Proposals for Control Data Objects in ANDECS 2.0

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Abstract. In this working note we introduce the system theoretical definitions of the *control data objects* (CDOs) intended to be implemented in Version 2.0 of ANDECS. Besides the already supported CDOs: LS - Linear State Space Model, RS - Real Signals and FR - Frequency Responses, several new CDOs are defined to handle other systems descriptions: GS - Generalized Linear State Space (Descriptor) Model, PM - Polynomial Differential State Space Model, TM - Transfer Matrix Model. The proposed set of CDOs for systems representations is considerably more general than those employed presently, including information on dead-times structures as well on parametric uncertainty structures modeled by LFTs. CDOs subclasses are also defined for particular model types (as for example models without dead-times) and a standard nomenclature for these subclasses is introduced. In parallel with the description of the new CDOs, the necessary model transformations applicable to each model class or subclass are specified.

1 LS – Linear State Space Model

Definition:

$$\begin{aligned}\lambda x(t) &= \sum_{i=0}^{k_A} A_i(p)x(t-\tau_i) + \sum_{i=0}^{k_B} B_i(p)u(t-\tau_i)\\ y(t) &= \sum_{i=0}^{k_C} C_i(p)x(t-\tau_i) + \sum_{i=0}^{k_D} D_i(p)u(t-\tau_i)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$, $p \in \mathbb{R}^q$, $\tau_0 = 0$ and $\tau_i > 0,\ i = 1, \ldots, \max\{k_A, k_B, k_C, k_D\}$.

Notations:

$\mathbf{LS}_\tau$	:= *Linear System with Delays:*	$\max\{k_A, k_B, k_C, k_D\} > 0$
$\mathbf{LS}_0$	:= *Linear System without Delays:*	$k_A = k_B = k_C = k_D = 0$
$\mathbf{LS}_{c0}$ or $\mathbf{LS}_{c\tau}$	:= *Continuous Linear System:*	$\lambda x(t) = \dot{x}(t)$
$\mathbf{LS}_{d0}$ or $\mathbf{LS}_{d\tau}$	:= *Discrete Linear System:*	$\lambda x(t) = x(t+T)$, $\tau_i = h_i T$ (def: $T = 1$).

1.1 General Transformations

- $\mathbf{LS}_\tau \to \mathbf{RS}$: Simulation for $\{u(t),\ t \in [t_o,\ t_f],\ x(t_0) = x_0,\ p = p_0\}$
- $\mathbf{LS}_\tau \to \mathbf{FR}$: Frequency response for $\{\omega \in [\omega_{min},\ \omega_{max}],\ p = p_0\}$

  (a) $\mathbf{LS}_{c\tau} \to \mathbf{FR}$: Evaluation in s-Domain

$$G(j\omega) = \left(\sum_{i=0}^{k_C} C_i e^{-j\omega\tau_i}\right)\left(j\omega I - \sum_{i=0}^{k_A} A_i e^{-j\omega\tau_i}\right)^{-1}\left(\sum_{i=0}^{k_B} B_i e^{-j\omega\tau_i}\right) + \sum_{i=0}^{k_D} D_i e^{-j\omega\tau_i}$$

  (b) $\mathbf{LS}_{d\tau} \to \mathbf{FR}$: Evaluation in z-Domain

$$G(z) = \left(\sum_{i=0}^{k_C} C_i z^{-h_i}\right)\left(zI - \sum_{i=0}^{k_A} A_i z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{k_B} B_i z^{-h_i}\right) + \sum_{i=0}^{k_D} D_i z^{-h_i}, \quad z = e^{j\omega T}$$

- $\mathbf{LS} \to \mathbf{GS}$: Transformation to generalized state space (descriptor) form ($E = I$)
- $\mathbf{LS}_{d\tau} \to \mathbf{LS}_{d0}$: Transformation of discrete systems to representations without delays

  Simplified notation:

  (a) $k_1 = k_A = k_C$, $k_2 = k_B = k_D$;

  (b) $0 = h_0 \le h_1 \le \cdots \le h_k$.

  Resulting system (without delays):

$$\begin{aligned}\tilde{x}(kT+T) &= \tilde{A}\tilde{x}(kT) + \tilde{B}u(kT)\\ y(kT) &= \tilde{C}\tilde{x}(kT) + \tilde{D}u(kT)\end{aligned}$$

  where $\tilde{x}(t) = [\,x(t)\ x(t-T)\ \cdots\ x(t-h_{k_1}T)\ u(t-T)\ \cdots\ u(t-h_{k_2}T)\,]^T$. The matrices of the extended system are:

$$\widetilde{A} = \begin{bmatrix} A_0 & 0 & \cdots & A_1 & 0 & \cdots & A_{k_1} & 0 & \cdots & B_1 & 0 & \cdots & B_{k_2} \\ 0 & I_n & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & I_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_m & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & I_m \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\widetilde{C} = \begin{bmatrix} C_0 & 0 & \cdots & C_1 & 0 & \cdots & C_{k_1} & 0 & \cdots & D_1 & 0 & \cdots & D_{k_2} \end{bmatrix}, \quad \widetilde{D} = \begin{bmatrix} D_0 \end{bmatrix}$$

1.2 Transformations with Restrictions on Delays

$$\begin{aligned} \lambda x(t) &= Ax(t) + \sum_{i=0}^{k_B} B_i u(t-\tau_i) \\ y(t) &= \sum_{i=0}^{k_C} C_i x(t-\tau_i) + \sum_{i=0}^{k_D} D_i u(t-\tau_i) \end{aligned}$$

- $\mathbf{LS}_\tau \rightarrow \mathbf{TM}$: Evaluation of the transfer-function matrix

$$\begin{aligned} \mathbf{LS}_{c\tau} \ : G(s) &= \sum_{j=0}^{k_C} \sum_{i=0}^{k_B} C_j (sI-A)^{-1} B_i e^{-s(\tau_i+\tau_j)} + \sum_{i=0}^{k_D} D_i e^{-s\tau_i} := \sum_{i=0}^{k_B k_C + k_D} \widetilde{G}_i(s) e^{-s\tilde{\tau}_i} \\ \mathbf{LS}_{d\tau} \ : G(z) &= \sum_{j=0}^{k_C} \sum_{i=0}^{k_B} C_j (zI-A)^{-1} B_i z^{-(h_i+h_j)} + \sum_{i=0}^{k_D} D_i z^{-h_i} := \sum_{i=0}^{k_B k_C + k_D} \widetilde{G}_i(z) z^{-\tilde{h}_i} \end{aligned}$$

- $\mathbf{LS}_{c\tau} \rightarrow \mathbf{LS}_{d\tau}$: Discretization with a sampling period T

  Let h_i, $i = 0, 1, \ldots, \max(k_B, k_C, k_D)$ be positive integers such that $\tau_i = h_i T + \lambda_i$, $0 \le \lambda_i < T$ and $\lambda_i = 0$ for $i = 0, \ldots, k_C$. The last conditions on λ_i should be satisfied by an appropriate choice of the sampling period T. The resulting discretized system is

$$\begin{aligned} x(kT+T) &= Fx(kT) + \sum_{i=0}^{k_B} H_i' u(kT-h_iT) + \sum_{i=1}^{k_B} H_i'' u(kT-h_iT-T) \\ y(kT) &= \sum_{i=0}^{k_C} C_i x(kT-h_iT) + \sum_{i=0}^{k_C} D_i u(kT-h_iT) + \sum_{i=k_C+1}^{k_D} D_i u(kT-h_iT-T) \end{aligned}$$

where

$$F = e^{AT}, \quad H_i' = \int_0^{T-\lambda_i} e^{At} B_i dt, \quad i = 0, \ldots, k_B$$

$$H_i'' = \int_0^{\lambda_i} e^{At} B_i dt, \quad i = 1, \ldots, k_B$$

1.3 Input-Output Equivalence Transformations on Systems without Delays

- $\mathbf{LS}_0 \to \mathbf{LS}_0$: Coordinate transformations

  Original system:

$$\begin{aligned} \lambda x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

  Coordinate transformation: $x(t) = T\widetilde{x}(t),\ u(t) = V\widetilde{u}(t),\ y(t) = W\widetilde{y}(t)$

  Resulting system:

$$\begin{aligned} \lambda \widetilde{x}(t) &= \widetilde{A}\widetilde{x}(t) + \widetilde{B}\widetilde{u}(t) \\ \widetilde{y}(t) &= \widetilde{C}\widetilde{x}(t) + \widetilde{D}\widetilde{u}(t) \end{aligned}$$

  where

$$(\widetilde{A},\ \widetilde{B},\ \widetilde{C},\ \widetilde{D}) := (T^{-1}AT,\ T^{-1}BV,\ W^{-1}CT, W^{-1}DV)$$

  - General coordinate transformation
    T, V and W general invertible matrices
  - General orthogonal coordinate transformation
    T, V and W general orthogonal matrices
  - Scalling of system matrices
    T, V and W diagonal matrices
  - Balancing transformation
    T non-orthogonal, $V = I,\ W = I$
  - Reduction of A to block-diagonal form
    T non-orthogonal but well conditioned, $V = I,\ W = I$
  - Reduction of A to Hessenberg form
    T orthogonal, $V = I,\ W = I$
  - Reduction of A to real Schur form or ordered real Schur form
    T orthogonal, $V = I,\ W = I$
  - Reduction of system matrices to controllability or observability forms
    T orthogonal, $V = I,\ W = I$

- $\mathbf{LS}_0 \to \mathbf{LS}_0$: Minimal state space realization

  Original system of order n:

$$\begin{aligned} \lambda x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

  Resulting system of minimal order n':

$$\begin{aligned} \lambda \widetilde{x}(t) &= \widetilde{A}\widetilde{x}(t) + \widetilde{B}u(t) \\ y(t) &= \widetilde{C}\widetilde{x}(t) + Du(t) \end{aligned}$$

  such that

$$\widetilde{C}(\lambda I - \widetilde{A})^{-1}\widetilde{B} = C(\lambda I - A)^{-1}B$$

- $\mathbf{LS}_{c0} \leftrightarrow \mathbf{LS}_{d0}$: *Continuous to discrete* and *discrete to continuous* bilinear transformations

$$\widetilde{G}(z) = G(s)\big|_{s=\frac{az+b}{cz+d}}$$

1.4 Building an LFT Uncertainty System Model: $\text{LS}_0 \rightarrow$LFT

Consider a partitioned rational matrix

$$M(s) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{R}(s)^{(p_1+p_2)\times(q_1+q_2)}$$

and a rational matrix $\Delta \in \mathbb{R}(s)^{q_1 \times p_1}$ and define the *upper linear fractional transformation* (LFT) $\mathcal{F}_u(M,\Delta)$ as

$$\mathcal{F}_u(M,\Delta) = M_{22} + M_{21}(I - \Delta M_{11})^{-1}\Delta M_{12}.$$

Any parametric uncertainty in the elements of matrices A, B, C or D of the form $p \in [p_{min}, p_{max}]$ can be expressed as a local LFT uncertainty model with constant matrices

$$p = \mathcal{F}_u\left(\begin{bmatrix} 0 & s_0 \\ 1 & p_0 \end{bmatrix}, \delta\right),$$

where $p_0 = (p_{min} + p_{max})/2$ and $s_0 = (p_{max} - p_{min})/2$. It is easy to see that $p = p_0 + s_o\delta$ with $|\delta| \leq 1$. By using elementary coupling operations with LFTs, for each of system matrices, LFT uncertainty models can be generated in the forms

$$\begin{aligned} A(p) &= \mathcal{F}_u\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_0 \end{bmatrix}, \Delta_A\right), & B(p) &= \mathcal{F}_u\left(\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_0 \end{bmatrix}, \Delta_B\right), \\ C(p) &= \mathcal{F}_u\left(\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_0 \end{bmatrix}, \Delta_C\right), & D(p) &= \mathcal{F}_u\left(\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_0 \end{bmatrix}, \Delta_D\right) \end{aligned}$$

where Δ_A, Δ_B, Δ_C and Δ_D are diagonal matrices having on the diagonal the normalized uncertainty parameters δ_1, δ_2, ... Note that A_0, B_0, C_0 and D_0 can be viewed as nominal values for the respective matrices (for all δ_i set to zero). The parametric uncertainties at component level can be transformed to *structured uncertainties at the system level* by using the properties of LFTs. The LFT uncertainty system model can be expressed as

$$G_p(\lambda) = \mathcal{F}_u(G(\lambda), \Delta)$$

where $\Delta = \text{diag}(\Delta_A, \Delta_B, \Delta_C, \Delta_D)$ and $G(\lambda)$ is the following partitioned transfer function matrix with the corresponding state space realization

$$\begin{aligned} G(\lambda) &= \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_0(\lambda) \end{bmatrix} \\ &:= \left[\begin{array}{cccc|c} A_{11} & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & B_{12} \\ 0 & 0 & C_{11} & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} \\ \hline 0 & 0 & C_{21} & D_{21} & D_0 \end{array}\right] - \left[\begin{array}{c} A_{12} \\ 0 \\ C_{12} \\ 0 \\ \hline C_0 \end{array}\right] (\lambda I - A_0)^{-1} \left[\begin{array}{cccc|c} A_{21} & B_{21} & 0 & 0 & B_0 \end{array}\right] \end{aligned}$$

Note that $G_0(\lambda) = C_0(\lambda I - A_0)^{-1}B_0 + D_0$ is the *nominal* transfer-function matrix.

2 GS – Generalized Linear State Space (Descriptor) Model

Definition:

$$\begin{aligned}\sum_{i=0}^{k_E} E_i(p)\lambda x(t-\tau_i) &= \sum_{i=0}^{k_A} A_i(p)x(t-\tau_i) + \sum_{i=0}^{k_B} B_i(p)u(t-\tau_i)\\ y(t) &= \sum_{i=0}^{k_C} C_i(p)x(t-\tau_i) + \sum_{i=0}^{k_D} D_i(p)u(t-\tau_i)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$, $p \in \mathbb{R}^q$, $\tau_0 = 0$ and $\tau_i > 0,\ i = 1, \ldots, \max\{k_E, k_A, k_B, k_C, k_D\}$.

Notations:

$\mathbf{GS}_\tau$	:= *Generalized Linear System with Delays:*	$\max\{k_E, k_A, k_B, k_C, k_D\} > 0$
$\mathbf{GS}_0$	:= *Generalized Linear System without Delays:*	$k_E = k_A = k_B = k_C = k_D = 0$
$\mathbf{GS}_{c0}$ or $\mathbf{GS}_{c\tau}$	:= *Continuous Generalized Linear System:*	$\lambda x(t) = \dot{x}(t)$
$\mathbf{GS}_{d0}$ or $\mathbf{GS}_{d\tau}$	:= *Discrete Generalized Linear System:*	$\lambda x(t) = x(t+T)$, $\tau_i = h_i T$ (def: $T = 1$).

2.1 General Transformations

- $\mathbf{GS}_\tau \to \mathbf{RS}$: Simulation for $\{u(t),\ t \in [t_o, t_f],\ x(t_0) = x_0,\ p = p_0\}$
- $\mathbf{GS}_\tau \to \mathbf{FR}$: Frequency response for $\{\omega \in [\omega_{min}, \omega_{max}],\ p = p_0\}$

  (a) $\mathbf{GS}_{c\tau} \to \mathbf{FR}$: Evaluation in s-Domain

$$G(j\omega) = \left(\sum_{i=0}^{k_C} C_i e^{-j\omega\tau_i}\right)\left(\sum_{i=0}^{k_E} E_i e^{-j\omega\tau_i} j\omega - \sum_{i=0}^{k_A} A_i e^{-j\omega\tau_i}\right)^{-1}\left(\sum_{i=0}^{k_B} B_i e^{-j\omega\tau_i}\right) + \sum_{i=0}^{k_D} D_i e^{-j\omega\tau_i}$$

  (b) $\mathbf{GS}_{d\tau} \to \mathbf{FR}$: Evaluation in z-Domain

$$G(z) = \left(\sum_{i=0}^{k_C} C_i z^{-h_i}\right)\left(\sum_{i=0}^{k_E} E_i z^{-h_i+1} - \sum_{i=0}^{k_A} A_i z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{k_B} B_i z^{-h_i}\right) + \sum_{i=0}^{k_D} D_i z^{-h_i}, \quad z = e^{j\omega T}$$

- $\mathbf{GS}_{d\tau} \to \mathbf{GS}_{d0}$: Transformation of discrete systems to representations without delays

  Simplified notation:

  (a) $k_1 = k_A = k_E = k_C$, $k_2 = k_B = k_D$;

  (b) $0 = h_0 \leq h_1 \leq \cdots \leq h_k$.

  Resulting system (without delays):

$$\begin{aligned}\widetilde{E}\widetilde{x}(kT+T) &= \widetilde{A}\widetilde{x}(kT) + \widetilde{B}u(kT)\\ y(kT) &= \widetilde{C}\widetilde{x}(kT) + \widetilde{D}u(kT)\end{aligned}$$

  where $\widetilde{x}(t) = [\,x(t)\ x(t-T)\ \cdots\ x(t-h_{k_1}T)\ u(t-T)\ \cdots\ u(t-h_{k_2}T)\,]^T$. The matrices of the extended system are:

$$
\widetilde{E} = \begin{bmatrix} E_0 & 0 & \cdots & E_1 & 0 & \cdots & E_{k_1} & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & I_n & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}
$$

$$
\widetilde{A} = \begin{bmatrix} A_0 & 0 & \cdots & A_1 & 0 & \cdots & A_{k_1} & 0 & \cdots & B_1 & 0 & \cdots & B_{k_2} \\ 0 & I_n & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & I_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_m & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & I_m \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}
$$

$$
\widetilde{C} = \begin{bmatrix} C_0 & 0 & \cdots & C_1 & 0 & \cdots & C_{k_1} & 0 & \cdots & D_1 & 0 & \cdots & D_{k_2} \end{bmatrix}, \quad \widetilde{D} = \begin{bmatrix} D_0 \end{bmatrix}
$$

2.2 Transformations with Restrictions on Delays

$$
\begin{aligned}
E\lambda x(t) &= Ax(t) + \sum_{i=0}^{k_B} B_i u(t-\tau_i) \\
y(t) &= \sum_{i=0}^{k_C} C_i x(t-\tau_i) + \sum_{i=0}^{k_D} D_i u(t-\tau_i)
\end{aligned}
$$

- **GS$_\tau$→TM** : Evaluation of the transfer-function matrix

$$
\begin{aligned}
\mathbf{GS}_{c\tau} \ : G(s) &= \sum_{j=0}^{k_C}\sum_{i=0}^{k_B} C_j(sE-A)^{-1}B_i e^{-s(\tau_i+\tau_j)} + \sum_{i=0}^{k_D} D_i e^{-s\tau_i} := \sum_{i=0}^{k_B k_C + k_D} \widetilde{G}_i(s) e^{-s\tilde{\tau}_i} \\
\mathbf{GS}_{d\tau} \ : G(z) &= \sum_{j=0}^{k_C}\sum_{i=0}^{k_B} C_j(zE-A)^{-1}B_i z^{-(h_i+h_j)} + \sum_{i=0}^{k_D} D_i z^{-h_i} := \sum_{i=0}^{k_B k_C + k_D} \widetilde{G}_i(z) z^{-\tilde{h}_i}
\end{aligned}
$$

2.3 Input-Output Equivalence Transformations on Systems without Delays

- **GS$_0$→GS$_0$** : Coordinate transformations

  Original system:

$$
\begin{aligned}
E\lambda x(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

  Coordinate transformation and left multiplication:
  $x(t) = Z\widetilde{x}(t),\ u(t) = V\widetilde{u}(t),\ y(t) = W\widetilde{y}(t)$, left multiplication matrix Q

Resulting system:

$$\begin{aligned} \widetilde{E}\lambda\widetilde{x}(t) &= \widetilde{A}\widetilde{x}(t) + \widetilde{B}\widetilde{u}(t) \\ \widetilde{y}(t) &= \widetilde{C}\widetilde{x}(t) + \widetilde{D}\widetilde{u}(t) \end{aligned}$$

where

$$(\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C},\ \widetilde{D}) := (QEZ,\ QAZ,\ QBV,\ W^{-1}CZ, W^{-1}DV)$$

- General coordinate transformation
  Q, Z, V and W general invertible matrices
- General orthogonal coordinate transformation
  Q, Z, V and W general orthogonal matrices
- Scalling of system matrices
  Q, Z, V and W diagonal matrices
- Reduction of pair (E, A) to block-diagonal form (two blocks)
  Q, Z non-orthogonal but well conditioned, $V = I$, $W = I$
- Reduction of the pair (E, A) to generalized Hessenberg form
  Q, Z orthogonal, $V = I$, $W = I$
- Reduction of pair (E, A) to generalized real Schur form or ordered generalized real Schur form
  Q, Z orthogonal, $V = I$, $W = I$
- Reduction of system matrices to controllability or observability forms
  Q, Z orthogonal, $V = I$, $W = I$

- $\mathbf{GS}_0 \rightarrow \mathbf{GS}_0$: Minimal state space realization

  Original system of order n:

$$\begin{aligned} E\lambda x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

  Resulting system of minimal order n':

$$\begin{aligned} \widetilde{E}\lambda\widetilde{x}(t) &= \widetilde{A}\widetilde{x}(t) + \widetilde{B}u(t) \\ y(t) &= \widetilde{C}\widetilde{x}(t) + Du(t) \end{aligned}$$

  such that

$$\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} = C(\lambda E - A)^{-1}B$$

- $\mathbf{GS}_0 \rightarrow \mathbf{LS}_0$: Reduction to standard state-space representation

  Condition to be fulfilled: number of finite poles equals rank(E).

2.4 Building an LFT Uncertainty System Model: $GS_0 \to$LFT

Consider the LFT uncertainty models of the system matrices

$$
\begin{aligned}
E(p) &= \mathcal{F}_u(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_0 \end{bmatrix}, \Delta_E), \\
A(p) &= \mathcal{F}_u(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_0 \end{bmatrix}, \Delta_A), \quad B(p) = \mathcal{F}_u(\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_0 \end{bmatrix}, \Delta_B), \\
C(p) &= \mathcal{F}_u(\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_0 \end{bmatrix}, \Delta_C), \quad D(p) = \mathcal{F}_u(\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_0 \end{bmatrix}, \Delta_D)
\end{aligned}
$$

where Δ_E, Δ_A, Δ_B, Δ_C and Δ_D are diagonal matrices having on the diagonal the normalized uncertainty parameters δ_1, δ_2, $\ldots$. Note that E_0, A_0, B_0, C_0 and D_0 can be viewed as nominal values for the respective matrices (for all δ_i set to zero). The parametric uncertainties at component level can be transformed to *structured uncertainties at the system level* by using the properties of LFTs. The LFT uncertainty system model can be expressed as

$$G_p(\lambda) = \mathcal{F}_u(G(\lambda), \Delta)$$

where $\Delta = \mathrm{diag}(\Delta_E, \Delta_A, \Delta_B, \Delta_C, \Delta_D)$ and $G(\lambda)$ is the following partitioned transfer function matrix with the corresponding state space realization

$$
\begin{aligned}
G(\lambda) &= \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_0(\lambda) \end{bmatrix} \\
&:= \left[\begin{array}{ccccc|c} E_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{11} & 0 & 0 & B_{12} \\ 0 & 0 & 0 & C_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{11} & D_{12} \\ \hline 0 & 0 & 0 & C_{21} & D_{21} & D_0 \end{array}\right] - \left[\begin{array}{c} \lambda E_{12} \\ A_{12} \\ 0 \\ C_{12} \\ 0 \\ \hline C_0 \end{array}\right] (\lambda E_0 - A_0)^{-1} \left[\begin{array}{ccccc|c} E_{21} & A_{21} & B_{21} & 0 & 0 & B_0 \end{array}\right]
\end{aligned}
$$

Note that $G_0(\lambda) = C_0(\lambda E_0 - A_0)^{-1}B_0 + D_0$ is the *nominal* transfer-function matrix.

3 PM – Polynomial Differential State Space Model

Definition:

$$\begin{aligned}\sum_{i=0}^{k_P} P_i(\lambda;p)x(t-\tau_i) &= \sum_{i=0}^{k_Q} Q_i(\lambda;p)u(t-\tau_i)\\ y(t) &= \sum_{i=0}^{k_V} V_i(\lambda;p)x(t-\tau_i)+\sum_{i=0}^{k_W} W_i(\lambda;p)u(t-\tau_i)\end{aligned}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^r$, $p\in\mathbb{R}^q$, $\tau_0=0$ and $\tau_i>0,\ i=1,\ldots,\max\{k_P,k_Q,k_V,k_W\}$; $P(\lambda;p)\in\mathbb{R}^{n\times n}[\lambda]$, $Q(\lambda;p)\in\mathbb{R}^{n\times m}[\lambda]$, $V(\lambda;p)\in\mathbb{R}^{r\times n}[\lambda]$, $W(\lambda;p)\in\mathbb{R}^{r\times m}[\lambda]$ are polynomial matrices.

Notations for polynomial model representation:

$\mathbf{PM}^{(m)}(\lambda)$	:=	*Polynomial Model in Matrix Polynomial Representation in* λ $Z_i(\lambda)=\sum_{j=0}^{nz_j} Z_{ij}(p)\lambda^j$ for $Z=P,\ Q,\ V,\ W$.
$\mathbf{PM}^{(m)}(\lambda^{-1})$	:=	*Polynomial Model in Matrix Polynomial Representation in* λ^{-1} $Z_i(\lambda^{-1})=\sum_{j=0}^{nz_j} Z_{ij}(p)\lambda^{-j}$ for $Z=P,\ Q,\ V,\ W$.
$\mathbf{PM}^{(c)}(\lambda)$	:=	*Polynomial Model with Coefficient Representation of Polynomials in* λ $Z_k(\lambda):=[\,z_{ij}^k(\lambda)\,]$ for $Z=P,\ Q,\ V,\ W$, where each $z_{ij}^k(\lambda)$ has the form $c_l(p)\lambda^l+\ldots+c_1(p)\lambda+c_0(p)$.
$\mathbf{PM}^{(c)}(\lambda^{-1})$	:=	*Polynomial Model with Coefficient Representation of Polynomials in* λ^{-1} $Z_k(\lambda^{-1}):=[\,z_{ij}^k(\lambda^{-1})\,]$ for $Z=P,\ Q,\ V,\ W$, where each $z_{ij}^k(\lambda^{-1})$ has the form $c_l(p)\lambda^{-l}+\ldots+c_1(p)\lambda^{-1}+c_0(p)$.
$\mathbf{PM}^{(r)}(\lambda)$	:=	*Polynomial Model with Factored Representation of Polynomials in* λ $Z_k(\lambda):=[\,z_{ij}^k(\lambda)\,]$ for $Z=P,\ Q,\ V,\ W$, where each $z_{ij}^k(\lambda)$ has the form $g(\lambda-r_1(p))\ldots(\lambda-r_l(p))$.
$\mathbf{PM}^{(r)}(\lambda^{-1})$	:=	*Polynomial Model with Factored Representation of Polynomials in* λ^{-1} $Z_k(\lambda^{-1}):=[\,z_{ij}^k(\lambda^{-1})\,]$ for $Z=P,\ Q,\ V,\ W$, where each $z_{ij}^k(\lambda^{-1})$ has the form $g(\lambda^{-1}-r_1(p))\ldots(\lambda^{-1}-r_l(p))$.

Notations for polynomial model classes:

$\mathbf{PM}_\tau$	:= *Polynomial Model with Delays:*	$\max\{k_P,k_Q,k_V,k_W\}>0$
$\mathbf{PM}_0$	:= *Polynomial Model without Delays:*	$k_P=k_Q=k_V=k_W=0$
$\mathbf{PM}_{c0}$ or $\mathbf{PM}_{c\tau}$	:= *Continuous Polynomial Model:*	$\lambda z(t)=\dot z(t)$, for $z=x,\ u,\ y$.
$\mathbf{PM}_{d0}$ or $\mathbf{PM}_{d\tau}$	:= *Discrete Polynomial Model:*	$\lambda z(t)=z(t+T)$, for $z=x,\ u,\ y$; $\tau_i=h_iT$ (def: $T=1$).

3.1 General Transformations

- $\mathbf{PM}_\tau{}^{(m)}\leftrightarrow\mathbf{PM}_\tau{}^{(c)}\leftrightarrow\mathbf{PM}_\tau{}^{(r)}\leftrightarrow\mathbf{PM}_\tau{}^{(m)}$
- $\mathbf{PM}_{d\tau}{}^{(z)}(\lambda)\leftrightarrow\mathbf{PM}_{d\tau}{}^{(z)}(\lambda^{-1})$, $z=m,\ c,\ r,$.
- $\mathbf{PM}_\tau\to\mathbf{FR}$: Frequency response for $\{\omega\in[\omega_{min},\ \omega_{max}],\ p=p_0\}$

(a) **PM$_{c\tau}$→FR**: Evaluation in s-Domain

$$G(j\omega) = \left(\sum_{i=0}^{k_V} V_i(j\omega)e^{-j\omega\tau_i}\right)\left(\sum_{i=0}^{k_P} P_i(j\omega)e^{-j\omega\tau_i}\right)^{-1}\left(\sum_{i=0}^{k_Q} Q_i(j\omega)e^{-j\omega\tau_i}\right) + \sum_{i=0}^{k_W} W_i(j\omega)e^{-j\omega\tau_i}$$

(b) **PM$_{d\tau}$→FR**: Evaluation in z-Domain

$$G(z) = \left(\sum_{i=0}^{k_V} V_i(z)z^{-h_i}\right)\left(\sum_{i=0}^{k_P} P_i(z)z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{k_Q} Q_i(z)z^{-h_i}\right) + \sum_{i=0}^{k_W} W_i(z)z^{-h_i}, \quad z = e^{j\omega T}$$

- **PM$_{d\tau}$→PM$_{d0}$** : Transformation of discrete systems to representations without delays

  Let $h = \max_i\{h_i\}$ and define

$$\widehat{P}(z) = \sum_{i=0}^{k_P} P_i(z)z^{h-h_i}, \quad \widehat{Q}(z) = \sum_{i=0}^{k_Q} Q_i(z)z^{h-h_i}, \quad \widehat{V}(z) = \sum_{i=0}^{k_V} V_i(z)z^{h-h_i}, \quad \widehat{W}(z) = \sum_{i=0}^{k_W} W_i(z)z^{h-h_i}$$

  Resulting system:

$$\begin{aligned} \widetilde{P}(z)\widetilde{x}(kT) &= \widetilde{Q}(z)\widetilde{u}(kT) \\ y(kT) &= \widetilde{V}(z)\widetilde{x}(kT) + \widetilde{W}(z)u(kT) \end{aligned}$$

  where $\widetilde{x}(t) = [\,v(t)\ x(t)\,]^T$ and the matrices of the system without delays are:

$$\widetilde{P}(z) = \begin{bmatrix} z^h I_r & -\widehat{V}(z) \\ 0 & \widehat{P}(z) \end{bmatrix}, \quad \widetilde{Q}(z) = \begin{bmatrix} \widehat{W}(z) \\ \widehat{Q}(z) \end{bmatrix}, \quad \widetilde{V}(z) = \begin{bmatrix} I_r & 0 \end{bmatrix}, \quad \widetilde{W}(z) = 0.$$

3.2 Input-Output Equivalent Transformations on Systems without Delays

- **PM$_0$→GS$_0$** : Minimal state space realization

  Original polynomial system of order n:

$$\begin{aligned} P(\lambda)x(t) &= Q(\lambda)u(t) \\ y(t) &= V(\lambda)x(t) + W(\lambda)u(t) \end{aligned}$$

  Resulting generalized state space system of minimal order n':

$$\begin{aligned} E\lambda\widetilde{x}(t) &= A\widetilde{x}(t) + Bu(t) \\ y(t) &= C\widetilde{x}(t) + Du(t) \end{aligned}$$

  such that

$$C(\lambda E - A)^{-1}B + D = V(\lambda)P(\lambda)^{-1}Q(\lambda) + W(\lambda)$$

- **PM$_0$→LS$_0$** : Reduction to standard state-space representation

  Condition to be fulfilled: number of finite poles equals rank(E).

3.3 Building an LFT Uncertainty System Model: $PM_0 \rightarrow LFT$

Parametric uncertainties in the coefficients or roots of polynomials in the system matrices can be expressed as LFT uncertainty models of the forms

$$\begin{aligned} P(\lambda,p) &= \mathcal{F}_u\left(\begin{bmatrix} P_{11} & P_{12}(\lambda) \\ P_{21} & P_0(\lambda) \end{bmatrix}, \Delta_P\right), & Q(\lambda,p) &= \mathcal{F}_u\left(\begin{bmatrix} Q_{11} & Q_{12}(\lambda) \\ Q_{21} & Q_0(\lambda) \end{bmatrix}, \Delta_Q\right), \\ V(\lambda,p) &= \mathcal{F}_u\left(\begin{bmatrix} V_{11} & V_{12}(\lambda) \\ V_{21} & V_0(\lambda) \end{bmatrix}, \Delta_V\right), & W(\lambda,p) &= \mathcal{F}_u\left(\begin{bmatrix} W_{11} & W_{12}(\lambda) \\ W_{21} & W_0(\lambda) \end{bmatrix}, \Delta_W\right) \end{aligned}$$

where Δ_P, Δ_Q, Δ_V and Δ_W are diagonal matrices having on the diagonal the normalized uncertainty parameters δ_1, δ_2, $\ldots$. Note that $P_0(\lambda)$, $Q_0(\lambda)$, $V_0(\lambda)$ and $W_0(\lambda)$ can be viewed as nominal values for the respective matrices (for all δ_i set to zero). The parametric uncertainties at component level can be transformed to *structured uncertainties at the system level* by using the properties of LFTs. The LFT uncertainty system model can be expressed as

$$G_p(\lambda) = \mathcal{F}_u(G(\lambda), \Delta)$$

where $\Delta = \text{diag}(\Delta_P, \Delta_Q, \Delta_V, \Delta_W)$ and $G(\lambda)$ is the following partitioned transfer function matrix with the corresponding state space realization

$$\begin{aligned} G(\lambda) &= \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_0(\lambda) \end{bmatrix} \\ &:= \left[\begin{array}{cccc|c} P_{11} & 0 & 0 & 0 & 0 \\ 0 & Q_{11} & 0 & 0 & Q_{12}(\lambda) \\ 0 & 0 & V_{11} & 0 & 0 \\ 0 & 0 & 0 & W_{11} & W_{12}(\lambda) \\ \hline 0 & 0 & V_{21} & W_{21} & W_0(\lambda) \end{array}\right] - \left[\begin{array}{c} P_{12}(\lambda) \\ 0 \\ V_{12}(\lambda) \\ 0 \\ \hline V_0(\lambda) \end{array}\right] P_0(\lambda)^{-1} \left[\begin{array}{cccc|c} P_{21} & Q_{21} & 0 & 0 & Q_0(\lambda) \end{array}\right] \end{aligned}$$

Note that $G_0(\lambda) = V_0(\lambda)P_0(\lambda)^{-1}Q_0(\lambda) + W_0(\lambda)$ is the *nominal* transfer-function matrix.

4 TM – Transfer Matrix Model

Definition of continuous-time transfer matrix model:

$$Y(s) = G(s;p)U(s) = \sum_{i=0}^{k_G} G_i(s;p)e^{-s\tau_i}U(s)$$

where $Y(s)$ and $U(s)$ are the Laplace-transforms of $y(t) \in \mathbb{R}^r$ and $u(t) \in \mathbb{R}^m$, respectively, $p \in \mathbb{R}^q$, $\tau_0 = 0$ and $G_i(s;p) \in \mathbb{R}^{r\times m}(s)$, $i = 0,\ldots,k_G$ are rational matrices.

Definition of discrete-time transfer matrix model:

$$Y(z) = G(z;p)U(z) = \sum_{i=0}^{k_G} G_i(z;p)z^{-h_i}U(z)$$

where $Y(z)$ and $U(z)$ are the Z-transforms of $y(t) \in \mathbb{R}^r$ and $u(t) \in \mathbb{R}^m$, respectively, $p \in \mathbb{R}^q$, $h_0 = 0$ and $G_i(z;p) \in \mathbb{R}^{r\times m}(z)$, $i = 0,\ldots,k_G$ are rational matrices.

Notations for transfer matrix model representation:

$\mathbf{TM}^{(c)}(\lambda)$	:=	*Transfer Matrix Model with Coefficient Representation of Polynomials in* λ $G_k(\lambda) := [\,q_{ij}^k(\lambda)/t_{ij}^k(\lambda)\,]$ where each $q_{ij}^k(\lambda)$ and $t_{ij}^k(\lambda)$ has the form $c_l(p)\lambda + \ldots + c_1(p)\lambda + c_0(p)$.
$\mathbf{TM}^{(c)}(\lambda^{-1})$	:=	*Transfer Matrix Model with Coefficient Representation of Polynomials in* λ^{-1} $G_k(\lambda^{-1}) := [\,q_{ij}^k(\lambda^{-1})/t_{ij}^k(\lambda^{-1})\,]$ where each $q_{ij}^k(\lambda^{-1})$ and $t_{ij}^k(\lambda^{-1})$ has the form $c_l(p)\lambda^{-l} + \ldots + c_1(p)\lambda^{-1} + c_0(p)$.
$\mathbf{TM}^{(r)}(\lambda)$	:=	*Transfer Matrix Model with Factored Representation of Polynomials in* λ $G_k(\lambda) := [\,q_{ij}^k(\lambda)/t_{ij}^k(\lambda)\,]$ where each $q_{ij}^k(\lambda)/t_{ij}^k(\lambda)$ has the form $g(\lambda - r_1(p))\ldots(\lambda - r_l(p))/[(\lambda - z_1(p))\ldots(\lambda - z_l(p))]$.
$\mathbf{TM}^{(r)}(\lambda^{-1})$	:=	*Transfer Matrix Model with Factored Representation of Polynomials in* λ^{-1} $G_k(\lambda^{-1}) := [\,q_{ij}^k(\lambda^{-1})/t_{ij}^k(\lambda^{-1})\,]$ where each $q_{ij}^k(\lambda^{-1})/t_{ij}^k(\lambda^{-1})$ has the form $g(\lambda^{-1} - r_1(p))\ldots(\lambda^{-1} - r_l(p))/[(\lambda^{-1} - z_1(p))\ldots(\lambda^{-1} - z_l(p))]$.

Notations for transfer matrix model classes:

$\mathbf{TM}_\tau$	:= *Transfer Matrix Model with Delays:*	$k_G > 0$
$\mathbf{TM}_0$	:= *Transfer Matrix Model without Delays:*	$k_G = 0$
$\mathbf{TM}_{c0}$ or $\mathbf{TM}_{c\tau}$	:= *Continuous Transfer Matrix Model*	
$\mathbf{TM}_{d0}$ or $\mathbf{TM}_{d\tau}$	:= *Discrete Transfer Matrix Model .*	

4.1 General Transformations

- $\mathbf{TM}_{d\tau}{}^{(c)}(\lambda^{-1}) \rightarrow \mathbf{RS}$: Time response of discrete filters
- $\mathbf{TM}_\tau{}^{(c)} \leftrightarrow \mathbf{TM}_\tau{}^{(r)} \leftrightarrow \mathbf{TM}_\tau{}^{(m)}$
- $\mathbf{TM}_{d\tau}{}^{(z)}(\lambda) \leftrightarrow \mathbf{TM}_{d\tau}{}^{(z)}(\lambda^{-1})$, $z = c, r$.
- $\mathbf{TM}_\tau \rightarrow \mathbf{FR}$: Frequency response for $\{\omega \in [\omega_{min}, \omega_{max}],\ p = p_0\}$

  (a) $\mathbf{TM}_{c\tau} \rightarrow \mathbf{FR}$: Evaluation in s-Domain

$$G(j\omega) = \sum_{i=0}^{k_G} G_i(j\omega)e^{-j\omega\tau_i}$$

(b) $\mathbf{TM}_{d\tau}\to\mathbf{FR}$: Evaluation in z-Domain

$$G(z) = \sum_{i=0}^{k_G} G_i(z)e^{-h_i}, \quad z = e^{j\omega T}$$

- $\mathbf{TM}_{c\tau}\to\mathbf{TM}_{d0}$: Discretization of continuous transfer matrix models
  Restriction: proper systems

- $\mathbf{TM}_{c\tau}\to\mathbf{TM}_{c0}$: Transformation of continuous-time systems to representations without delays
  Method: Approximating the irrational terms with Padé rational approximations, as for instance
  $$e^{-s\tau_i} \approx \frac{1 - s\tau_i/2}{1 - s\tau_i/2}$$

- $\mathbf{TM}_{d\tau}\to\mathbf{TM}_{d0}$: Transformation of discrete-time systems to representations without delays

- $\mathbf{TM}_{\tau}\to\mathbf{GS}_{\tau}$: Minimal generalized state space realization

  Resulting generalized state space system of minimal order:

  $$\begin{aligned} E\lambda x(t) &= Ax(t) + \sum_{i=0}^{k_G} B_i u(t-\tau_i) \\ y(t) &= Cx(t) + \sum_{i=0}^{k_G} D_i u(t-\tau_i) \end{aligned}$$

  such that
  $$G_i(\lambda) = C(\lambda E - A)^{-1}B_i + D_i, \; i = 1, \ldots, k_G$$

- $\mathbf{TM}_{\tau}\to\mathbf{LS}_{\tau}$: Minimal standard state space realization

  Restriction: Proper systems

  Resulting state space system of minimal order:

  $$\begin{aligned} \lambda x(t) &= Ax(t) + \sum_{i=0}^{k_G} B_i u(t-\tau_i) \\ y(t) &= Cx(t) + \sum_{i=0}^{k_G} D_i u(t-\tau_i) \end{aligned}$$

  such that
  $$G_i(\lambda) = C(\lambda I - A)^{-1}B_i + D_i, \; i = 1, \ldots, k_G$$

- $\mathbf{TM}_{c0}\leftrightarrow\mathbf{TM}_{d0}$: *Continuous to discrete* and *discrete to continuous* bilinear transformations

  $$\widetilde{G}(z) = G(s)\big|_{s=\frac{az+b}{cz+d}}$$

4.2 Building an LFT Uncertainty System Model: $TM_0 \rightarrow LFT$

Parametric uncertainties in the coefficients or roots of numerator and denominator polynomials in the transfer matrix can be expressed as an LFT uncertainty model of the form

$$G(\lambda;p) = \mathcal{F}_u(\begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_0(\lambda) \end{bmatrix}, \Delta)$$

where $\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_k)$ and $G_0(\lambda)$ is the *nominal* transfer matrix.

5 FR - Frequency Responses

Definition:

$$\{ (G_i; \omega_i), \quad i = 1, \ldots, N \}$$

where $G_i \in \mathbb{C}^{r \times m}$.

Notations for frequency responses:

$\mathbf{FR}^{(Re,Im)}$	:=	*Real and Imaginary Part Representation of Complex Matrices*
$\mathbf{FR}^{(A,\varphi)}$	:=	*Amplitude-Phase Representation of Complex Matrices*

Notations for frequency response classes:

$\mathbf{FR}_c$	:= *Laplace Transform Frequency Response:*	$G_i = G_i(\omega_i)$
$\mathbf{FR}_d$	:= *Z-Transform Frequency Response:*	$G_i = G_i(e^{j\omega_i T})$

5.1 Transformations

- **FR→RS** : Inverse fast Fourier transform
- $\mathbf{FR}^{(Re,Im)} \leftrightarrow \mathbf{FR}^{(A,\varphi)}$: Real-Imaginary to Amplitude-Phase transformation

6 RS - Real Signals

Definition:

$$\{ (Y_i; t_i), \quad i = 1, \dots, N \}$$

where $Y_i \in \mathbb{R}^r$.

6.1 Transformations

- **RS→FR** : Fast Fourier transform
- **RS→LS** or **GS** : Time Domain Identification