# Factorizations Methods of Rational Matrices

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**Abstract.** We propose numerically reliable state space algorithms for computing the following rational stable coprime factorizations of rational matrices: 1) factorizations with least order stable denominators; 2) factorizations with inner and $J$-inner denominators; and 3) factorizations with proper stable factors. The new algorithms are based on a recursive generalized Schur algorithm for pole assignment. They are generally applicable regardless the original descriptor state space representation is minimal or not, or is stabilizable/detectable or not. The proposed algorithms are useful in solving various computational problems for both standard and descriptor system representations.

**Keywords:** Coprime factorization; descriptor systems; stabilization; pole assignment; numerical algorithms.

## 1 Introduction

Let $G(s)$ or $G(z)$ be a given $p \times m$ rational *transfer-function matrix* (TFM) of a linear time-invariant continuous-time or discrete-time descriptor system, respectively, and let $(E, A, B, C, D)$ an equivalent $n$th order *regular* descriptor representation satisfying $G(\lambda) = C(\lambda E - A)^{-1}B + D$, where $\lambda$ is either the complex variable $s$ or $z$, depending on the type of the system. The regularity assumption means that $\det(\lambda E - A) \not\equiv 0$. If the TFM $G$ is not proper then the matrix $E$ is singular and let $r = \mathrm{rank}(E)$. We say that $G$ is *stable* if all its finite poles are in $\mathbb{C}^-$, where $\mathbb{C}^-$ denotes the stability region of the complex plane $\mathbb{C}$ and is either the left open complex half-plane for a continuous-time system or the interior of the unit circle for a discrete-time system. The instability region $\mathbb{C}^+$ is the complement of $\mathbb{C}^-$ with respect to $\mathbb{C}$. If the state space realization $(E, A, B, C, D)$ is minimal, then the poles of $G$ (finite and infinite) are the generalized eigenvalues of the pair $(E, A)$ denoted as $\Lambda(A, E)$.

A proper and stable TFM $G$ is *inner* if $G^*G = I$, where $G^*(s) = G^T(-s)$ in continuous-time and $G^*(z) = G^T(1/z)$ in discrete-time. Consider the inertia matrix $J = \mathrm{diag}(I_{m_1}, -I_{m_2})$ with $m_1 + m_2 = m$. A TFM $G$ is *J-inner* if $G^*JG = J$. A fractional representation of $G$ in the form $G = NM^{-1}$ with $N$ and $M$ stable rational matrices, is called a *right coprime factorization* (RCF) if there exist stable rational matrices $U$ and $V$ such that $UN + VM = I$. Analogously, a fractional representation of $G$ in the form $G = M^{-1}N$ with $N$ and $M$ stable rational matrices, is called a *left coprime factorization* (LCF) if there exist stable rational matrices $U$ and $V$ such that $NU + MV = I$. Several special factorizations could be of interest in particular applications.

The simplest factorization to obtain is when $M$ is proper and $N$ is proper or improper depending on if the original $G$ is proper or not. This factorization, with $M$ having possibly least order, is useful as a preliminary or as a final step in computing some other factorizations. A particular case of this factorization is when $M$ is *inner*. This factorization has several important applications in evaluating norms of TFMs or in computing spectral factors of TFMs. Provided $G$ is square ($p = m$), coprime factorizations of its inverse $G^{-1}$ are useful to compute alternative factorizations of rational TFMs, as for instance factorizations with minimum-phase factors or inner-outer factorizations. Factorizations in which both $N$ and $M$ are proper rational matrices can be viewed as alternative representations of rational matrices. Factorizations in which both $N$ and $M$ are polynomial matrices or both are proper rational matrices can be viewed as alternative representations of rational matrices. This factorization is potentially useful in performing order reduction of descriptor systems by using the coprime factors reduction approach analogously as in case of standard systems [4], [15]. Moreover, this factorization can ben used to compute factorizations in which both factors are polynomial matrices.

In this paper we propose numerically reliable algorithms for computing three of the above mentioned RCFs of rational TFMs, namely the factorizations with $M$ proper having least order, the factorization with $M$ inner and the factorization with both $M$ and $N$ proper. The same algorithms can be also used to compute LCFs by applying them to the dual TFM $G^T$. The proposed algorithms represent generalizations of similar algorithms for standard systems [15], [16] and are based on a recursive generalized Schur technique for pole assignment of descriptor systems [18]. The new procedures are generally applicable regardless the original descriptor state space representation is minimal or not, or is stabilizable/detectable. They are well suited for robust software implementations. The presented techniques can be also seen as extensions of the general recursive factorization approach introduced by Van Dooren [10].

## 2 Fractional representations: basic facts

The factorization algorithms proposed in this paper rely on simple facts concerning fractional representations.

**Fact 1.** *Any rational matrix $G$ with a stabilizable state-space realization $(E, A, B, C, D)$ has a RCF given by the following choice of the factors* [19]

$$\begin{array}{rcl} N & = & (E,\, A+BF,\, BW,\, C+DF,\, DW) \\ M & = & (E,\, A+BF,\, BW,\, F,\, W) \end{array} \qquad (1)$$

*where $F$ is chosen such that all finite eigenvalues of the pair $(E, A+BF)$ (at most $r$) are stable, the pencil $A + BF - \lambda E$ is regular and $W$ is an arbitrary invertible matrix.*

Particular factorizations with special properties, as for instance with inner denominator or with proper factors, can be determined by suitably choosing the matrix pair $(F, W)$. In some factorizations $W$ can be simply chosen as $W = I$. If necessary, $\widetilde{M} = M^{-1}$ can be explicitly evaluated as

$$\widetilde{M} = (E,\, A,\, B,\, -W^{-1}F,\, W^{-1})$$

The pair $(F, G)$ can be viewed as the free parameters which determines a particular factorization. The algorithms proposed in this paper use implicitly the more general expressions for the factors

$$\begin{array}{rcl} N & = & (UEV,\, U(A+BF)V,\, UBW,\, (C+DF)V,\, DW) \\ M & = & (UEV,\, U(A+BF)V,\, UBW,\, FV,\, W) \end{array} \tag{2}$$

where $U$ and $V$ are orthogonal transformation matrices (usually not accumulated). Although general, non-singular matrices $U$ and $V$ could be also considered as additional free parameters of RCFs, their role in the proposed algorithms is only to allow to obtain the resulting matrices in particular condensed forms or to preserve convenient condensed forms of matrices which lead to efficient implementation of the algorithms.

**Remark.** If the TFM $G$ is square, any algorithm to compute RCFs can be used to compute a LCF $G = M^{-1}N$ in which both factors are minimum-phase. This can be done by applying the algorithm to the inverse system

$$G^{-1} = \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix}, [\, 0 \; -I \,], 0 \right) \tag{3}$$

to compute the RCF $G^{-1} = \widetilde{N}\widetilde{M}^{-1}$ in the form (1) by using a feedback matrix partitioned as $F = [\, F_1 \;\; F_2 \,]$. It easy to verify that the factors of the minimum-phase LCF of $G$ are

$$\begin{array}{rcl} N & = & (\, E,\, A,\, B,\, W^{-1}(C+F_1),\, W^{-1}(D+F_2)\, ) \\ M & = & \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix}, -[\, W^{-1}F_1 \;\; W^{-1}F_2 \,], W^{-1} \right) \end{array}$$

with both $N = \widetilde{N}^{-1}$ and $M = \widetilde{M}^{-1}$ having zeros in $\mathbb{C}^-$. Note that in some factorizations, $G$ shonld be expressed as $G = \widetilde{M}N$. In this case we use directly the expression of $\widetilde{M}$ resulted from (1)

$$\widetilde{M} = \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C+F_1 & D+F_2 \end{bmatrix}, \begin{bmatrix} 0 \\ W \end{bmatrix}, [F_1 \; F_2], W \right)$$

□

**Fact 2.** *If $G = N_1M_1^{-1}$ and $N_1 = N_2M_2^{-1}$, then $G$ has the fractional representation $G = NM^{-1}$, where $N = N_2$ and $M = M_1M_2$.*

This simple fact allows us to obtain explicit formulas to update partial factorizations by using simple state space formulas. Let $N_1$ and $M_1$ be the factors computed as

$$\begin{array}{rcl} N_1 & = & (E,\, A+BF_1,\, BW_1,\, C+DF_1,\, DW_1) \\ M_1 & = & (E,\, A+BF_1,\, BW_1,\, F_1,\, W_1) \end{array} \tag{4}$$

and let $N_2$ and $M_2$ be the factors of $N_1$ computed as

$$\begin{array}{rcl} N_2 & = & (E,\, A+BF,\, BW,\, C+DF,\, DW) \\ M_2 & = & (E,\, A+BF,\, BW,\, F_2,\, W_2) \end{array} \tag{5}$$

where

$$\begin{array}{rcl} F & = & F_1 + W_1F_2 \\ W & = & W_1W_2 \end{array} \tag{6}$$

It easy to verify that the product $M_1M_2$ is given by

$$M_1M_2 = (E,\ A + BF,\ BW,\ F,\ W) \tag{7}$$

and thus equations (6) serve as explicit updating formulas of fractional representations. These formulas can be extended in a straightforward way to include arbitrary coordinate transformation matrices. If we denote $\widetilde{A} = A + BF_1$, $\widetilde{B} = BW_1$, $\widetilde{C} = C + DF_1$ and $\widetilde{B} = BW_1$, then the following formulas can be used simultaneously to update $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$:

$$\widetilde{A} \leftarrow \widetilde{A} + \widetilde{B}F_2, \quad \widetilde{B} \leftarrow \widetilde{B}W_2, \quad \widetilde{C} \leftarrow \widetilde{C} + \widetilde{D}F_2, \quad \widetilde{D} \leftarrow \widetilde{D}W_2 \tag{8}$$

All factorization algorithms presented in the paper rely on the use of such updating formulas. If $W_1 = I$ and $W_2 = I$, then the updating formulas (6) reduce to a very simple form

$$F = F_1 + F_2, \tag{9}$$

which is used in some of proposed algorithms. □

**Fact 3.** *An implicit updating technique of fractional representations is based on the following evident identities:*

$$\begin{bmatrix} G \\ I \end{bmatrix} = \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} M_1^{-1} = \begin{bmatrix} N_2 \\ M_1M_2 \end{bmatrix} (M_1M_2)^{-1} \tag{10}$$

It can be easily seen that the two successive numerator factors $\begin{bmatrix} N_1 \\ M_1 \end{bmatrix}$ and $\begin{bmatrix} N_2 \\ M_1M_2 \end{bmatrix}$ of the extended TFM $\begin{bmatrix} G \\ I \end{bmatrix}$ contain the elements of the successive factorizations $G = N_1M_1^{-1} = N_2(M_1M_2)^{-1}$. This implicit updating procedure is especially useful when combining different factorization algorithms because it is applicable even if the factors computed by different algorithms have different orders or if coordinate transformations are present in the representations of factors. Notice that the use of the updating formulas (6) requires that the two successive state space representations (4) and (5) have the same order. Otherwise it is not possible to obtain explicit updating formula as in (6) for the state feedback matrix. □

# 3 RCF with least order denominator

In this section we propose an algorithm to compute a RCF of $G$ with a least order $M$. The new algorithm has guaranteed numerical reliability and additionally it can handle even the case when the original descriptor system representation is not stabilizable. The basis for our algorithm is a pole assignment procedure described in [13] (see also [18]). This algorithm has the ability to keep unaltered the stable eigenvalues of the pair $(E, A)$ and to move only the unstable ones to stable locations by choosing an appropriate feedback matrix F. An additional useful feature of this algorithm is that simultaneously with the stabilizing F, it determines the generalized Schur form of the pair $(E, A + BF)$. This makes possible to extract easily a minimal realization for the denominator factor $M$.

The main steps of the generalized Schur algorithm are shortly explained below. Assume that the pair $(E, A)$ is already in a *generalized real Schur form* (GRSF), and the matrices $E$, $A$ and $B$ are partitioned conformally as

$$E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \ A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{11}$$

where the pair $(E_{22}, A_{22})$ has only unstable generalized eigenvalues. By choosing a feedback matrix of the form

$$F = [\,0\ F_2\,] \tag{12}$$

partitioned conformally with the matrices in (11), we see that

$$A + BF = \begin{bmatrix} A_{11} & A_{12} + B_1F_2 \\ 0 & A_{22} + B_2F_2 \end{bmatrix}$$

and thus the feedback perturbs only the generalized eigenvalues of the pair $(E_{22}, A_{22})$, the rest of generalized eigenvalues of the pair $(E, A)$ remaining unperturbed. In particular, if $E_{22}$ and $A_{22}$ are the last diagonal blocks in the GRSF (of order one or two), then the pair $(E, A + BF)$ is still in a GRSF. Provided $B_2 \neq 0$, the generalized eigenvalues of the pair $(E_{22}, A_{22} + B_2F_2)$ can be arbitrarily modified by suitably choosing $F_2$.

The stabilization of a given system can be performed by iteratively modifying the generalized eigenvalues of the pair $(E, A)$ as in the following conceptual algorithm:

1. Reduce the system matrices by using orthogonal similarity transformations such that the pair $(E, A)$ is in an ordered GRSF (11) with the pair $(E_{22}, A_{22})$ having only unstable generalized eigenvalues and the pair $(E_{11}, A_{11})$ having only stable or infinite generalized eigenvalues.

2. Determine a stabilizing feedback $F$ of the form (12) which moves the generalized eigenvalues of the last diagonal blocks of the pair $(E, A)$ into stable positions.

3. Update $A$ as $A+BF$; by using orthogonal similarity transformations, bring another pair of diagonal blocks with unstable eigenvalues in the last diagonal position of the pair $(E, A)$ and resume the previous step.

To ensure complete generality, a deflation mechanism can be included into the factorization algorithm to remove automatically the unstabilizable part of the system. Such deflation is possible by observing that if the generalized eigenvalues corresponding to the last diagonal blocks $E_{22}$ and $A_{22}$ are not controllable, then the corresponding $B_2$ should be zero. If we partition C accordingly with the matrices in (11) as

$$C = [\,C_1\ C_2\,]$$

then we can replace the original system $(E, A, B, C, D)$ with an input-output equivalent realization of lower order $(E_{11}, A_{11}, B_1, C_1, D)$ by simply deleting the rows and columns in matrices $E$, $A$, $B$ and $C$ which corresponds to the unstabilizable part. In this case the resulting coprime factorization has order less than $n$.

The following implementable state space algorithm to compute a RCF of a rational TFM $G$ materialises the above ideas.

**GRCF Algorithm.**

1. Reduce the pair $(E, A)$ by an orthogonal similarity transformation, to the ordered GRSF [6], [9]
$$\widetilde{E} = QEZ = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \; \widetilde{A} = QAZ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{13}$$
where $E_{11}, A_{11} \in \mathbb{R}^{q \times q}$, $Q$ and $Z$ are orthogonal matrices, $\Lambda(A_{11}, E_{11}) \subset \mathbb{C}^- \cup \{\infty\}$ and $\Lambda(A_{22}, E_{22}) \subset \mathbb{C}^+$. Compute $\widetilde{B} = QB$, $\widetilde{C} = CZ$ and set $\widetilde{F} = 0$.

2. If $q = n$, go to 8.

3. Let $\alpha$ and $\delta$ be the $k \times k$ last diagonal blocks of $\widetilde{A}$ and $\widetilde{E}$ respectively, and let $\beta$ the $k \times m$ matrix formed from the last $k$ rows of $\widetilde{B}$. If $\|\beta\| \leq \epsilon$ (a given tolerance), then $n \leftarrow n - k$ and go to 2.

4. Choose a $k \times k$ matrix $\gamma$ such that $\Lambda(\gamma) \subset \mathbb{C}^-$ and compute $\varphi = \beta^+(\delta\gamma - \alpha)$.

5. Compute $\widetilde{A} \leftarrow \widetilde{A} + \widetilde{B}[0 \; \varphi]$, $\widetilde{F} \leftarrow \widetilde{F} + [0 \; \varphi]$ .

6. Compute the orthogonal similarity transformation matrices $\widetilde{Q}$ and $\widetilde{Z}$ which move the last $k \times k$ blocks of $\widetilde{A}$ and $\widetilde{E}$ to positions $(q+1, q+1)$ by interchanging the diagonal blocks of the GRSF; apply the transformations: $\widetilde{E} \leftarrow \widetilde{Q}\widetilde{E}\widetilde{Z}$, $\widetilde{A} \leftarrow \widetilde{Q}\widetilde{A}\widetilde{Z}$, $\widetilde{B} \leftarrow \widetilde{Q}\widetilde{B}$, $\widetilde{C} \leftarrow \widetilde{C}\widetilde{Z}$, $\widetilde{F} \leftarrow \widetilde{F}\widetilde{Z}$.

7. Put $q \leftarrow q + k$ and go to 2.

8. Put $N = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C} + D\widetilde{F}, D)$, $M = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{F}, I)$.

The resulting pair $(\widetilde{E}, \widetilde{A})$ is in a GRSF. If the original system is stabilizable, then $\widetilde{E}$ and $\widetilde{A}$ contain the matrices $UEV$ and $U(A+BF)V$, respectively, where $U$ and $V$ are the accumulated orthogonal transformations performed at steps 1 and 6 of the algorithm, and $F$ is the stabilizing feedback matrix $\widetilde{F}V^T$. If the original system is not stabilizable, then the unstabilizable blocks are detected at step 3 and the corresponding unstabilizable parts are deflated by simply decreasing the order of system with $k$. If unstabilizable blocks are detected by the algorithm then the resulting factors have order less than $n$.

One of the advantages of the resulting form of matrices of the computed factors is that a minimal realization of $M$ can be easily determined. The resulting $\widetilde{F}$ has always the form
$$\widetilde{F} = [0 \; \widetilde{F}_2], \tag{14}$$
where the number of columns of $\widetilde{F}_2$ equals the number of unstable controllable generalized eigenvalues of the pair $(E, A)$. By partitioning accordingly the resulting $\widetilde{E}$, $\widetilde{A}$ and $\widetilde{B}$
$$\widetilde{E} = \begin{bmatrix} \widetilde{E}_{11} & \widetilde{E}_{12} \\ 0 & \widetilde{E}_{22} \end{bmatrix}, \; \widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ 0 & \widetilde{A}_{22} \end{bmatrix}, \; \widetilde{B} = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{bmatrix}, \tag{15}$$
then $(\widetilde{E}_{22}, \widetilde{A}_{22}, \widetilde{B}_2, \widetilde{F}_2, I)$ is a minimal realization of $M$. Because $\widetilde{E}_{22}$ is invertible, the TFM $M$ is always proper. However generally the factor $N$ is not proper if the system has impulsive modes.

**Remark 1.** If in the descriptor representation of $G$ the unstable controllable eigenvalues of the pair $(E, A)$ are observable, then the order of the minimal realization of $M$ is simultaneously the least order of all possible proper denominators in a RCF of $G$. Notice however that although the resulting descriptor representation of $M$ is always minimal, the order of $M$ can be higher then the least possible order if some unstable eigenvalues of $(E, A)$ are controllable but not observable. For example consider the non-minimal descriptor representation of the transfer function $G(s) = 1/(s+1)$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

The least order proper denominator of a RCF of $G$ is evidently $M = 1$. By choosing a feedback matrix $F = [0 \ -1]$ which assigns the eigenvalues of the pair $(E, A + BF)$ in $\{-1, \ -1\}$, we obtain the following factors of the RCF of $G$:

$$N(s) = \frac{s}{(s+1)^2}, \quad M(s) = \frac{s}{s+1}$$

The increase of the order happens because the chosen feedback makes the modified eigenvalue observable. □

**Remark 2.** The above remark leads us to a very simple procedure to determine a factorization with an arbitrary order greater than the least possible order. First we apply Algorithm GRCF till completion but without performing its last step (step 8) which assembles the elements of the resulting factors. Assume that the minimal realization of the denominator which would result has the least possible order. (This is always possible to arrange by removing before applying Algorithm GRCF, all unobservable finite eigenvalues. A very efficient numerically stable procedure proposed in [14] can be used for this purpose.) Instead performing the last step of Algorithm GRCF, we replace the system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, D)$ by the following system of order increased by one having the same TFM:

$$\left( \begin{bmatrix} \widetilde{E} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \widetilde{B} \\ \beta \end{bmatrix}, \begin{bmatrix} \widetilde{C} & 0 \end{bmatrix} \right)$$

The unstable eigenvalue in origin is not observable, but is controllable provided we choose a non-zero last row $\beta$. We replace $\widetilde{F}$ by $[\widetilde{F} \ \ 0]$ and perform again steps 3-7 of Algorithm GRCF. The resulting $M$ after this computation has now the order increased by one. By repeating several times this procedure, we can determine the denominator with any desired order. □

**Remark 3.** The choice of eigenvalues to be assigned at step 4 can be performed in several ways. A convenient choice for software implementations is to ensure for the resulting factors a prescribed stability degree. For instance, in the continuous-time case a stability degree $\delta < 0$ can be ensured by moving all eigenvalues lying outside the stability region $\mathbb{C}^-$ to positions with real parts equal to $\delta$ and unmodified imaginary parts. In the discrete-time case, a stability degree $\delta$, with $0 < \delta < 1$, can be ensured by assigning the unstable eigenvalues to values with moduli equal to $\delta$. The algorithm can be also implemented to factorize a given rational matrix with respect to an arbitrary symmetric region of the complex plan. This is always possible because of the freedom offered by the pole assignment technique in moving the controllable eigenvalues to arbitrary locations in the complex plane. □

**Remark 4.** The GRCF Algorithm is based on a generalization of a pole assignment algorithm for standard systems [11]. The roundoff error analysis of that algorithm [12] revealed that if each partial feedback matrix of the form $K = [0 \ \varphi]$, computed at step 5 satisfies the condition $||K|| \leq ||A||/||B||$, then the pole assignment algorithm is numerically backward stable. This condition is also applicable in our case, because it is independent of the presence of the E matrix. We note however that unfortunately this condition can not be always fulfilled if large gains are necessary to stabilize the system. This can arise either if the unstable poles are too "far" from the stable region or if these poles are weekly controllable. □

# 4 RCF with inner denominator

We assume in this section that $G$ has no poles on the imaginary axis in continuous-time case or on the unit circle in the discrete-time case. The algorithm to compute the *right coprime factorization with inner denominator* (RCFID) of a rational TFM $G$ use recursively the following formulas to compute the RCFID of a particular class of systems.

**Fact 4.** *Let $G = (E, A, B, C, D)$ a controllable descriptor representation with $E$ non-singular and $\Lambda(E, A) \in \mathbb{C}^+$. Then $M = (E, A + BF,\ BW,\ F,\ W)$ is* inner *by choosing $F$ and $W$ as:*

$$\left.\begin{array}{l} F = -B^T(YE^T)^{-1}, \quad W = I \\ AYE^T + EYA^T - BB^T = 0 \end{array}\right\} \quad (continuous - time)$$

$$\left.\begin{array}{l} F = -B^T(EYE^T + BB^T)^{-1}A \\ W = (I + B^T(EYE^T)^{-1}B)^{-1/2} \\ AYA^T - BB^T = EYE^T \end{array}\right\} \quad (discrete - time)$$

The above expressions represent straightforward transcriptions of analogous formulas for standard systems [16]. In the following algorithm, we use these formulas (at step 4) to compute inner denominators for simple systems of orders at most two.

**GRCFID Algorithm.**

1. Find orthogonal matrices $Q$ and $Z$ to reduce the pair $(E, A)$ to the ordered GRSF (13), where $E_{11}, A_{11} \in \mathbb{R}^{q\times q}$, $Q$ and $Z$ are orthogonal matrices, $\Lambda(A_{11}, E_{11}) \subset \mathbb{C}^- \cup \{\infty\}$ and $\Lambda(A_{22}, E_{22}) \subset \mathbb{C}^+$. Compute $\widetilde{B} = QB$, $\widetilde{C} = CZ$. Set $\widetilde{F} = 0$, $\widetilde{W} = I$.
2. If $q = n$, go to 7.
3. Let $(\delta, \alpha)$ be the last diagonal blocks of $(\widetilde{E}, \widetilde{A})$ of order $k$ and let $\beta$ be the $k \times m$ matrix formed from the last $k$ rows of $\widetilde{B}$. If $||\beta|| \leq \epsilon$ (a given tolerance), then $n \leftarrow n - k$ and go to 2.
4. For the system $(\delta, \alpha, \beta, *, *)$ compute $\varphi$ and $V$ such that $(\delta, \alpha + \beta\varphi,\ \beta V, \varphi, V)$ is inner. Set $K = [\,0\ \varphi\,]$.
5. Compute $\widetilde{A} \leftarrow \widetilde{A} + \widetilde{B}K$, $\widetilde{F} \leftarrow \widetilde{F} + \widetilde{W}K$, $\widetilde{W} \leftarrow \widetilde{W}V$.
6. Compute the orthogonal matrices $\widetilde{Q}$ and $\widetilde{Z}$ to move the last blocks of $(\widetilde{E}, \widetilde{A})$ to positions $(q+1, q+1)$ by interchanging the diagonal blocks of the GRSF. Compute $\widetilde{E} \leftarrow \widetilde{Q}\widetilde{E}\widetilde{Z}$, $\widetilde{A} \leftarrow \widetilde{Q}\widetilde{A}\widetilde{Z}$, $\widetilde{B} \leftarrow \widetilde{Q}\widetilde{B}$, $\widetilde{C} \leftarrow \widetilde{C}\widetilde{Z}$, $\widetilde{F} \leftarrow \widetilde{F}\widetilde{Z}$. Put $q \leftarrow q + k$ and go to 2.
7. $N = (\widetilde{E}, \widetilde{A}, \widetilde{B}\widetilde{W}, \widetilde{C} + D\widetilde{F}, D\widetilde{W})$, $M = (\widetilde{E}, \widetilde{A}, \widetilde{B}\widetilde{W}, \widetilde{F}, \widetilde{W})$.

A minimal realization for the inner factor $M$ is given by $(\widetilde{E}_{22}, \widetilde{A}_{22}, \widetilde{B}_2\widetilde{W}, \widetilde{F}_2, \widetilde{W})$ and can be determined from the partitioning (14) and (15) of the resulting $\widetilde{F}$, $\widetilde{E}$, $\widetilde{A}$ and $\widetilde{B}$.

The above algorithm relies exclusively on reliable numerical techniques. It can be viewed as a pole assignment algorithm which assigns the unstable poles in symmetrical positions with respect to the imaginary axis in the continuous-time case or the unit circle in the discrete-time case. Because practically there is no freedom in assigning the poles, it is to be expected that the algorithm perform in a numerically stable way only if the norms of the elementary feedback matrices $K$ computed at step 4 are not too high.

A similar algorithm can be devised to compute a RCF with J-inner denominator, a generalization of the RCFID. Let

$$J = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}$$

be an innertia matrix such that $m_1 + m_2 = m$. We have the following fact generalizing the results of **Fact 4.**

**Fact 5.** *Let $G = (E, A, B, C, D)$ a controllable descriptor representation with $E$ non-singular and $\Lambda(E, A) \in \mathbb{C}^+$. Then $M = (E, A + BF, BW, F, W)$ is* J-inner *by choosing $F$ and $W$ as:*

$$\left.\begin{array}{l} F = -JB^T(YE^T)^{-1}, \quad W = I \\ AYE^T + EYA^T - BJB^T = 0, \quad Y \geq 0 \end{array}\right\} \quad (continuous - time)$$

$$\left.\begin{array}{l} F = -JB^T(EYE^T + BJB^T)^{-1}A \\ W^T(J + B^T(EYE^T)^{-1}B)W = J \\ AYA^T - BJB^T = EYE^T, \quad Y \geq 0 \end{array}\right\} \quad (discrete - time)$$

These formulas can be used at step 4 of the GRCFID Algorithm to compute the RCF with J-inner denominator of a TFM. Notice that at each iteration, the positive definiteness of the solution $Y$ of above Lyapunov equation should be additionally checked. If the positivity check fails, then the given TFM has no RCF with J-inner denominator. For the discrete-time case the matrix $W$ can be computed in the form

$$W = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix}$$

where with $X = (EYE^T)^{-1}$ and $B = [\, B_1 \;\; B_2 \,]$

$$\begin{array}{rcl} W_{11} & = & (I + B_1^T X B_1)^{-\frac{1}{2}} \\ W_{22} & = & (I - B_2^T X B_2 + B_2^T X B_1 W_{11}^2 B_1^T X B_2)^{-\frac{1}{2}} \\ W_{12} & = & -W_{11}^2 B_1^T X B_2 W_{22} \end{array}$$

**Remark.** If the TFM $G$ is square and stable, the GRCFID Algorithm can be used to compute an *inner-outer* (*J-inner-outer*) factorization $G = MN$, where $M$ is inner (J-inner) and $N$ is outer (minimum-phase and stable). This can be done by applying the algorithm to the inverse system (3). This algorithm can be also used in the case of a rectangular $G$ to compute the inner (J-inner) denominator factor $M$ of a suitable left or right inverse of $G$ and then to compute the outer factor $N$ as $N = M^{-1}G$. The method is described in details in a forthcoming paper [17]. □

## 5 RCF with proper factors

If the given system $(E, A, B, C, D)$ has impulsive modes, that is, the finite generalized eigenvalues are fewer than $r = \text{rank}(E)$, then the $N$ factor resulting from the GRCF Algorithm has impulsive modes too and therefore is not proper. A trivial example is when $G$ is a polynomial matrix in which case the factors are simply $N = G$ and $M = I$. If however $G$ is impulse free, the numerator factor $N$ computed by the GRCF Algorithm is also impulse free. This observation leads to the following conceptual approach to compute a *proper right comprime factorization* (PRCF), that is a factorization in which both factors are proper and stable:

1. Compute a factorization of $G$ in the form $G = N_1M_1^{-1}$, where both factors are proper but possibly unstable.
2. Compute a RCF of $\begin{bmatrix} N_1 \\ M_1 \end{bmatrix}$ in the form $\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \widetilde{M_2}^{-1}$ by using the GRCF Algorithm and define $N = N_2$ and $M = M_2$.

It is easy to see that $G = NM^{-1}$ is the desired PRCF.

The above two steps can be related to the two main steps of an S-stabilization (*strong-stabilization*) algorithm proposed recently in [18]. Assume for the moment that the given descriptor representation of $G$ is *strongly stabilizable*, that is, $\text{rank}([\,\lambda E - A\ B\,]) = n$ for all finite $\lambda \in \mathbb{C}^+$ and $\text{rank}([\,E\ AS_\infty\ B\,]) = n$, where the columns of $S_\infty$ span the null space of $E$. In the mentioned algorithm a preliminary state feedback $F_1$ is determined to move all impulsive modes to finite locations. Then a second partial feedback $F_2$ is used to perform the stabilization of perturbed system. These partial feedback matrices can be used then to define the PRCF of $G$ according to (2).

We sketch shortly the procedure to compute $F_1$ which defines the factors $N_1$ and $M_1$ at the first step. We can assume that the given system has no uncontrollable infinite poles, that is, $\text{rank}([\,E\ B\,]) = n$. If this condition is not fulfilled, then the given descriptor representation of the TFM $G$ is not minimal and contains uncontrollable infinite poles. These poles can be removed by using the Algorithm 1 presented in the Appendix. The computation of a preliminary feedback $F_1$ which moves the impulsive modes to finite locations is based on reducing the system matrices $E$, $A$ and $B$ to special condensed forms by using transformations of the form $\widetilde{E} = UEV$, $\widetilde{A} = UAV$, $\widetilde{B} = UB$ with $U$ and $V$ orthogonal matrices. Specifically, $U$ and $V$ can be determined such that the matrices of the transformed system have the forms

$$\widetilde{E} = UEV = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \widetilde{A} = UAV = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & 0 & 0 \end{bmatrix}, \ \widetilde{B} = UB = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{16}$$

where $E_{11} \in \mathbb{R}^{r,r}$ is non-singular, $\begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$ has maximal row rank, and $A_{22}$ is also non-singular. This reduction is always possible if the condition $\text{rank}([\,E\ B\,]) = n$ is fulfilled, that is the system is controllable at infinity. The state feedback matrix $F_1$ can be computed as $F_1 = [\,0\ 0\ F_{13}\,]V^T$, where $F_{13}$ is chosen such that the matrix $B_3F_{13}$ is non-singular. We have immediately the condition $\text{rank}([\,E\ (A + BF_1)S_\infty\,]) = n$, satisfied and thus the system $(E, A + BF_1, B)$ is regular and has $r$ finite and $n - r$ infinite poles [1].

**PGRCF Algorithm.**

1. Find orthogonal matrices $U_1$ and $V_1$ such that

$$E \leftarrow U_1 E V_1 := \begin{bmatrix} E_{11}^1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $E_{11}^1 \in \mathbb{R}^{r,r}$ is non-singular and upper-triangular, and set

$$A \leftarrow U_1 A V_1 := \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}, \quad B \leftarrow U_1 B := \begin{bmatrix} B_1^1 \\ B_2^1 \end{bmatrix}, \quad C \leftarrow CV_1.$$

2. If $B_2^1$ has no full row rank, perform Algorithm 1 (from Appendix) on the system $(E, A, B, C, D)$ and determine a reduced order system controllable at infinity with the matrices having the same form as at step 1.

3. Find orthogonal matrices $U_2$ and $V_2$ such that $U_2 A_{22}^1 V_2 = \begin{bmatrix} A_{22}^2 & A_{23}^2 \\ 0 & 0 \end{bmatrix}$, where $A_{22}^2$ is non-singular, and set

$$E := \begin{bmatrix} E_{11}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A \leftarrow \begin{bmatrix} I_r & 0 \\ 0 & U_2 \end{bmatrix} A \begin{bmatrix} I_r & 0 \\ 0 & V_2 \end{bmatrix} := \begin{bmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 \\ A_{21}^2 & A_{22}^2 & A_{23}^2 \\ A_{31}^2 & 0 & 0 \end{bmatrix},$$

$$B \leftarrow \begin{bmatrix} I_r & 0 \\ 0 & U_2 \end{bmatrix} B := \begin{bmatrix} B_1^2 \\ B_2^2 \\ B_3^2 \end{bmatrix}, \quad C \leftarrow C \begin{bmatrix} I_r & 0 \\ 0 & V_B \end{bmatrix}$$

4. Determine $F_1 = [\,0\ 0\ F_{13}\,]$ such that $B_3^2 F_{13}$ is non-singular and set

$$A \leftarrow A + BF_1 = \begin{bmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 + B_1^2 F_{13} \\ A_{21}^2 & A_{22}^2 & A_{23}^2 + B_2^2 F_{13} \\ A_{31}^2 & 0 & B_3^2 F_{13} \end{bmatrix}$$

5. Apply the GRCF Algorithm to $\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} := \left( E,\ A,\ B, \begin{bmatrix} C + DF_1 \\ F_1 \end{bmatrix}, \begin{bmatrix} D \\ I \end{bmatrix} \right)$ to compute the factors

$$\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} = \left( \widetilde{E},\ \widetilde{A},\ \widetilde{B}, \begin{bmatrix} \widetilde{C} \\ \widetilde{F} \end{bmatrix}, \begin{bmatrix} D \\ I \end{bmatrix} \right), \quad \widetilde{M_2} = (\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{F},\ I)$$

6. Put $N = (\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C},\ D)$ and $M = (\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{F},\ I)$.

The algorithm uses at steps 1 and 2 exclusively orthogonal transformations. The details of step 1 are discussed in the Appendix. The row compressions at step 3 can be performed by using the rank revealing QR-decomposition [2].

The procedure is intended for an efficient modularized implementation. Many algorithmic details, as for example exploiting and preserving particular (triangular) shapes of various submatrices can improve suplimentary the efficiency of computations. For example, before performing step 5, it is possible to reduce first the pencil $\lambda E - A$ to a block upper triangular form by annihilating the submatrices $A_{21}^2$ and $A_{31}^2$ of $A$ and by preserving simultaneously the upper triangular shape of $E$. This form allows to extract immediately the non-dynamic part of the system by including it into an appropriate feedthrough matrix. Step 5 can be then performed on a descriptor representation of smaller order. This approach can be particularly useful when the computed PRCF is intended to be used for coprime factors model reduction [4].

**Remark.** From the matrices computed by the PRCF Algorithm it is not possible to extract immediately a least order minimal realization for $M$. If such a factorization is of interest, then an alternative procedure is more appropriate. We sketch only the main steps of this procedure.

1. Apply Algorithm 2 (see Appendix 2) to reduce the pair $(E, A)$ by using the orthogonal transformation matrices $Q$ and $Z$, to the block upper triangular form

$$\bar{E} = QEZ = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \; \bar{A} = QAZ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

   where the pair $(E_{11}, A_{11})$ has only finite generalized eigenvalues and the pair $(E_{22}, A_{22})$ has only infinite generalized eigenvalues. Compute $\bar{B} = QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $\bar{C} = CZ = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

2. Compute $F_2$ such that the pair $(E_{22}, A_{22} + B_2F_2)$ has rank$(E_{22})$ stable finite eigenvalues and the pencil $A_{22} + B_2F_2 - \lambda E_{22}$ is regular. (This is always possible if the descriptor system is strongly stabilizable.)

3. Apply the GRCF Algorithm to the descriptor system

$$\left( \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} + B_1F_2 \\ 0 & A_{22} + B_2F_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 + DF_2 \\ 0 & F_2 \end{bmatrix}, \begin{bmatrix} D \\ I \end{bmatrix} \right)$$

   to compute the factors

$$\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} = \left( \widetilde{E}, \widetilde{A}, \widetilde{B}, \begin{bmatrix} \widetilde{C} \\ \widetilde{F} \end{bmatrix}, \begin{bmatrix} D \\ I \end{bmatrix} \right), \quad \widetilde{M}_2 = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{F}, I)$$

4. Put $N = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, D)$ and $M = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{F}, I)$.

This algorithm is less efficient than the PRCF Algorithm. However the computed form of matrices is appropriate to obtain a minimal realization of a least order factor $M$.

# 6 Conclusions

Efficient numerically reliable algorithms for computing several RCFs have been proposed. They are well suited for robust and modular software implementations. The algorithms are based on a recursive generalized Schur technique for pole assignment by using proportional state-feedback. This technique can be extended in a straightforward way to use derivative state-feedback too, leading to alternative algorithms for computing RCFs. The derivative feedback also allows to compute other useful factorizations as for instance RCFs with polynomial factors. It is still an open question the existence of general recursive algorithms for computing other factorizations as for example the inner–outer factorization for non-square systems, the normalized coprime factorization, the spectral and J–lossless factorizations.

# Appendix 1.

Let $(E, A, B, C, D)$ be a given $n$-th order *regular* descriptor system with the system matrices $E$, $A$ and $B$ having the conformally partitioned forms

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $E_{11} \in \mathbb{R}^{r\times r}$ is non-singular and upper-triangular. The following algorithm determines a reduced $n'$-th order descriptor representation $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, D)$ with the same TFM as the given system, with no uncontrollable infinite poles. The matrices $\widetilde{E}$, $\widetilde{A}$ and $\widetilde{B}$ have the following conformally partitioned forms

$$\widetilde{E} = \begin{bmatrix} \widetilde{E}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{bmatrix},$$

where $\widetilde{E}_{11} \in \mathbb{R}^{r'\times r'}$ is non-singular and upper-triangular and $\widetilde{B}_2$ has full row rank. We have that $\operatorname{rank}([\,\widetilde{E}\ \widetilde{B}\,]) = n'$, that is, the resulting system is controllable at infinity.

**Algorithm 1.**

1. Set up the system matrices as

$$E := \begin{bmatrix} E_{11}^1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}, \quad B := \begin{bmatrix} B_1^1 \\ B_2^1 \end{bmatrix}.$$

2. Find an orthogonal matrix $U_1$ such that $U_1 B_2^1 = \begin{bmatrix} B_2^2 \\ 0 \end{bmatrix}$, where $B_2^2 \in \mathbb{R}^{q\times m}$ has full row rank $q$. If $q = n - r$, then *Stop*; else set

$$E := \begin{bmatrix} E_{11}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A \leftarrow \begin{bmatrix} I_r & 0 \\ 0 & U_1 \end{bmatrix} A := \begin{bmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 \\ A_{21}^2 & A_{22}^2 & A_{23}^2 \\ A_{31}^2 & A_{32}^2 & A_{33}^2 \end{bmatrix}, \quad B \leftarrow \begin{bmatrix} I_r & 0 \\ 0 & U_1 \end{bmatrix} B := \begin{bmatrix} B_1^2 \\ B_2^2 \\ 0 \end{bmatrix}.$$

3. Find orthogonal matrices $U_2$ and $V_2$ such that $U_2[A_{32}^2\ A_{33}^2]V_2 = \begin{bmatrix} 0 & 0 \\ 0 & A_{44}^3 \end{bmatrix}$, whith $A_{44}^3 \in \mathbb{R}^{s\times s}$ non-singular, and set

$$A \leftarrow \begin{bmatrix} I_{r+q} & 0 \\ 0 & U_2 \end{bmatrix} A \begin{bmatrix} I_r & 0 \\ 0 & V_2 \end{bmatrix} := \begin{bmatrix} A_{11}^3 & A_{12}^3 & A_{13}^3 & A_{14}^3 \\ A_{21}^3 & A_{22}^3 & A_{23}^3 & A_{24}^3 \\ A_{31}^3 & 0 & 0 & 0 \\ A_{41}^3 & 0 & 0 & A_{44}^3 \end{bmatrix},$$

$$E := \begin{bmatrix} E_{11}^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} B_1^3 \\ B_2^3 \\ 0 \\ 0 \end{bmatrix}, \quad C \leftarrow C \begin{bmatrix} I_r & 0 \\ 0 & V_2 \end{bmatrix}.$$

4. Find orthogonal $U_3$ and $V_3$ such that $[A_{41}^3\ 0\ 0\ A_{44}^3]V_3 = [0\ 0\ 0\ A_{44}^4]$ with $A_{44}^4$ non-singular and $U_3[E_{11}^3\ 0\ 0\ 0]V_3 = [E_{11}^4\ 0\ 0\ 0\ E_{14}^4]$ with $E_{11}^4$ non-singular and upper-triangular. Set

$$E \leftarrow \begin{bmatrix} U_3 & 0 \\ 0 & I_{n-r} \end{bmatrix} EV_3 := \begin{bmatrix} E_{11}^4 & 0 & 0 & E_{14}^4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A \leftarrow \begin{bmatrix} U_3 & 0 \\ 0 & I_{n-r} \end{bmatrix} AV_3 := \begin{bmatrix} A_{11}^4 & A_{12}^4 & A_{13}^4 & A_{14}^4 \\ A_{21}^4 & A_{22}^4 & A_{23}^4 & A_{24}^4 \\ A_{31}^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44}^4 \end{bmatrix},$$

$$B \leftarrow \begin{bmatrix} U_3 & 0 \\ 0 & I_{n-r} \end{bmatrix} B := \begin{bmatrix} B_1^4 \\ B_2^4 \\ 0 \\ 0 \end{bmatrix}, \quad C \leftarrow CV_3 := [C_1^4\ C_2^4\ C_3^4\ C_4^4]$$

5. Remove the uncontrollable part: $n \leftarrow n - s$ and set

$$E := \begin{bmatrix} E_{11}^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} A_{11}^4 & A_{12}^4 & A_{13}^4 \\ A_{21}^4 & A_{22}^4 & A_{23}^4 \\ A_{31}^4 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} B_1^4 \\ B_2^4 \\ 0 \end{bmatrix}, \quad C := [C_1^4\ C_2^4\ C_3^4]$$

If $n = r + q$ then *Stop.*

6. Find orthogonal matrices $U_4$ and $V_4$ such that $A_{31}^4 V_4 = [0\ A_{42}^5]$ with $A_{42}^5$ non-singular and $U_4 E_{11}^4 V_4$ upper triangular. Set

$$E \leftarrow \begin{bmatrix} U_4 & 0 \\ 0 & I_{n-r} \end{bmatrix} E \begin{bmatrix} V_4 & 0 \\ 0 & I_{n-r} \end{bmatrix} := \begin{bmatrix} E_{11}^5 & E_{12}^5 & 0 & 0 \\ 0 & E_{22}^5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A \leftarrow \begin{bmatrix} U_4 & 0 \\ 0 & I_{n-r} \end{bmatrix} A \begin{bmatrix} V_4 & 0 \\ 0 & I_{n-r} \end{bmatrix} := \begin{bmatrix} A_{11}^5 & A_{12}^5 & A_{13}^5 & A_{14}^5 \\ A_{21}^5 & A_{22}^5 & A_{23}^5 & A_{24}^5 \\ A_{31}^5 & A_{32}^5 & A_{33}^5 & A_{34}^5 \\ 0 & A_{42}^5 & 0 & 0 \end{bmatrix},$$

$$B \leftarrow \begin{bmatrix} U_4 & 0 \\ 0 & I_{n-r} \end{bmatrix} B := \begin{bmatrix} B_1^5 \\ B_2^5 \\ B_3^5 \\ 0 \end{bmatrix}, \quad C \leftarrow C \begin{bmatrix} V_4 & 0 \\ 0 & I_{n-r} \end{bmatrix} := [C_1^5\; C_2^5\; C_3^5\; C_4^5].$$

7. Remove the uncontrollable part: $t = n - r - q$, $n \leftarrow n - t$, $r \leftarrow r - t$, set

$$E := \left[\begin{array}{c|c} E_{11}^1 & 0 \\ \hline 0 & 0 \end{array}\right] := \left[\begin{array}{c|cc} E_{11}^5 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

$$A := \left[\begin{array}{c|c} A_{11}^1 & A_{12}^1 \\ \hline A_{21}^1 & A_{22}^1 \end{array}\right] := \left[\begin{array}{c|cc} A_{11}^5 & A_{13}^5 & A_{14}^5 \\ \hline A_{21}^5 & A_{23}^5 & A_{24}^5 \\ A_{31}^5 & A_{33}^5 & A_{34}^5 \end{array}\right], \quad B := \left[\begin{array}{c} B_1^1 \\ \hline B_2^1 \end{array}\right] := \left[\begin{array}{c} B_1^5 \\ \hline B_2^5 \\ B_3^5 \end{array}\right], \quad C := [C_1^5\; C_3^5\; C_4^5]$$

and go to 2.

The matrices of the reduced order descriptor system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, D)$ result in the place of the corresponding matrices of the original system. The dimension $n$ is updated accordingly. No computational overhead occurs when using Algorithm 1 on a system with no uncontrollable infinite poles. In such a case, the algorithm exits at step 2 and determines the system matrices in the required form for Algorithm PGRCF. Note that Algorithm 1 applied to the system $(A, E, B, C, D)$ (notice that $A$ and $E$ are interchanged) can be used to remove the uncontrollable poles in the origin.

This algorithm uses exclusively orthogonal transformations and therefore is numerically backward stable. The rank revealing orthogonal decompositions, as for instance the complete orthogonal decompositions at steps 1 and 3 or the row compression at step 2 can be computed either by using the singular value decomposition or the rank revealing QR-decomposition [2] combined if necessary with RQ-decomposition. The latter alternative is substantially cheaper than the first one and usually possesses the same reliability in determining the ranks of matrices [3]. The special column compressions at step 4 and 6 in which simultaneously the upper triangular shape of $E$ is preserved can be performed by using a technique similar to that described in detail in [14].

**Remark.** Algorithm 1 can be thought as a particularization of the procedure proposed in [5] to compute the zeros of the system matrix

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]$$

In our case, $\mathcal{S}(\lambda)$ has the particular form

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \end{array}\right]$$

and only the zeros at infinity (the uncontrollable infinite poles) are computed. Additionally to [5], Algorithm 1 applies the performed orthogonal transformation to the matrix $C$ too. The computational complexity of the algorithm is $0(n^3)$. By specialization of the algorithm of [5] to the pencil

$$\mathcal{S}(\lambda) = \left[\begin{array}{c} A - \lambda E \\ \hline C \end{array}\right]$$

a similar procedure can be developed to remove the unobservable infinite poles. □

# Appendix 2.

Let $(E, A, B, C, D)$ an $n$-th order descriptor system with the matrices $E$ and $A$ be conformally partitioned in the forms

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $E_{11} \in \mathbb{R}^{r,r}$ is non-singular and upper-triangular. We assume that the pencil $A - \lambda E$ is *regular*. The algorithm given below determines the orthogonal transformation matrices $Q$ and $Z$ such that the orthogonally similar system

$$(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, D) := (QEZ,\ QAZ,\ QB,\ CZ,\ D)$$

has the matrices $\widetilde{E}$ and $\widetilde{A}$ in the form

$$\widetilde{E} = \begin{bmatrix} \widetilde{E}_{11} & \widetilde{E}_{12} \\ 0 & \widetilde{E}_{22} \end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ 0 & \widetilde{A}_{22} \end{bmatrix},$$

where the pair $(\widetilde{E}_{11}, \widetilde{A}_{11})$ has only finite generalized eigenvalues and the pair $(\widetilde{E}_{22}, \widetilde{A}_{22})$ has only infinite generalized eigenvalues.

**Algorithm 2.**

1. Set $j = 1$, $n_1 = n$, $r_1 = r$, $E_{11}^{(1)} = E_{11}$, $A_{11}^{(1)} = A_{11}$, $A_{12}^{(1)} = A_{12}$, $A_{21}^{(1)} = A_{21}$, $A_{22}^{(1)} = A_{22}$.

2. Compute the orthogonal matrices $Q_j$ and $Z_j$ such that

$$Q_j \begin{bmatrix} E_{11}^{(j)} & 0 \\ 0 & 0 \end{bmatrix} Z_j = \left[\begin{array}{cc|c} E_{11}^{(j+1)} & 0 & E_{13}^{(j+1)} \\ 0 & 0 & E_{23}^{(j+1)} \\ \hline 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\, r_{j+1} \\ \}\, n_j - r_j \\ \}\, \mu_j \end{array}$$

$$Q_j \begin{bmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{bmatrix} Z_j = \left[\begin{array}{cc|c} A_{11}^{(j+1)} & A_{12}^{(j+1)} & A_{13}^{(j+1)} \\ A_{21}^{(j+1)} & A_{22}^{(j+1)} & A_{23}^{(j+1)} \\ \hline 0 & 0 & A_{33}^{(j+1)} \end{array}\right] \begin{array}{l} \}\, r_{j+1} \\ \}\, n_j - r_j \\ \}\, \mu_j \end{array}$$

   where $E_{11}^{(j+1)}$ and $A_{33}^{(j+1)}$ are non-singular and upper-triangular and $E_{23}^{(j+1)}$ has full column rank.

3. Set

$$E \leftarrow \begin{bmatrix} Q_j & 0 \\ 0 & I_{n-n_j} \end{bmatrix} E \begin{bmatrix} Z_j & 0 \\ 0 & I_{n-n_j} \end{bmatrix},$$

$$A \leftarrow \begin{bmatrix} Q_j & 0 \\ 0 & I_{n-n_j} \end{bmatrix} A \begin{bmatrix} Z_j & 0 \\ 0 & I_{n-n_j} \end{bmatrix}, \quad B \leftarrow \begin{bmatrix} Q_j & 0 \\ 0 & I_{n-n_j} \end{bmatrix} B, \quad C \leftarrow C \begin{bmatrix} Z_j & 0 \\ 0 & I_{n-n_j} \end{bmatrix}$$

4. If $\mu_j > 0$, then $n_{j+1} = n_j - \mu_j$, $r_{j+1} = r_j - \mu_j$, $j \leftarrow j+1$, and go to 2; else, go to 5.

5. Find orthogonal matrices $U_f$ and $V_f$ such that $\begin{bmatrix} A_{21}^{(j+1)} & A_{22}^{(j+1)} \end{bmatrix} V_f = \begin{bmatrix} 0 & \widehat{A}_{22} \end{bmatrix}$, where $\widehat{A}_{22}$ is non-singular, and $U_f \begin{bmatrix} E_{11}^{(j+1)} & 0 \end{bmatrix} V_f = \begin{bmatrix} \widetilde{E}_{11} & \widehat{E}_{12} \end{bmatrix}$, where $\widetilde{E}_{11}$ is non-singular and upper-triangular. Set

$$\widetilde{E} = \begin{bmatrix} U_f & 0 \\ 0 & I_{n-r_j} \end{bmatrix} E \begin{bmatrix} V_f & 0 \\ 0 & I_{n-n_j} \end{bmatrix} := \begin{bmatrix} \widetilde{E}_{11} & \widetilde{E}_{12} \\ 0 & \widetilde{E}_{22} \end{bmatrix},$$

$$\widetilde{A} = \begin{bmatrix} U_f & 0 \\ 0 & I_{n-r_j} \end{bmatrix} A \begin{bmatrix} V_f & 0 \\ 0 & I_{n-n_j} \end{bmatrix} := \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ 0 & \widetilde{A}_{22} \end{bmatrix},$$

$$\widetilde{B} = \begin{bmatrix} U_f & 0 \\ 0 & I_{n-r_j} \end{bmatrix} B, \quad \widetilde{C} = C \begin{bmatrix} V_f & 0 \\ 0 & I_{n-n_j} \end{bmatrix}$$

The reduction to be performed at step 2 can be more easily explained if we introduce the following notation: $\bar{E} = E_{11}^{(j)}$, $\bar{A} = A_{11}^{(j)}$, $\bar{B} = A_{12}^{(j)}$, $\bar{C} = A_{21}^{(j)}$, and $D = A_{22}^{(j)}$. The descriptor system $(\bar{E}, \bar{A}, \bar{B}, \bar{C}, \bar{D})$ has order $r_j$, and $n_j - r_j$ inputs and the same number of outputs. Moreover, the matrix $\bar{E}$ is non-singular and upper-triangular. The reduction is performed in two steps. First we determine an orthogonal $W$ to compress the rows of $\bar{D}$ such that

$$W[\bar{C}\ \bar{D}] = \begin{bmatrix} \bar{C}_1 & \bar{D}_1 \\ \bar{C}_2 & 0 \end{bmatrix} \begin{matrix} \}\,\rho_j \\ \}\,\tau_j \end{matrix}$$

where $\bar{D}_1 \in \mathbb{R}^{\rho_j \times (n_j - r_j)}$ has full row rank $\rho_j$, where $\rho_j = n_j - r_j - \tau_j$. If $\tau_j = 0$, then we set $\mu_j = 0$, $Q_j = I_{n_j}$, $Z_j = I_{n_j}$ and we finished. Otherwise, we determine the orthogonal matrices $U$ and $V$ such that $\bar{C}_2 V = [\,0\ \ \bar{C}_{22}\,]$, with $\bar{C}_{22} \in \mathbb{R}^{\mu_j \times \mu_j}$ non-singular, and $UEV$ is further upper-triangular. We partition compatibly the transformed matrices as follows

$$UEV = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} \\ 0 & \bar{E}_{22} \end{bmatrix}, \quad UAV = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$$

$$UB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} V = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} \\ 0 & \bar{C}_{22} \end{bmatrix}.$$

Notice that $\bar{C}_{22}$ results square because we assumed that the pair $(E, A)$ is regular. With the matrices computed above we can define the snbmatrices computed at step 2 of the Algorithm 2 as:

$$E_{11}^{(j+1)} = \bar{E}_{11},\ E_{13}^{(j+1)} = \bar{E}_{12},\ E_{23}^{(j+1)} = \begin{bmatrix} \bar{E}_{22} \\ 0_{\rho_j \times \mu_j} \end{bmatrix},\ A_{11}^{(j+1)} = \bar{A}_{11},\ A_{12}^{(j+1)} = \bar{B}_1,\ A_{13}^{(j+1)} = \bar{A}_{12}, \tag{17}$$

$$A_{21}^{(j+1)} = \begin{bmatrix} A_{21} \\ C_{11} \end{bmatrix},\ A_{22}^{(j+1)} = \begin{bmatrix} \bar{B}_2 \\ \bar{D}_1 \end{bmatrix},\ A_{23}^{(j+1)} = \begin{bmatrix} \bar{A}_{22} \\ \bar{C}_{12} \end{bmatrix},\ A_{33}^{(j+1)} = \bar{C}_{22}. \tag{18}$$

The transformation matrices $Q_j$ and $Z_j$ can be assembled as

$$Q_j = \begin{bmatrix} U & 0 \\ 0 & W \end{bmatrix}, \quad Z_j = \begin{bmatrix} V & 0 \\ 0 & I_{r_j} \end{bmatrix} \begin{bmatrix} I_{r_j - \mu_j} & 0 & 0 \\ 0 & 0 & I_{n_j - r_j} \\ 0 & I_{\mu_j} & 0 \end{bmatrix}.$$

Algorithm 2 is a more efficient version of computational complexity $0(n^3)$ of an algorithm initially proposed by Van Dooren [8]. The reduction technique is similar to that used in the recently developed algorithm for computing the zeros of descriptor systems [5]. If the algorithm stops at step $k$ then at the end of the algorithm the submatrices $\widetilde{E}_{22}$ and $\widetilde{A}_{22}$ have the following forms

$$\widetilde{E}_{22} = \begin{bmatrix} 0 & E_{k,k-1} & \cdots & E_{k,1} \\ 0 & 0 & \cdots & E_{k-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \widetilde{A}_{22} = \begin{bmatrix} A_{k,k} & A_{k,k-1} & \cdots & A_{k,1} \\ 0 & A_{k-1,k-1} & \cdots & A_{k-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{1,1} \end{bmatrix},$$

where the diagonal matrices $A_{i,i} \in \mathbb{R}^{\mu_i \times \mu_i}$, $i = 1, \ldots, k$ are invertible and the principal superdiagonal matrices $E_{i,i+1} \in \mathbb{R}^{\mu_i \times \mu_{i+1}}$, $i = 1, \ldots, k-1$ have full row rank. This last property can be easily seen by observing that at step $(j+1)$ the matrix $A_{22}^{(j+1)}$ has the structure in (18) where $\bar{D}_1$ has full row rank $\rho_j$. The row compression performed on $A_{22}^{(j+1)}$ is simultaneously applied to the matrix $E_{23}^{(j+1)}$ which has the form in (17), where $\bar{E}_{22}$ of order $\mu_j$ is invertible. Thus the matrix $[\,A_{22}^{(j+1)} \;\; E_{23}^{(j+1)}\,]$ has full row rank and the row compression of $A_{22}^{(j+1)}$ produces a full row rank matrix in the last $\tau_{j+1}$ rows of the transformed $E_{23}^{(j+1)}$.

By defining $\mu_{k+1} = 0$, from the structure of the pencil $\widetilde{A}_{22} - \lambda\widetilde{E}_{22}$ we have that [7] the pencil $A - \lambda E$ has $\mu_i - \mu_{i+1}$ inifinite elementary divisors of degree $i$, $i = 1, \ldots, k$.

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