Computational Methods for Stabilization of Descriptor Systems

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1 Introduction

Consider the descriptor system (E, A, B) described by the equation

$$E\lambda x(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, E, $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, and $\lambda x(t) = \dot{x}(t)$ for a continuous-time system or $\lambda x(t) = x(t+1)$ for a discrete-time system. The following stabilization problems are considered.

1. *S-Stabilization Problem* (SSP): For the system (1) determine a state feedback matrix $F \in \mathbb{R}^{m,n}$ such that the closed-loop system $(E, A + BF, B)$ is regular and has exactly $r = \text{rank}(E)$ stable finite poles.

2. *R-Stabilization Problem* (RSP): For the regular system (1) determine a state feedback matrix $F \in \mathbb{R}^{m,n}$ such that the closed-loop system $(E, A + BF, B)$ is regular and all its finite poles are stable.

Procedures for S- and R-stabilization can be viewed as simple synthesis methods for designing controllers or observers for descriptor systems [3]. The solution of the SSP can be used for the initialization of Newton's method to solve descriptor Riccati equations [5]. Necessary and sufficient conditions for the existence of solutions of the formulated stabilization problems are given in the following theorems.

Theorem 1 [5] *A solution F of the SSP exists iff the system (E, A, B) is strongly stabilizable, that is: 1)* $\text{rank}([\,\lambda E - A\ B\,]) = n$ *for all finite $\lambda \in \mathbb{C}^+$; and 2)* $\text{rank}([\,E\ AS_\infty\ B\,]) = n$*, where the columns of S_∞ span the null space of E.*

Theorem 2 [3] *A solution F of the RSP exists iff the system (E, A, B) is stabilizable, that is,* $\text{rank}([\,\lambda E - A\ B\,]) = n$ *for all finite $\lambda \in \mathbb{C}^+$.*

2 S-stabilization procedure

For the solution of the SSP a procedure in two steps can be used: 1) Determine F_1 such that the system $(E, A + BF_1, B)$ is regular and impulse free; 2) Find F_2 such that the system $(E, A + BF, B)$ with $F = F_1 + F_2$ has exactly r stable finite poles. A realization of these steps is the following algorithm to solve the SSP.

Algorithm SSDS.

1. Find orthogonal matrices U and V such that

$$UEV := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_{33} \end{bmatrix}, \quad UAV := \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad UB := \begin{bmatrix} 0 \\ B_{21} \\ B_{31} \end{bmatrix},$$

   where $E_{33} \in \mathbb{R}^{r,r}$ is non-singular, A_{11} is non-singular and B_{21} has full row rank.

2. Determine F_{12} such that $A_{22} + B_{21}F_{12}$ is non-singular and set $F_1 = [\,0\ F_{12}\ 0\,]V^T$.
   Comment. The system $(E, A + BF_1, B)$ is now regular and impulse free.

3. Find an orthogonal matrix Q such that

$$Q \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} + B_{21}F_{12} \\ A_{31} & A_{32} + B_{31}F_{12} \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix},$$

   where S is non-singular of order $n - r$ and define the new partitioning

$$QUEV := \begin{bmatrix} 0 & E_{12} \\ 0 & E_{22} \end{bmatrix}, \quad QU(A + BF_1)V := \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad QUB := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

   where $A_{11} := R$.
   Comment. E_{22} is non-singular because the pencil $\lambda E - A - BF_1$ is regular.

4. Put $F = F_1 + [\,0\ F_{22}\,]V^T$, where F_{22} is such that $\lambda(A_{22} + B_2F_{22}, E_{22}) \in \mathbb{C}^-$.

The computations at steps 1 and 3 are based on using orthogonal decompositions (singular value decompositions, QR- or RQ-decompositions). Algorithmic details can be found in [11]. Algorithms to solve the RSP at step 4 are presented in the next sections. The whole procedure is well suited for an efficient computer implementation.

3 R-stabilization by direct methods

At step 4 of the SSDS Algorithm we have to solve the following particular RSP: For the system (E, A, B) with E *non-singular*, compute F such that $\lambda(A + BF, E) \in \mathbb{C}^-$. An obvious way to solve this problem is to transform it into a stabilization problem for the pair $(E^{-1}A, E^{-1}B)$ and to use an appropriate stabilization method for standard systems [8]. In this paper we propose algorithms which avoid the inversion of E and thus prevent potential accuracy degradations if E is ill-conditioned. The following approach to solve the RSP is generally applicable regardless E is invertible or not.

Algorithm RSDS.

1. Reduce the pair (A, E) by an orthogonal similarity transformation, to the ordered *generalized real Schur form* (GRSF) [6], [7]

$$QEZ = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \; QAZ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \; QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $E_{11}, A_{11} \in \mathbb{R}^{q,q}$, Q and Z are orthogonal matrices, $\lambda(A_{11}, E_{11}) \subset \mathbb{C}^- \cup \{\infty\}$ and $\lambda(A_{22}, E_{22}) \subset \mathbb{C}^+$.

2. Set $F = [\,0\ F_2\,]Z^T$, where F_2 is such that $\lambda(A_{22} + B_2F_2, E_{22}) \subset \mathbb{C}^-$.

The stabilization problem to be actually solved at step 2 is the following simpler problem: For the *controllable* system (E, A, B) with E *non-singular* and $\lambda(A, E) \subset \mathbb{C}^+$, compute F such that $\lambda(A+BF, E) \subset \mathbb{C}^-$. We can further assume that the pair (A, E) is already in a GRSF. Two direct stabilization methods, based on the following theorems, provide the solution of this simpler problem. They are generalizations of the methods of *Bass-Armstrong* [1] and *Armstrong-Rublein* [2] for the stabilization of continuous-time and discrete-time standard systems, respectively.

Theorem 3 *For a continuous-time controllable system* (E, A, B) *with* $\mathrm{Re}\lambda_i(A, E) \geq 0$, *a stabilizing* F *is given by* $F = -B^T(XE^T)^{-1}$, *where for* $\beta > 0$, X *satisfies the continuous generalized Lyapunov equation*

$$(A + \beta E)XE^T + EX(A + \beta E)^T = 2BB^T \tag{2}$$

Theorem 4 *For a discrete-time controllable system* (E, A, B), *with* $|\ \lambda_i(A, E)| \geq 1$, *a stabilizing* F *is given by* $F = -B^T(EXE^T + BB^T)^{-1}A$ *where for* $0 < \beta < 1$, X *satisfies the discrete generalized Lyapunov equation*

$$AXA^T = \beta^2 EXE^T + 2BB^T \tag{3}$$

The generalized Lyapunov equations (2) and (3) can be reliably solved by using the generalized Bartels-Stewart algorithms proposed in [4]. Note that in our problem the pairs $(A+\beta E, E)$ or $(A, \beta E)$ are already in a GRSF and thus the costs of solving the generalized Lyapunov equations are negligible in comparison with the cost of computing the GRSF of the pair (A, E).

4 R-stabilization by pole assignment

A numerically more satisfactory approach for solving the RSP can be based on pole assignment techniques. The following procedure, based on an efficient technique for updating and reordering the GRSF of the pair (A, E), moves sequentially the unstable poles in the stable region by using partial state feedback matrices which do not perturb the locations of already stable poles. The algorithm given below is an extension of an earlier method proposed in [10].

Algorithm RSPA.

1. Reduce the pair (A, E) by an orthogonal similarity transformation, to the ordered GRSF

$$QEZ = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix},\ QAZ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where $E_{11}, A_{11} \in \mathbb{R}^{q,q}$, Q and Z are orthogonal matrices, $\lambda(A_{11}, E_{11}) \subset \mathbb{C}^- \cup \{\infty\}$ and $\lambda(A_{22}, E_{22}) \subset \mathbb{C}^+$. Compute $E \leftarrow QEZ, A \leftarrow QAZ, B \leftarrow QB$ and set $F = 0$.

2. If $q = n$, then *Stop*; else let α and δ be the $k \times k$ last diagonal blocks of A and E respectively, and let β be the $k \times m$ matrix formed from the last k rows of B. Choose a $k \times k$ matrix γ such that $\lambda(\gamma) \subset \mathbb{C}^-$ and compute $\varphi = \beta^{\#}(\delta\gamma - \alpha)$.

3. Compute $A \leftarrow A + B[\,0\ \varphi\,]$, $F \leftarrow F + [\,0\ \varphi\,]Z^T$.

4. Move the last $k \times k$ blocks of A and E to positions $(q+1, q+1)$ by using an orthogonal similarity transformation $E \leftarrow \tilde{Q}E\tilde{Z}$, $A \leftarrow \tilde{Q}A\tilde{Z}$ and compute $B \leftarrow \tilde{Q}B$, $Z \leftarrow Z\tilde{Z}$, $Q \leftarrow \tilde{Q}Q$. Put $q \leftarrow q + k$ and go to step 2.

The RSPA algorithm is a generalization of a pole assignment algorithm for standard systems [9]. By extending the roundoff error analysis of that algorithm, it follows that if each partial feedback matrix of the form $K = [\,0\ \varphi\,]$ satisfies $\|K\| \leq \|A\|/\|B\|$, then the proposed pole assignment algorithm is numerically backward stable.

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