G(s) \bigg|_{s = \frac{az + b}{cz + d}}$$
4.2 Building an LFT Uncertainty System Model: $TM_0$-$\rightarrow$LFT

Parametric uncertainties in the coefficients or roots of numerator and denominator polynomials in the transfer matrix can be expressed as an LFT uncertainty model of the form

$$G(\lambda; p) = \mathcal{F}_d\left(\begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_0(\lambda) \end{bmatrix}, \Delta \right)$$

where $\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_k)$ and $G_0(\lambda)$ is the nominal transfer matrix.
5 FR - Frequency Responses

Definition: \[
\{ (G_i; \omega_i), \quad i = 1, \ldots, N \}
\]

where \( G_i \in \mathbb{C}^{r \times m} \).

Notations for frequency responses:

\[\begin{align*}
\text{FR}^{(\text{Re,Im})} & : \text{ Real and Imaginary Part Representation of Complex Matrices} \\
\text{FR}^{(A,\phi)} & : \text{ Amplitude-Phase Representation of Complex Matrices}
\end{align*}\]

Notations for frequency response classes:

\[\begin{align*}
\text{FR}_c & : \text{ Laplace Transform Frequency Response: } G_i = G_i(\omega_i) \\
\text{FR}_d & : \text{ Z-Transform Frequency Response: } G_i = G_i(e^{j\omega_i T})
\end{align*}\]

5.1 Transformations

- FR\(\rightarrow\)RS : Inverse fast Fourier transform

- FR\(^{(\text{Re,Im})}\) \(\leftrightarrow\) FR\(^{(A,\phi)}\) : Real-Imaginary to Amplitude-Phase transformation
6 RS - Real Signals

Definition:
\[ \{ (Y_i; t_i), \quad i = 1, \ldots, N \} \]
where \( Y_i \in \mathbb{R}^r \).

6.1 Transformations

- **RS→FR**: Fast Fourier transform
- **RS→LS** or **GS**: Time Domain Identification