



Advanced nonlinear modeling and control stability analysis of sloshing dynamics in liquid-propelled reusable launch vehicles

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Abstract

This paper presents a modeling approach for the liquid dynamics inside the tanks of liquid-propelled vehicles. The proposed formulation is applicable to high- g environments and planetary atmospheric flight, encompassing expendable and reusable launch vehicles as well as planetary landers. The oscillatory behavior of the liquid free surface is modeled as a superposition of n swinging pendula, each corresponding to a harmonic mode of sloshing.

The coupling between the rigid body and the pendula is formulated using a classical multibody dynamics methodology. Specifically, the iterative Newton–Euler approach is applied to a general spacecraft configuration with m tanks and n associated sloshing modes. A detailed derivation of a symbolic three-degree-of-freedom pendulum model is presented, followed by its reduction to a two-degree-of-freedom planar model. While the full model is suitable for time-domain simulations, the reduced version is tailored for control design and stability analysis.

Symbolic computations are performed with the *Symbolic Multibody Dynamics* toolbox developed by the Guidance, Navigation and Control Systems Department at the German Aerospace Center. This library enables analytical derivation of equations of motion for a wide range of multibody space systems and supports seamless export to simulation frameworks for the verification and validation of guidance, navigation and control systems.

The model parameters for a representative space mission were derived from the geometry of a reusable launch vehicle's propellant and oxidizer tanks. A stability analysis of the complex pole–zero pairs associated with each sloshing mode was conducted, and the locations of the equivalent sloshing masses relative to critical stability zones were examined. Finally, a closed-loop analysis was performed using an $\mathcal{H}_\infty$ -based controller, and the resulting stability margins were evaluated along a reference trajectory and via nonlinear simulations.

Keywords Launch vehicle · RLV · VTVL · Sloshing · Multibody · Ascent · Control

List of symbols

Latin symbols

$\mathbf{a}, a$	Acceleration vector and norm [m s^{-2}]
a_x, a_y, a_z	Components of the acceleration vector [m s^{-2}]
D	Diameter [m]
$\mathbf{f}, f$	Force vector and norm [N]
f	Frequency [Hz]
f_x, f_y, f_z	Components of the force vector [N]

$\mathbf{g}, g$	Gravity acceleration vector and norm [m s^{-2}]
h	Height [m]
$\mathbf{I}$	Identity matrix [–]
$\hat{\mathbf{i}}_F, \hat{\mathbf{j}}_F, \hat{\mathbf{k}}_F$	Basis (unit) vectors of reference frame F [–]
$\mathbf{J}, J$	Moment of inertia matrix and element [kg m^{-2}]
ℓ	Lever arm [m]
m	Mass [kg]
p, q, r	Components of the angular velocity vector [rad s^{-1}]
$\mathcal{R}$	Rotation matrix [–]
$\mathbf{r}, r$	Position vector and norm [m]
$\mathbf{S}$	Skew-symmetric matrix operator [–]
$\mathbf{o}_F$	Origin of frame F [–]
T	Thrust magnitude [N]
$\mathbf{u}, \mathbf{v}, \mathbf{w}$	Components of the velocity vector [m s^{-1}]

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$\mathbf{v}, v$	Velocity vector and norm [m s ⁻¹]
$\mathbf{v}_W, v_W$	Wind velocity vector and norm [m s ⁻¹]
X, Y, Z	State-space force coefficients [–]
x, y, z	Components of the position vector [m]

Greek symbols

α	Angular acceleration vector [rad s ⁻²]
α	Angle of attack [rad]
$\alpha_1, \alpha_2, \alpha_3$	Components of the angular acceleration vector [rad s ⁻²]
Γ	Non-gravitational longitudinal acceleration [m s ⁻²]
δ_y, δ_z	TVC deflections [rad]
$\delta_\phi, \delta_\theta, \delta_\psi$	Fin deflections [rad]
$\boldsymbol{\varepsilon}$	Euler angles vector [rad]
ζ	Damping coefficient [–]
μ_x, μ_y, μ_z	State-space moment coefficients [–]
ξ	Bessel function [–]
ρ	Density [kg m ⁻³]
$\boldsymbol{\tau}, \tau$	Torque vector and norm [N m]
τ_x, τ_y, τ_z	Components of the torque vector [N m]
ϕ, θ, ψ	Roll, pitch, and yaw angles [rad]
$\boldsymbol{\omega}$	Angular velocity vector [rad s ⁻¹]
ω_n	Natural frequency [rad s ⁻¹]

Special symbols

★	Generic placeholder symbol. Denotes a translational/rotational quantity when used in the main position; denotes a reference frame when used as a subscript or superscript
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Frames

B	Body frame
EFUEN	Earth-fixed Up–East–North frame
I	Inertial frame
S	Structural frame
SH(<i>i</i>)	Sloshing: hinge point frame (<i>i</i> indicates the <i>i</i> th element)
SL(<i>j</i>)	Sloshing: liquid frame (<i>j</i> indicates the <i>j</i> th element)
SM(<i>i</i>)	Sloshing: mass frame (<i>i</i> indicates the <i>i</i> th element)
ST(<i>j</i>)	Sloshing: tank structural frame (<i>j</i> indicates the <i>j</i> th element)

Subscript text

★ ₀ , ★ _{SM0}	Static liquid mass
★ _B	Rigid body
★ _{SH}	Hinge point
★ _{<i>i</i>} , ★ _{SM(<i>i</i>)}	Sloshing mass (<i>i</i> indicates the <i>i</i> th element)
★ _{liq} , ★ _{SL}	Liquid mass within a tank
★ _{surf}	Liquid surface within a tank

★ _{tot}	Total multibody assembly
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Indexing

Translational quantities:

^{F₃} ★ _{O_{F₂}, O_{F₁}}	★ = { <i>r</i> , <i>v</i> , <i>a</i> } of point <i>o</i> _{F₁} with respect to point <i>o</i> _{F₂} , expressed in frame F ₃
^{F₁} ★	★ = { <i>r</i> , <i>v</i> } with the first time derivative taken in frame F ₁
^{F₂, F₁} ★	★ = { <i>r</i> , <i>v</i> , <i>a</i> } with the first time derivative taken in frame F ₁ and the second time derivative taken in frame F ₂
^{F₃} <i>J</i> _{O_{F₂}, m₁}	Moment of inertia of mass <i>m</i> ₁ with respect to point <i>o</i> _{F₂} , expressed in frame F ₃

Rotational quantities:

^{F₂} ★ _{F₁}	★ = { <i>R</i> , <i>ε</i> } of F ₁ relative to F ₂ , or ★ = { <i>R</i> , <i>ε</i> } to rotate from F ₂ to F ₁
^{F₃} ★ _{F₂, F₁}	★ = { <i>ω</i> , <i>α</i> } of F ₁ relative to F ₂ expressed in F ₃

Abbreviations

2D	Two dimensional
3D	Three dimensional
CALLISTO	Cooperative Action Leading to Launcher Innovation for Stage Tossback Operation
CFD	Computational fluid dynamics
CNES	French National Center for Space Studies
CoG	Center of gravity
CoM	Center of mass
DLR	German Aerospace Center
DoF	Degrees of freedom
EoM	Equations of motion
GNC	Guidance, Navigation and Control
JAXA	Japan Aerospace Exploration Agency
LH2	Liquid hydrogen
LOx	Liquid oxygen
LV	Launch vehicle
MoI	Moment of inertia
RHS	Right-hand side
RLV	Reusable launch vehicle
RTLS	Return-to-launch-site
TVC	Thrust vector control
V&V	Verification and validation

1 Introduction

1.1 Motivation

Liquid sloshing in partially filled tanks is a well-known phenomenon that degrades the stability and control of liquid-propelled spacecraft and atmospheric vehicles, particularly at liftoff when loaded propellant masses are large. For reusable

launch vehicles (RLVs) and planetary landers, retropropulsive burns require substantial onboard propellant, typically stored in separate fuel and oxidizer tanks. Accurate Guidance, Navigation and Control (GNC) design demands models that capture the coupling between slosh and the vehicle dynamics with sufficient fidelity, while remaining tractable across the mission life cycle; this applies from concept through design to verification and validation (V&V).

Insufficient treatment of sloshing has contributed to well-documented flight instabilities and losses in spacecraft and launch vehicles. For instance, the ATS-V satellite launched in 1969 entered an uncontrolled spin due to underestimated sloshing effects [1]. Furthermore, during the flight of the Saturn SA-1 rocket, a slosh instability that transitioned from linear to rotational motion led to a premature engine cut-off [2]. More recently, during the second test flight of the Falcon 1 in 2007, oscillations about the pitch and yaw axes due to sloshing in the liquid oxygen (LOx) tank caused a complete loss of the second stage. In the latter two cases, subsequent configurations incorporated additional baffling in critical regions to mitigate these issues [3]. Space agencies consistently stress that, because liquid sloshing significantly reduces stability margins during ascent and descent, it warrants explicit analysis, especially for safety-critical missions [4–6].

1.2 Related work and gaps

Addressing sloshing for spacecraft from a GNC perspective entails two core tasks: (i) accurate simulation of the effect on the vehicle dynamics for V&V; and (ii) integration of slosh dynamics into the synthesis of control laws and stability assessment across the flight envelope. Hahn [7] classified state-of-the-art routes to address (i). Here, that taxonomy is extended to encompass (ii) as well; the expanded pathways are summarized in Fig. 1.

The first attempts to approximate the motion of liquid in a cylindrical tank date back to Poisson and Rayleigh, who pioneered harmonic-oscillator solutions (with Rayleigh introducing Bessel functions) [8]. During the 1960s, Saturn-era developments produced the now-canonical mechanical analogs that approximate sloshing as either a set of spring-mass systems or swinging pendula. Within their validity regimes, these analogs have been applied successfully to a variety of launch vehicles over several decades [8–11].

Mechanical models for simulation can be implemented directly from the governing equations, or via multibody/physical-modeling environments (e.g., Simscape [12], Modelica [13], or similar), which internally encode the equivalent analytical formulations.

For control design, the nonlinear equations of the mechanical models are trimmed along the reference trajectory to obtain simplified models (nonlinear or linear time-invariants

(LTIs)), and these models are used for synthesis and stability analysis.

The parameters for the mechanical models may be obtained from (a) analytical formulations (for standard geometries); (b) experiments on full-scale or scaled tanks [14]; or (c) high-fidelity solutions of the Navier–Stokes equations.

In the latter case, one either (i) derives the parameters for the equivalent mechanical models directly from the computational fluid dynamics (CFD) datasets [15], or (ii) employs a lower-resolution grid [7]. Other workflows couple external CFD solvers to MATLAB/Simulink via co-simulation for V&V [16, 17].

More recent approaches compute surrogate models [18] or train black-box models using machine learning for faster simulation of sloshing [19]. System identification of linear parameter-varying (LPV) models for control design can also be achieved with recent advances [20] in such techniques.

Since the work presented in this article requires a model that covers points (i) and (ii) in Fig. 1 for the entire lifespan of the Cooperative Action Leading to Launcher Innovation for Stage Tossback Operation (CALLISTO) mission, the analytical formulations for the mechanical model are derived. Several methods exist for the derivation of symbolic equations of motions (EoMs) [21–23]. The iterative Newton–Euler method has the advantage of producing formulations directly in vector form, which also eases modularity. It also maps the forces and torques acting on the different bodies directly at their points of application. Effects like damping or aerodynamic excitations do not require mapping into generalized coordinates. The same analytical sparse structure is used efficiently in simulation frameworks and propagates accordingly when linearized. In contrast, the minimal-coordinate approach of the Lagrangian gives compact energy-based equations for a few bodies, but becomes algebraically harder to handle as the number of bodies increases and requires forces and torques expressed as generalized quantities via the Jacobian. Kane’s method is naturally strong in analytical derivations, but also requires generalized forms and does not exploit sparsity as efficiently as the Newton–Euler method for this class of systems. The pendulum-based slosh parameters integrate naturally into the Newton–Euler framework, maintaining full transparency from identification to simulation of the space mission.

1.3 Contributions of this work

This paper makes the following contributions:

- A list of parameters needed to model sloshing dynamics and their approximations via analytical formulations that can be used during early design phases of a space mission.

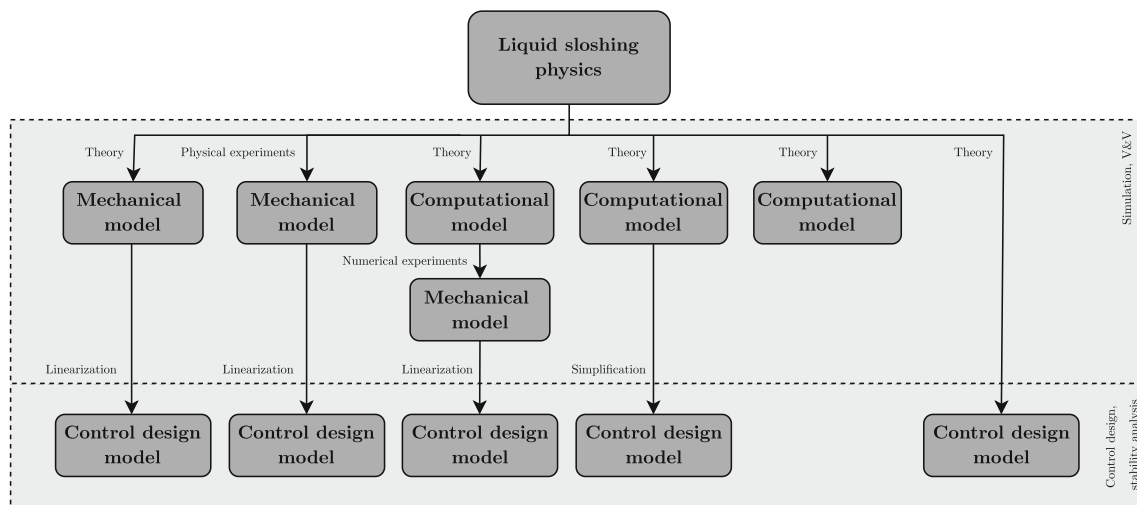


Fig. 1 Sloshing modeling and analysis methods for GNC systems, adapted from Hahn [7]

- An analytical multibody derivation for a rigid body with m tanks and n slosh modes (pendula) that accounts for the 6-degrees of freedom (DoF) of the rigid body and the 5-DoF. Two rotational states (pitch and yaw) and three translational (x, y, z) of each pendulum, and is suitable for simulation frameworks, such as MATLAB- or C++-based implementations. This derivation does not require external interfaces with other software or toolboxes.
- A simplification of the full-order nonlinear multibody model into a reduced nonlinear form, and consequently an even simpler LTI state-space representation. The latter, together with the uncertainties in the parameters of the vehicle and sloshing modes, leads to linear fractional transformations (LFTs) or, potentially, LPVs models by introducing dependence on time or other variables.
- A linearization yielding a state-space model (and equivalent attitude transfer functions) that exposes the complex pole-zero pair per mode clarifies the role of the center of percussion and supports the controller's bandwidth placement and quantifies its impact on stability margins.
- An application of all previous points to a real RLV mission: parameter computation for LOx and liquid hydrogen (LH₂) tanks; comparison of analytical predictions with CFD values for the mission; the evolution of the danger zone for a classic return-to-launch-site (RTL) trajectory; closed-loop robust $\mathcal{H}_\infty$ control with verified stability margins; and validation of the models and control via Monte Carlo (MC) simulations.

1.4 Paper organization

Section 2 summarizes tank geometry and slosh-parameter computation (with assumptions and validity ranges). Section 3 derives the three-dimensional (3D) multibody equa-

tions, embeds slosh forces/torques into the rigid-body dynamics, and presents the planar reduction and linearization; the resulting control-oriented model and stability interpretation (poles/zeros and center of percussion) are highlighted. Section 4 applies the framework to CALLISTO, including CFD comparison, danger zone analysis, and $\mathcal{H}_\infty$ closed-loop results. Section 5 concludes and outlines extensions to multi-tank/multi-mode coupling and experimental validation.

A preliminary version of this work was presented at the CEAS/EUCASS Conference [24]. The present manuscript extends that work by providing revised results and expanded analysis.

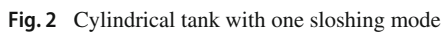
2 Parameters characterization

The most common arrangement for launch vehicle tanks consists of one oxidizer tank and one fuel tank per stage, with solid boosters mounted on each side [25]. This configuration characterizes most of the launch vehicles in service. A few alternatives, such as the Delta IV and the Falcon rocket [26], both in their Heavy configurations, employ liquid side boosters.

Conversely, both conceptual designs [27] and flight-tested landers [28] often include a series of tanks distributed around the longitudinal axis in a ring configuration, primarily for dimensional reasons.

The focus of this work is the analysis of the fuel and oxidizer tanks, which constitute most of the wet mass of a vehicle during launch. The liquid needed for cold gas thrusters and the pressurization systems is usually much smaller and is therefore considered negligible.

Common spacecraft tank geometries include cylindrical tanks, cylindrical tanks with elliptical domes, spherical tanks,



Several studies provide formulas to convert domed-tank parameters to an equivalent standard cylindrical geometry [30]. This conversion yields an effective diameter and tank height. It is assumed that the dome-to-barrel height ratio is small ($h_{\text{dome}}/h_{\text{barrel}} < 10\%$), which makes the formulations in [8, 29, 31, 32] applicable to this study.

Some of the key considerations for this analysis are:

- The total liquid mass contained in each j th tank with n associated modes and one static mass is given by¹

¹ Index j is omitted for notational simplicity.

For the computation of the MoI with respect to the center of mass (CoM) of the unperturbed liquid, see Eq. (6). Also note that the following relationships hold: $\mathbf{J}_0 = \mathbf{J}_{\text{OSM}0, \mathbf{m}_0}$, $\mathbf{J}_i = \mathbf{J}_{\text{OSM}i, \mathbf{m}_i}$ and $\mathbf{J}_{\text{liq}} = \mathbf{J}_{\text{OSL}, \mathbf{m}_{\text{liq}}}$.

For the computation of the specific parameters associated with each of the modes, the roots of the Bessel function of the first kind are required for the given tank geometry. They can be found in the work of Bauer [8] and are repeated here for simplicity as follows: $\xi_1 = 1.8412$, $\xi_2 = 5.3314$, $\xi_3 = 8.5393$, $\xi_4 = 11.7060$, and $\xi_i \approx \xi_{i-1} + \pi$. These roots depend on tank geometry rather than fluid properties, making them applicable to any liquid with a sufficiently large Bond number.

For the description and analysis of the motion of the i th pendulum with respect to the vehicle's CoM, some important frames, in addition to the classic body (B) and Earth-Fixed Up-East-North (EFUEN) frames, must be defined. For a flat and non-rotating Earth, the EFUEN frame is considered the inertial reference and is thus denoted as $\mathbb{I}$. The definitions of the axes and rotation matrices are omitted here to maintain

² The star ★ left superscript indicates that these vectors can be expressed in any frame, e.g., I, B, SM or SH frame by making use of the correct rotation matrix $\mathcal{R}$.

Table 1 Summary of tank and sloshing mode parameters

Parameter	Equation	Units
Total liquid mass	$m_{\text{liq}} = \frac{1}{4}\pi\rho_{\text{liq}}D^2h_{\text{surf}}$ (3)	[kg]
Total liquid CoM height ^a	$h_{\text{SL}} = \frac{1}{2}h_{\text{surf}}$ (4)	[m]
MoI of the solidified liquid	$J_0 = m_{\text{liq}}\left(\frac{1}{12}h_{\text{surf}}^2 + \frac{1}{16}D^2\right)$ (5)	[kg m ²]
Total liquid MoI w.r.t. liquid CoM	$J_{\text{liq}} = J_0 + m_0\left(\frac{1}{2}h_{\text{surf}} + h_{\text{SM}_0}\right)^2 + \sum_{i=1}^n m_i\left(\frac{1}{2}h_{\text{surf}} + h_{\text{SM}_i}\right)^2$ (6)	[kg m ²]
i th mode mass	$m_i = \frac{D \tanh\left(\frac{2\xi_i h_{\text{surf}}}{D}\right) m_{\text{liq}}}{\xi_i (\xi_i^2 - 1) h_{\text{surf}}}$ (7)	[kg]
Surface height w.r.t. i th hinge point ^b	$h_{\text{SH}_i, \text{surf}} = -\frac{D}{\xi_i \sinh\left(\frac{2\xi_i h_{\text{surf}}}{D}\right)}$ (8)	[m]
Hinge point height w.r.t. tank bottom	$h_{\text{SH}_i} = h_{\text{surf}} - h_{\text{SH}_i, \text{surf}}$ (9)	[m]
Pendulum lever arm	$\ell_{\text{SM}_i} = \frac{D}{2\xi_i \tanh\left(\frac{\xi_i h_{\text{surf}}}{D}\right)}$ (10)	[m]
Surface height w.r.t. i th mass	$h_{\text{SM}_i, \text{surf}} = h_{\text{SH}_i, \text{surf}} + \ell_{\text{SM}_i}$ (11)	[m]
Mass height w.r.t. tank bottom ^c	$h_{\text{SM}_i} = h_{\text{surf}} - h_{\text{SM}_i, \text{surf}}$ (12)	[m]
Mass height w.r.t. liquid CoM ^c	$h_{\text{SL}, \text{SM}_i} = \frac{h_{\text{surf}}}{2} - h_{\text{SM}_i, \text{surf}}$ (13)	[m]
Sloshing mode natural frequency ^d	$f_{\text{SM}_i} = \frac{1}{2\pi} \sqrt{\frac{B_{\ddot{x}} - B_{g_x}}{\ell_{\text{SM}_i}}} = \frac{1}{2\pi} \sqrt{\frac{\Gamma}{\ell_{\text{SM}_i}}}$ (14)	[Hz]
Static mass	$m_0 = m_{\text{liq}} - \sum_{i=1}^n m_i$ (15)	[kg]
Surface height w.r.t. static mass ^e	$h_{\text{SM}_0, \text{surf}} = h_{\text{surf}} - h_{\text{SM}_0}$ (16)	[m]
Static mass height w.r.t. liquid CoM	$h_{\text{SL}, \text{SM}_0} = \sum_{i=1}^n \frac{m_i}{m_0} h_{\text{SL}, \text{SM}_i}$ (17)	[m]
Static mass height w.r.t. bottom of the tank	$h_{\text{SM}_0} = \frac{h_{\text{surf}}}{2} + h_{\text{SL}, \text{SM}_0}$ (18)	[m]

^aThe position of the fluid's CoM is found at half the distance of the total liquid height, and is obtained by assuming that the liquid behaves as a solid under unperturbed conditions

^bTo respect the conservation of momentum, the computation of the static mass is done using the relative masses height with respect to the CoM of the solidified liquid. However, here they are presented relative to the bottom of the tank for easier understanding

^cThese assumptions hold true assuming the pendulum angle with respect to the tank longitudinal axis is zero

^dElement $\Gamma = ((T - A)/m_{\text{tot}})$ is the longitudinal component of the non-gravitational acceleration, also known as sensed acceleration. For a static tank, the frequency depends only on the gravity

^eUsually, the position of the hinge is located below the hinge point for filling level above 50%. Below this limit this value becomes negative and its limit tends toward zero as the filling level approaches zero

conciseness. They can, however, be found in earlier investigations by the author [35, 36] or derived from standard flight dynamics references [37–39].

Remark 2 The frames presented below are defined for a single tank and a single mode/sloshing mass. For a practical implementation, appropriate indexing must be chosen according to the actual number of tanks and modes.

1. Tank frame (ST):

- o_{ST} : Located at the bottom of the j th tank.
- $\hat{i}_{\text{ST}}$: Parallel to the j th tank's longitudinal axis. Positive in the direction from 0% to 100% filling level.
- $\hat{j}_{\text{ST}}$: Pointing rightward. Parallel to the tank's lateral axis.

- $\hat{k}_{\text{ST}}$: Forms an orthonormal system. Parallel to the tank's normal axis.

This frame is used for the representation of the equations presented in Table 1; all quantities are referenced to its origin. These parameters are subsequently used to define the relative positions with respect to the vehicle's CoM. For this application, this frame is assumed to be aligned with the vehicle's structural (S) frame and, therefore, with the body frame

$${}^{\text{ST}}\mathcal{R}_{\text{S}} = {}^{\text{ST}}\mathcal{R}_{\text{B}} = \mathbf{I}_{3 \times 3}. \quad (19)$$

2. Pendulum–hinge frame (SH):

- o_{SH} : Located at the pendulum hinge point at the specific filling ratio.

- $\hat{i}_{SH}$: Parallel to the vector $\star r_{O_B, O_{SH}}$, positive in the direction from O_B to O_{SH} .
- $\hat{j}_{SH}$: Pointing rightward. This axis is parallel to the vehicle's lateral axis when θ_{SH} and ψ_{SH} are zero.
- $\hat{k}_{SH}$: Forms an orthonormal system. It is parallel to the vehicle's directional axis when both angles are zero.

The attitude of this frame coincides with those of the structural (S) and body (B) frames if and only if the tank is placed along the vehicle's longitudinal axis. It changes its origin as the liquid mass is depleted. The rotation matrix is defined as

$${}^{SH}\mathcal{R}_S = {}^{SH}\mathcal{R}_B = \mathcal{R}_2(-\theta_{SH})\mathcal{R}_3(-\psi_{SH}), \quad (20)$$

with the explicit formulation given in Appendix B.

3. Pendulum-mass frame (SM):

- O_{SM} : Located at the CoM of the i th sloshing mass.
- $\hat{i}_{SM}$: Parallel to the vector $\star r_{O_{SM}, O_{SH}}$, positive in the direction from O_{SM} to O_{SH} .
- $\hat{j}_{SM}$: Lying in the vehicle pitch plane, coincident with the vehicle's lateral axis when the liquid is unperturbed.
- $\hat{k}_{SM}$: Lying in the vehicle yaw plane, coincident with the vehicle's directional axis when the liquid is unperturbed.

This frame rotates and changes position as the mass moves (the liquid inside the tank sloshes). It also changes its origin as the liquid is depleted. The rotation matrix is defined as

$${}^{SM}\mathcal{R}_S = {}^{SM}\mathcal{R}_B = \mathcal{R}_2(\theta_{SM})\mathcal{R}_3(\psi_{SM}), \quad (21)$$

with the explicit formulation likewise given in Appendix B.

A graphical depiction of the axes and angles is provided in Fig. 3 in 3D coordinates, and in Fig. 4 for the planar configuration.

3.2 Pendulum dynamics

Remark 3 The modeling below is presented for a single mode and a single tank. However, it can be extended to multiple modes and multiple tanks.

As explained in Sect. 2, the static mass and the MoI are treated as part of the vehicle's dry mass. The mass distribution for a vehicle with a single tank and n modes is defined as

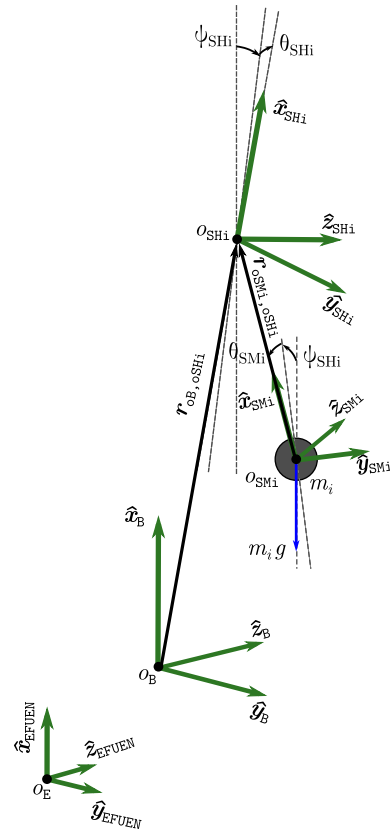


Fig. 3 Principal sloshing frames in 3D

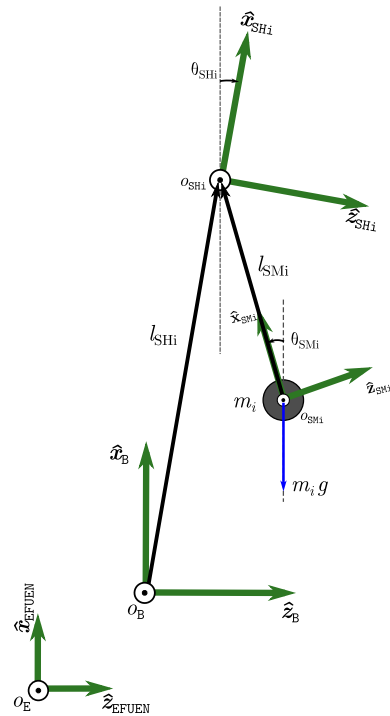


Fig. 4 Principal sloshing frames in 2D

follows:

$$m_{\text{tot}} = \underbrace{m_{\text{Bdry}} + m_0}_{m_B} + \sum_{i=1}^n m_i. \quad (22)$$

The position of the CoM of the total array can be derived from the expression

$$\star r_{o_{\text{ST}}, o_{\text{BT}}} = \frac{m_B \star r_{o_{\text{ST}}, o_{\text{B}}} + \sum_{i=1}^n m_i \star r_{o_{\text{ST}}, o_{\text{SM}i}}}{m_B + \sum_{i=1}^n m_i}. \quad (23)$$

Note that since the sum $\sum_{i=1}^n m_i$ is small relative to m_B , the points o_{BT} and o_{B} are typically considered coincident.

Following Remark 1, the total MoI of the tank includes only the MoI about the liquid CoM; the associated n modes are treated as inertia-less, and their effect depends only on the modal masses and lateral position shifts

$$\begin{aligned} {}^B J_{\text{tot}} &= \underbrace{{}^B J_{\text{Bdry}} + {}^B J_{o_{\text{B}}, m_0}}_{{}^B J_B} + \sum_{i=1}^n {}^B J_{o_{\text{B}}, m_i} \\ &= {}^B J_B + \sum_{i=1}^n \left(\overset{0}{\cancel{{}^B J_i}} - m_i \left[S({}^B r_{o_{\text{B}}, o_{\text{SM}i}}) \right]^2 \right). \end{aligned} \quad (24)$$

As the positions are expressed with respect to the origin of the B frame, the MoI is expressed with respect to this point ($J_{\text{tot}} = J_{o_{\text{B}}, m_{\text{tot}}}$). Shifting to the origin of the CoM of the array can be accomplished by applying the Huygens–Steiner theorem on the position in Eq. (23).

The considered single spherical pendulum has 2-DoF; its attitude is described by two angles: θ_{SM} , about the $\hat{j}_{\text{SM}}$ -axis, and ψ_{SM} , about the $\hat{k}_{\text{SM}}$ -axis.

The position of o_{SH} with respect to the i th mass origin o_{SM} is represented by

$${}^{\text{SM}} r_{o_{\text{SM}}, o_{\text{SH}}} = [\ell_{\text{SM}} \ 0 \ 0]^T, \quad (25)$$

and the hinge point position with respect to the vehicle's CoM by

$${}^{\text{SH}} r_{o_{\text{B}}, o_{\text{SH}}} = [\ell_{\text{SH}} \ 0 \ 0]^T. \quad (26)$$

Both position vectors have a single element in the first component when represented in frames SH and SM, respectively.

The position of o_{SM} with respect to o_{B} is denoted by

$$\star r_{o_{\text{B}}, o_{\text{SM}}} = \star r_{o_{\text{SH}}, o_{\text{SM}}} + \star r_{o_{\text{B}}, o_{\text{SH}}}, \quad (27)$$

in any given frame $\star$.

Regarding the linear velocity

$${}^{\text{B}} \dot{r}_{o_{\text{E}}, o_{\text{SM}}} = {}^{\text{B}} \dot{r}_{o_{\text{E}}, o_{\text{B}}} + {}^{\text{B}} \dot{r}_{o_{\text{B}}, o_{\text{SM}}}, \quad (28)$$

and the linear acceleration

$${}^{\text{B}} \ddot{r}_{o_{\text{E}}, o_{\text{SM}}} = {}^{\text{B}} \ddot{r}_{o_{\text{E}}, o_{\text{B}}} + {}^{\text{B}} \ddot{r}_{o_{\text{B}}, o_{\text{SM}}}, \quad (29)$$

computed from Eq. (27), the derivatives are performed in I frame; however, the vector is expressed in B frame. It is also possible to change the frame of reference via an appropriate rotation matrix ($\star \mathcal{R}_B$).

It is known that the vehicle's inertial acceleration in B-frame coordinates (${}^{\text{B}} \ddot{r}_{o_{\text{E}}, o_{\text{B}}} = {}^{\text{B}} \mathcal{R}_I^T a_{o_{\text{E}}, o_{\text{B}}} = [{}^{\text{B}} \ddot{x} \ {}^{\text{B}} \ddot{y} \ {}^{\text{B}} \ddot{z}]^T = {}^{\text{B}} \mathcal{R}_I^T [\ddot{x} \ \ddot{y} \ \ddot{z}]^T$) is given by the following relationship

$${}^{\text{B}} \ddot{r}_{o_{\text{E}}, o_{\text{B}}} = {}^{\text{B}} \ddot{r}_{o_{\text{E}}, o_{\text{B}}} + {}^{\text{B}} \omega_{I, B} \times {}^{\text{B}} \dot{r}_{o_{\text{E}}, o_{\text{B}}}, \quad (30)$$

where the inertial velocity in B-frame coordinates is defined by ${}^{\text{B}} \dot{r}_{o_{\text{E}}, o_{\text{B}}} = [u \ v \ w]^T$ and its derivative in B frame as ${}^{\text{B}} \ddot{r}_{o_{\text{E}}, o_{\text{B}}} = [\dot{u} \ \dot{v} \ \dot{w}]^T$.

For the computation of velocities on the right-hand side (RHS) of Eq. (28)

$${}^{\text{B}} \dot{r}_{o_{\text{B}}, o_{\text{SM}}} = {}^{\text{B}} \mathcal{R}_{\text{SH}}^{\text{SH}} \dot{r}_{o_{\text{B}}, o_{\text{SH}}} + {}^{\text{B}} \mathcal{R}_{\text{SM}}^{\text{SM}} \dot{r}_{o_{\text{SH}}, o_{\text{SM}}}, \quad (31)$$

and accelerations on the RHS of Eq. (29)

$${}^{\text{B}} \ddot{r}_{o_{\text{B}}, o_{\text{SM}}} = {}^{\text{B}} \mathcal{R}_{\text{SH}}^{\text{SH}} \ddot{r}_{o_{\text{B}}, o_{\text{SH}}} + {}^{\text{B}} \mathcal{R}_{\text{SM}}^{\text{SM}} \ddot{r}_{o_{\text{SH}}, o_{\text{SM}}}, \quad (32)$$

they are defined in terms of the first and second derivatives of positions in Eqs. (25) and (26). This step is crucial, as these position vectors have a single non-zero element; thus, the resulting derivatives are also single element when performed in the SH and SM frame, respectively.

Because the liquid level is assumed to decrease slowly, the rate of change of the attitude of the SH frame is close to zero. Consequently, the following angular velocity and angular acceleration of the SH frame with respect to the inertial reference can be expressed as follows:

$$\begin{aligned} {}^{\text{SH}} \star_{I, \text{SH}} &= {}^{\text{SH}} \star_{I, B} + \overset{0}{\cancel{{}^{\text{SH}} \star_{B, \text{SH}}}} = {}^{\text{SH}} \star_{I, B}, \\ \text{where } \star &= \{\omega, \alpha\}. \end{aligned} \quad (33)$$

Similarly, the angular velocity and angular acceleration of the mass with respect to the I frame are given by:

$$\begin{aligned} {}^{\text{SM}} \star_{I, \text{SM}} &= {}^{\text{SM}} \star_{I, \text{SH}} + {}^{\text{SM}} \star_{\text{SH}, \text{SM}}, \\ \text{where } \star &= \{\omega, \alpha\}. \end{aligned} \quad (34)$$

Recall that the body-frame angular velocity is given by ${}^B\omega_{I,B} = [p \ q \ r]^\top$, and the body-frame angular acceleration is ${}^B\alpha_{I,B} = [\alpha_1 \ \alpha_2 \ \alpha_3]^\top$. The angular velocity is

$${}^{SM}\omega_{B,SM} = {}^{SM}\mathcal{T}_I {}^{SM}\dot{\epsilon}_B, \quad (35)$$

and the angular acceleration is expressed as follows:

$${}^{SM}\alpha_{B,SM} = {}^{SM}\mathcal{T}_B {}^{SM}\ddot{\epsilon}_B + {}^{SM}\dot{\mathcal{T}}_B {}^{SM}\dot{\epsilon}_B, \quad (36)$$

as a function of the first and second derivatives of the pendulum's Euler angles.

The expanded forms of Eqs. (35) and (36) are provided in Appendix B.

Recall that ${}^{SM}\epsilon_B = [0 \ \theta_{SM} \ \psi_{SM}]^\top$ and ${}^{SH}\omega_{B,SH} = \mathbf{0}$ hold. Thus, ${}^{SH}\omega_{I,SH} = {}^{SH}\mathcal{R}_B [p \ q \ r]^\top$ follows. Moreover, since ${}^{SH}\alpha_{B,SH} = \mathbf{0}$ applies, ${}^{SH}\alpha_{I,SH} = {}^{SH}\mathcal{R}_B [\alpha_1 \ \alpha_2 \ \alpha_3]^\top$ also holds.

Under the earlier assumptions, the expanded expressions for the velocities on the RHS of Eq. (31) are

$${}^{SH}\dot{\mathbf{r}}_{I \ O_B, O_{SH}} = \underbrace{{}^{SH}\dot{\mathbf{r}}_{O_B, O_{SH}}}_{\mathbf{0}} + \underbrace{{}^{SH}\omega_{I,SH}}_{{}^{SH}\omega_{I,B}} \times {}^{SH}\mathbf{r}_{O_B, O_{SH}}, \quad (37)$$

and

$${}^{SM}\dot{\mathbf{r}}_{I \ O_{SH}, O_{SM}} = \underbrace{{}^{SM}\dot{\mathbf{r}}_{O_{SH}, O_{SM}}}_{\mathbf{0}} + {}^{SM}\omega_{I,SM} \times {}^{SM}\mathbf{r}_{O_{SH}, O_{SM}}. \quad (38)$$

Similarly, the accelerations on the RHS of Eq. (32) are given by

$$\begin{aligned} {}^{SH}\ddot{\mathbf{r}}_{I \ O_B, O_{SH}} &= \underbrace{{}^{SH}\ddot{\mathbf{r}}_{O_B, O_{SH}}}_{\mathbf{0}} + \underbrace{{}^{SH}\alpha_{B,SH}}_{{}^{SH}\alpha_{I,B}} \times {}^{SH}\mathbf{r}_{O_B, O_{SH}} \\ &\quad + 2 \underbrace{{}^{SH}\omega_{I,SH}}_{{}^{SH}\omega_{I,B}} \times \underbrace{{}^{SH}\dot{\mathbf{r}}_{O_B, O_{SH}}}_{\mathbf{0}} \\ &\quad + \underbrace{{}^{SH}\omega_{I,SH}}_{{}^{SH}\omega_{I,B}} \times \underbrace{{}^{SH}\omega_{I,SH}}_{{}^{SH}\omega_{I,B}} \times {}^{SH}\mathbf{r}_{O_B, O_{SH}}, \end{aligned} \quad (39)$$

and

$$\begin{aligned} {}^{SM}\ddot{\mathbf{r}}_{I \ O_{SH}, O_{SM}} &= \underbrace{{}^{SM}\ddot{\mathbf{r}}_{O_{SH}, O_{SM}}}_{\mathbf{0}} + {}^{SM}\alpha_{I,SM} \times {}^{SM}\mathbf{r}_{O_{SH}, O_{SM}} \\ &\quad + 2 \underbrace{{}^{SM}\omega_{I,SM}}_{{}^{SM}\omega_{I,SM}} \times \underbrace{{}^{SM}\dot{\mathbf{r}}_{O_{SH}, O_{SM}}}_{\mathbf{0}} \\ &\quad + {}^{SM}\omega_{I,SM} \times {}^{SM}\omega_{I,SM} \times {}^{SM}\mathbf{r}_{O_{SH}, O_{SM}}. \end{aligned} \quad (40)$$

Substituting Eqs. (37) and (38) into Eq. (31) yields the velocity

$${}^B\dot{\mathbf{r}}_{I \ O_B, O_{SM}} = {}^B\omega_{I,B} \times {}^B\mathbf{r}_{O_B, O_{SH}} + {}^B\omega_{B,SM} \times {}^B\mathbf{r}_{O_{SH}, O_{SM}}. \quad (41)$$

Similarly, substituting Eqs. (39) and (40) into Eq. (32) yields the acceleration

$$\begin{aligned} {}^{B,I}\ddot{\mathbf{r}}_{I \ O_B, O_{SM}} &= {}^B\alpha_{I,B} \times {}^B\mathbf{r}_{O_B, O_{SH}} + {}^B\omega_{I,B} \times {}^B\omega_{I,B} \\ &\quad \times {}^B\mathbf{r}_{O_B, O_{SH}} + {}^B\alpha_{B,SM} \times {}^B\mathbf{r}_{O_{SH}, O_{SM}} \\ &\quad + {}^B\omega_{B,SM} \times {}^B\omega_{B,SM} \times {}^B\mathbf{r}_{O_{SH}, O_{SM}}. \end{aligned} \quad (42)$$

After performing the simplifications for planar flight, the velocity of the i th sloshing mass in the pitch plane is expressed in B-frame coordinates as

$${}^B\dot{\mathbf{r}}_{I \ O_B, O_{SM}} = \begin{bmatrix} (\dot{\theta} + \dot{\theta}_{SM})\ell_{SM} \sin \theta_{SM} + \dot{\theta}\ell_{SH} \sin \theta_{SH} \\ 0 \\ (\dot{\theta} + \dot{\theta}_{SM})\ell_{SM} \cos \theta_{SM} - \dot{\theta}\ell_{SH} \cos \theta_{SH} \end{bmatrix}, \quad (43)$$

and the corresponding acceleration is

$$\begin{aligned} {}^{B,I}\ddot{\mathbf{r}}_{I \ O_B, O_{SM}} &= \\ &\begin{bmatrix} (\ddot{\theta} + \ddot{\theta}_{SM})\ell_{SM} \sin \theta_{SM} + (\dot{\theta} + \dot{\theta}_{SM})^2 \ell_{SM} \cos \theta_{SM} + \ddot{\theta}\ell_{SH} \sin \theta_{SH} - \dot{\theta}^2 \ell_{SH} \cos \theta_{SH} \\ 0 \\ (\ddot{\theta} + \ddot{\theta}_{SM})\ell_{SM} \cos \theta_{SM} - (\dot{\theta} + \dot{\theta}_{SM})^2 \ell_{SM} \sin \theta_{SM} - \ddot{\theta}\ell_{SH} \cos \theta_{SH} - \dot{\theta}^2 \ell_{SH} \sin \theta_{SH} \end{bmatrix}. \end{aligned} \quad (44)$$

3.3 Newton–Euler modeling

For a system with multiple pendula, deriving the equations can become quite complicated when performed analytically. This section focuses on obtaining the equations for a single pendulum connected to the rigid body; the analysis can later be extended to cover a system with multiple bodies.

The iterative Newton–Euler method [21, 22, 40] has seen success in the multibody dynamics and robotics fields [23, 41], where it is used to compute the EoMs of kinematic chains of systems with tree-like structures. In other words, it can be readily applied to the sloshing dynamics problem,³ which leads to a representative model that can be conveniently implemented in a simulation environment (e.g., Simulink®). Figure 5 depicts the simplest configuration for an RLV, with one tank and one sloshing mode. The pendulum is attached to the vehicle at the hinge point O_{SH} . The equations that govern the interaction between the two bodies are

$$\begin{aligned} m_B {}^I\mathbf{a}_{O_E, O_B} &= {}^I\mathbf{f}_{totB} + {}^I\mathbf{f}_r, \\ {}^I\mathbf{J}_B {}^I\alpha_{I,B} + {}^I\omega_{I,B} \times ({}^I\mathbf{J}_B {}^I\omega_{I,B}) \end{aligned} \quad (46)$$

³ This model can also be used for the calculation of engine-induced oscillations (Tail-Wag-Dog) with the proper modifications to account for the position of the engine relative to the CoM of the vehicle.

$$= {}^{\mathbb{I}}\boldsymbol{\tau}_{\text{tofSM}} - {}^{\mathbb{I}}\boldsymbol{\tau}_{\text{r}} - {}^{\mathbb{I}}\boldsymbol{r}_{\text{OSM,OSH}} \times {}^{\mathbb{I}}\boldsymbol{f}_{\text{r}}, \quad (49)$$
$$\star \tau_{\text{totSM}} = 0. \quad (53)$$
$${}^{\text{B}}f_{\text{sloshB}} = -m_i \begin{bmatrix} (\ddot{\theta} + \ddot{\theta}_{\text{SM}})\ell_{\text{SM}} \sin \theta_{\text{SM}} + (\dot{\theta} + \dot{\theta}_{\text{SM}})^2 \ell_{\text{SM}} \cos \theta_{\text{SM}} - \dot{\theta}^2 \ell_{\text{SH}} \\ 0 \\ (\ddot{\theta} + \ddot{\theta}_{\text{SM}})\ell_{\text{SM}} \cos \theta_{\text{SM}} - (\dot{\theta} + \dot{\theta}_{\text{SM}})^2 \ell_{\text{SM}} \sin \theta_{\text{SM}} - \ddot{\theta} \ell_{\text{SH}} \end{bmatrix}. \quad (61)$$

For the case where $\theta_{SH} \neq 0$, the reader is referred to Eq. (45) for the complete expression.

The computation of the damping coefficient ζ_{SM} for cylindrical tanks is discussed by Dodge [33], which provides formulations for approximating it using different criteria. For the purposes of this publication, the damping is integrated into the EoMs via the damping force and, consequently, the damping torque. The damping force, as seen by the sloshing mass, is

$${}^{SM}f_{dSM} = -\zeta_{SM} {}^{SM}\dot{v}_{B,SM}, \quad (62)$$

and, as seen by the rigid body

$${}^B f_{dB} = \zeta_{SM} {}^B v_{B,SM}. \quad (63)$$

Note that the first component of the damping force is assumed to be zero when expressed at the sloshing mass location (${}^{SM}f_{x_{dB}} = 0$ and ${}^{SM}f_{x_{dSM}} = 0$), because the damping is effectively only active on the lateral axes.

Consequently, the damping torque, as seen at the CoM of the sloshing mass, is defined as

$${}^{SM}\tau_{dSM} = {}^{SM}r_{SM,B} \times {}^B f_{dB}, \quad (64)$$

and at the CoM of the rigid body as

$${}^B\tau_{dB} = {}^B r_{B,SM} \times {}^B f_{dSM}. \quad (65)$$

The damping torque and force have opposite signs depending on whether they are expressed at the CoM of the rigid body or at the CoM of the sloshing mass, i.e., $\star f_{dB} = -\star f_{dSM}$, and $\star \tau_{dB} = -\star \tau_{dSM}$. The damping torque is not explicitly considered, as it is incorporated in the EoMs via the damping force.

Regarding the attitude dynamics of the multibody arrangement, the rotational dynamics of the pendulum are embedded into the rigid-body dynamics (Eq. 47) by means of the reaction torque (Eq. 55) and the reaction force (Eq. 54). The first step is to eliminate the pendulum's MoI, for the reasons explained in Remark 1. This approach is consistent with the methodologies proposed by other authors [42–44]. Secondly, as the focus is primarily on the dynamics of the vehicle in the B frame, the MoI is likewise expressed in B-frame coordinates,⁴ yielding the following expressions:

$${}^B J_B {}^B \alpha_{I,B}$$

⁴ For the frame transformation of the MoI one must pre- and post-multiply by the rotation matrix and its transpose, i.e., ${}^B J_{tot} = {}^B \mathcal{R}_I {}^I J_{tot} {}^B \mathcal{R}_I^T$.

$$+ {}^B \omega_{I,B} \times ({}^B J_B {}^B \omega_{I,B}) \\ = {}^B \tau_{totB} + {}^B \tau_r + {}^B r_{OB,OSH} \times {}^B f_r, \quad (66)$$

$$= {}^B \tau_{totB} + {}^B \tau_{totSM} + {}^B r_{OB,OSM} \\ \times \left(-m_i {}^B a_{OE,OSM} + {}^B f_{totSM} \right), \quad (67)$$

$$= {}^B \tau_{aero} + {}^B \tau_{rcs} + {}^B \tau_{tvc} \\ - \underbrace{{}^B r_{OB,OSM} \times \left(m_i \left({}^B a_{OE,OSM} - {}^B g \right) - {}^B f_{dSM} \right)}_{-{}^B \tau_{sloshB}}. \quad (68)$$

Accordingly, the sloshing torque is defined as

$${}^B \tau_{sloshB} = -{}^B r_{OB,OSM} \times \left(m_i \left({}^B a_{OE,OSM} - {}^B g \right) - {}^B f_{dSM} \right), \quad (69)$$

which for the planar case is expressed as ${}^B \tau_{sloshB} = \begin{bmatrix} 0 & \tau_{y_{sloshB}} & 0 \end{bmatrix}^T$, i.e.,

$${}^B \tau_{y_{sloshB}} = -m_i \left(\Gamma + {}^B \ddot{x}_{OB,OSM} \right) (\ell_{SM} \sin \theta_{SM} + \ell_{SH} \sin \theta_{SH}) \\ + {}^B \ddot{z}_{OB,OSM} (\ell_{SM} \cos \theta_{SM} - \ell_{SH} \cos \theta_{SH}) \\ - \zeta_{SM} \dot{\theta}_{SM} \ell_{SM} (\ell_{SM} - \ell_{SH} \cos(\theta_{SH} + \theta_{SM})). \quad (70)$$

The computation of the EoMs for the pendulum is a simpler process. The pendulum mass and arm are assumed to be decoupled from the rigid body. For the position dynamics, Eq. (48) is considered with the reaction force set to zero, yielding

$$m_i \left({}^{SM} \ddot{r}_{I,I} {}^{SM} r_{OB,OSM} - {}^{SM} g \right) - {}^{SM} f_{dSM} = 0. \quad (71)$$

The same assumptions apply to the rotational dynamics. In this case, Eq. (49) is simplified by neglecting the effect of the reaction torque while retaining the contribution of the reaction force

$$- {}^{SM} r_{OB,OSM} \times \left(m_i \left({}^{SM} \ddot{r}_{I,I} {}^{SM} r_{OB,OSM} - {}^{SM} g \right) - {}^{SM} f_{dSM} \right) = 0. \quad (72)$$

It is not necessary to include both the position and the rotation of the pendulum in the calculation of the linearized state-space model, as either is sufficient. Some authors prefer to include the lateral effects [44], whereas this article focuses on the rotational dynamics to leverage the proposed modeling framework. For the planar case, Eq. (72) can be further

simplified as follows:

$$\begin{aligned} m_i \left(\left(\Gamma + {}^{\text{B}}\ddot{x}_{\text{O}_B, \text{O}_{\text{SM}}} \right) (\ell_{\text{SM}} \sin \theta_{\text{SM}} + \ell_{\text{SH}} \sin \theta_{\text{SH}}) \right. \\ \left. + {}^{\text{B}}\ddot{z}_{\text{O}_B, \text{O}_{\text{SM}}} (\ell_{\text{SM}} \cos \theta_{\text{SM}} - \ell_{\text{SH}} \cos \theta_{\text{SH}}) \right) \\ + \zeta_{\text{SM}} \dot{\theta}_{\text{SM}} \ell_{\text{SM}} (\ell_{\text{SM}} - \ell_{\text{SH}} \cos(\theta_{\text{SH}} + \theta_{\text{SM}})) = 0. \end{aligned} \quad (73)$$

3.4 Summary of the equations

For clarity, Eqs. (59), (68), (71) and (72) are grouped below^{5,6}:

$$m_{\text{tot}} {}^{\text{B}}\ddot{\mathbf{r}}_{\text{O}_B, \text{O}_B} = m_{\text{tot}} {}^{\text{B}}\mathbf{g} + {}^{\text{B}}\mathbf{f}_{\text{aero}} + {}^{\text{B}}\mathbf{f}_{\text{tvc}} + {}^{\text{B}}\mathbf{f}_{\text{slosh}}, \quad (59)$$

$$\begin{aligned} {}^{\text{B}}\mathbf{J}_B {}^{\text{B}}\boldsymbol{\alpha}_{\text{I}, \text{B}} + {}^{\text{B}}\boldsymbol{\omega}_{\text{I}, \text{B}} \times ({}^{\text{B}}\mathbf{J}_B {}^{\text{B}}\boldsymbol{\omega}_{\text{I}, \text{B}}) \\ = {}^{\text{B}}\boldsymbol{\tau}_{\text{aero}} + {}^{\text{B}}\boldsymbol{\tau}_{\text{rcs}} + {}^{\text{B}}\boldsymbol{\tau}_{\text{tvc}} + {}^{\text{B}}\boldsymbol{\tau}_{\text{sloshB}}, \end{aligned} \quad (68)$$

$$0 = m_i \left({}^{\text{SM}}\ddot{\mathbf{r}}_{\text{I}, \text{I}}^{\text{SM}} - {}^{\text{SM}}\mathbf{g} \right) - {}^{\text{SM}}\mathbf{f}_{\text{dSM}}, \quad (71)$$

$$0 = {}^{\text{SM}}\mathbf{r}_{\text{O}_{\text{SM}}, \text{O}_{\text{SH}}} \times \left(m_i \left({}^{\text{SM}}\ddot{\mathbf{r}}_{\text{I}, \text{I}}^{\text{SM}} - {}^{\text{SM}}\mathbf{g} \right) - {}^{\text{SM}}\mathbf{f}_{\text{dSM}} \right). \quad (72)$$

These definitions can be implemented directly within a 6-DoF simulation environment. This is the primary reason for adopting the iterative Newton–Euler approach to derive the model, as it can be readily converted into code for numerical simulations. The pendulum model can also be viewed as an add-on component to an existing full rigid-body simulation; that is, only ${}^{\text{B}}\mathbf{f}_{\text{slosh}}$, ${}^{\text{B}}\boldsymbol{\tau}_{\text{slosh}}$, and Eq. (72) constitute the additional components. It should be noted that the equations have been validated against an alternative approach: the Lagrangian formulation, which is also implemented in the *Symbolic Multibody Dynamics* toolbox. Furthermore, when applying the forces and torques, they are considered to act at the total mass CoM. However, since the ratio m_i/m_B is small, it is reasonable to assume that they are applied at the same location, thereby avoiding unnecessary complexity in the equations (i.e., $\star \mathbf{r}_{\text{B}, \text{BT}} \approx 0$).

When deactivating sloshing (making sloshing angles zero), Eqs. (59) and (68) reduce to a set of interconnected static masses. The effects of mass variation are modeled by adding the extra terms on the right-hand side of Eqs. (59) and (68), they are discussed in Appendix C.

⁵ An alternative formulation of the position dynamics equation is $m_B {}^{\text{B}}\ddot{\mathbf{r}}_{\text{O}_B, \text{O}_B} = m_B {}^{\text{B}}\mathbf{g} + {}^{\text{B}}\mathbf{f}_{\text{aero}} + {}^{\text{B}}\mathbf{f}_{\text{tvc}} - m_i \left({}^{\text{B}}\ddot{\mathbf{r}}_{\text{O}_B, \text{O}_{\text{SM}}} - {}^{\text{B}}\mathbf{g} \right)$.

⁶ The longitudinal component of the non-gravitational acceleration is defined as $\Gamma = (\mathbf{T} - \mathbf{A})/m_{\text{tot}} = {}^{\text{B}}\ddot{x} + g \sin \theta$.

3.5 Linearization

For the linearization of the EoMs, attention is focused on Eqs. (59), (68), and (72). Several derivation steps have been omitted here to keep the length of this publication manageable:

1. Substitute the translational and rotational quantities presented in Sects. 3.1–3.2.
2. Since the primary interest lies in the pitch-directional motion (while the model for the yaw-lateral motion is equivalent), planar flight is assumed, and the yaw-lateral motion is therefore temporarily neglected.
3. Since the vehicle is assumed to be flying a gravity-turn trajectory, small perturbations about the reference solution are considered.
4. All trim angles are assumed to be small; thus, trigonometric functions can be linearized (for an angle a , $\sin a \approx a$ and $\cos a \approx 1$).
5. Products of two small quantities are assumed to be negligible ($ab \approx 0$).
6. The longitudinal non-gravitational acceleration has been aggregated into element Γ .
7. If one performs the simplifications and examines the resulting model for the coupled longitudinal axis, the first-order slosh contribution in the longitudinal channel cancels.
8. The mapping of the damping coefficient was defined as $\zeta_{\text{SM}} = 2\hat{\zeta}_{\text{SM}}\sqrt{\Gamma/\ell_{\text{SM}}}m_i$, where ζ_{SM} represents the true (physical) damping coefficient and $\hat{\zeta}_{\text{SM}}$ the modal damping ratio. It is also affected by the presence of baffles or is assumed to be zero when no value is provided.

The set of nonlinear simplified equations is denoted by

$$\begin{aligned} m_i(\ell_{\text{SM}} - \ell_{\text{SH}})\ddot{\theta} + (m_B + m_i)\Gamma\theta + (m_B + m_i){}^{\text{B}}\ddot{z} \\ + m_i\ell_{\text{SM}}\ddot{\theta}_{\text{SM}} = {}^{\text{B}}f_{\text{ztvc}} + {}^{\text{B}}f_{\text{zaero}}, \end{aligned} \quad (74)$$

$$\begin{aligned} \left(J_{yy_B} + m_i(\ell_{\text{SM}} - \ell_{\text{SH}})^2 \right) \ddot{\theta} + m_i\Gamma(\ell_{\text{SM}} - \ell_{\text{SH}})\theta \\ + m_i(\ell_{\text{SM}} - \ell_{\text{SH}}){}^{\text{B}}\ddot{z} \\ + m_i\ell_{\text{SM}}(\ell_{\text{SM}} - \ell_{\text{SH}})\ddot{\theta}_{\text{SM}} + \zeta_{\text{SM}}\ell_{\text{SM}}(\ell_{\text{SM}} - \ell_{\text{SH}})\dot{\theta}_{\text{SM}} \\ + \Gamma m_i\ell_{\text{SM}}\theta_{\text{SM}} = {}^{\text{B}}\tau_{\text{ytvc}} + {}^{\text{B}}\tau_{\text{yaero}}, \end{aligned} \quad (75)$$

$$\begin{aligned} m_i(\ell_{\text{SM}} - \ell_{\text{SH}})\ddot{\theta} + m_i\Gamma\theta + m_i{}^{\text{B}}\ddot{z} + m_i\ell_{\text{SM}}\ddot{\theta}_{\text{SM}} \\ + \zeta_{\text{SM}}\ell_{\text{SM}}\dot{\theta}_{\text{SM}} + m_i\Gamma\theta_{\text{SM}} = 0. \end{aligned} \quad (76)$$

A key finding in Eq. (76) is the emergence of a harmonic oscillator corresponding to the pendulum dynamics, representing a second-order system with a natural frequency given by $\omega_n^2 = \Gamma/\ell_{\text{SM}}$.

3.6 State-space representation

For the computation of the state-space equations, the linearized aerodynamic forces and torque can be incorporated into the equations as follows [35]:

$${}^B\bar{f}_{x_{aero}} = \frac{{}^B f_{x_{aero}}}{m_{tot}} = -X_\alpha \theta - X_{\dot{z}} \dot{z} - X_{\delta_\theta} \delta_\theta + X_{\dot{v}_W} v_W, \quad (77)$$

$${}^B\bar{f}_{z_{aero}} = \frac{{}^B f_{z_{aero}}}{m_{tot}} = -Z_\alpha \theta - Z_{\dot{z}} \dot{z} - Z_{\delta_\theta} \delta_\theta + Z_{\dot{v}_W} v_W, \quad (78)$$

$${}^B\bar{\tau}_{y_{aero}} = \frac{{}^B \tau_{y_{aero}}}{J_{yy_{tot}}} = \mu_\alpha \theta + \mu_{\dot{z}} \dot{z} + \mu_{\delta_\theta} \delta_\theta - \mu_{\dot{v}_W} v_W, \quad (79)$$

where δ_θ denotes the virtual fin deflection for the pitch channel. The linearized thrust vector control (TVC) dynamics are expressed as [35]

$${}^B\bar{f}_{x_{tvc}} = \frac{{}^B f_{x_{tvc}}}{m_{tot}} = -X_{\delta_y} \delta_y + X_T T, \quad (80)$$

$${}^B\bar{f}_{z_{tvc}} = \frac{{}^B f_{z_{tvc}}}{m_{tot}} = -Z_{\delta_y} \delta_y - Z_T T, \quad (81)$$

$${}^B\bar{\tau}_{y_{tvc}} = \frac{{}^B \tau_{y_{tvc}}}{J_{yy_{tot}}} = -\mu_{\delta_y} \delta_y - \mu_T T. \quad (82)$$

Here, δ_y is the pitch TVC deflection and T is the thrust magnitude. The definitions of the remaining coefficients can be found in Maces Hernandez et al. [35]. After rearranging Eqs. (74), (75), and (76) and subsequently taking the derivatives with respect to the states vector $\mathbf{x} = [\theta \ \dot{\theta} \ {}^B\dot{z} \ {}^B\ddot{z} \ \dot{\theta}_{SM} \ \ddot{\theta}_{SM}]^T$ and the control inputs vector $\mathbf{u} = [\delta_y \ \delta_\theta \ v_W]^T$, the state-space representation in the form $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ can be obtained

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ {}^B\dot{z} \\ {}^B\ddot{z} \\ \dot{\theta}_{SM} \\ \ddot{\theta}_{SM} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \mu_\alpha & 0 & 0 & \mu_{\dot{z}} & -\mu_{\ddot{\theta},\theta_{SM}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\Gamma - Z_\alpha & 0 & 0 & -Z_{\dot{z}} & Z_{\ddot{z},\theta_{SM}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \mu_{\ddot{\theta}_{SM},\theta} & 0 & 0 & \mu_{\ddot{\theta}_{SM},\dot{z}} & -\mu_{\ddot{\theta}_{SM},\theta_{SM}} & -\mu_{\ddot{\theta}_{SM},\dot{\theta}_{SM}} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ {}^B\dot{z} \\ {}^B\ddot{z} \\ \dot{\theta}_{SM} \\ \ddot{\theta}_{SM} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\mu_{\delta_y} & \mu_{\delta_\theta} & -\mu_{v_W} \\ 0 & 0 & 0 \\ -Z_{\delta_y} & -Z_{\delta_\theta} & Z_{v_W} \\ 0 & 0 & 0 \\ \mu_{\ddot{\theta}_{SM},\delta_y} & \mu_{\ddot{\theta}_{SM},\delta_\theta} & -\mu_{\ddot{\theta}_{SM},v_W} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_\theta \\ v_W \end{bmatrix}, \quad (83)$$

with

$$\mu_{\ddot{\theta},\theta_{SM}} = \frac{\Gamma \ell_{SH} m_i}{J_{yy_B}},$$

$$\begin{aligned} Z_{\ddot{z},\theta_{SM}} &= \frac{\Gamma m_i}{m_B}, \\ \mu_{\ddot{\theta}_{SM},\theta} &= \frac{Z_\alpha J_{yy_B} m_{tot} - \mu_\alpha J_{yy_{tot}} m_B (\ell_{SM} - \ell_{SH})}{\ell_{SM} J_{yy_B} m_B}, \\ \mu_{\ddot{\theta}_{SM},\dot{z}} &= \frac{Z_{\dot{z}} J_{yy_B} m_{tot} - \mu_{\dot{z}} J_{yy_{tot}} m_B (\ell_{SM} - \ell_{SH})}{\ell_{SM} J_{yy_B} m_B}, \\ \mu_{\ddot{\theta}_{SM},\theta_{SM}} &= \frac{\Gamma (J_{yy_B} m_{tot} - m_i m_B \ell_{SH} (\ell_{SM} - \ell_{SH}))}{\ell_{SM} J_{yy_B} m_B}, \\ \mu_{\ddot{\theta}_{SM},\dot{\theta}_{SM}} &= \frac{\zeta_{SM} (J_{yy_B} + m_i \ell_{SM} (\ell_{SM} - \ell_{SH}))}{J_{yy_B} m_i}, \\ \mu_{\ddot{\theta}_{SM},\delta_y} &= \frac{Z_{\delta_y} J_{yy_B} m_{tot} + \mu_{\delta_y} J_{yy_{tot}} m_B (\ell_{SM} - \ell_{SH})}{\ell_{SM} J_{yy_B} m_i}, \\ \mu_{\ddot{\theta}_{SM},\delta_\theta} &= \frac{Z_{\delta_\theta} J_{yy_B} m_{tot} - \mu_{\delta_\theta} J_{yy_{tot}} m_B (\ell_{SM} - \ell_{SH})}{\ell_{SM} J_{yy_B} m_i}, \\ \mu_{\ddot{\theta}_{SM},v_W} &= \frac{Z_{v_W} J_{yy_B} m_{tot} - \mu_{v_W} J_{yy_{tot}} m_B (\ell_{SM} - \ell_{SH})}{\ell_{SM} J_{yy_B} m_i}. \end{aligned}$$

3.7 Transfer functions for stability analysis

For all input–output combinations, attention is primarily focused on the attitude channels, namely $\theta(\mathcal{S})/\delta_y(\mathcal{S})$ and $\theta(\mathcal{S})/\delta_\theta(\mathcal{S})$, which are of principal interest for control design. The resulting attitude transfer function can be expressed as⁷

$$\frac{\theta}{\delta_j}(\mathcal{S}) = \mathcal{G}_{\text{rigid}}(\mathcal{S}) \left(\frac{\mathcal{S}^2 + 2\zeta_{z_i} \omega_{z_i} \mathcal{S} + \omega_{z_i}^2}{\mathcal{S}^2 + 2\zeta_{p_i} \omega_{p_i} \mathcal{S} + \omega_{p_i}^2} \right), \quad (84)$$

for a system with damping, and

$$\frac{\theta}{\delta_j}(\mathcal{S}) = \mathcal{G}_{\text{rigid}}(\mathcal{S}) \left(\frac{\mathcal{S}^2 + \omega_{z_i}^2}{\mathcal{S}^2 + \omega_{p_i}^2} \right), \quad (85)$$

when a more conservative analysis is done considering the damping zero, where

$$2\zeta_{z_i} \omega_{z_i} = 2\hat{\zeta}_{SM} \sqrt{\frac{\Gamma}{\ell_{SM}}} \left(\frac{J_{yy_B} + m_i \ell_{SM} (\ell_{SM} - \ell_{SH})}{J_{yy_B}} \right),$$

⁷ For the TVC channel, the classic full-order transfer function for each output can be derived from the state-space model in Eq. (83). The attitude channel is described by 1) $\mathcal{G}_{\text{rigid}}(\mathcal{S}) = -(\mu_{\delta_y} \mathcal{S} + Z_{\dot{z}} \mu_{\delta_y} + Z_{\delta_y} \mu_{\dot{z}}) / (\mathcal{S}^3 + Z_{\dot{z}} \mathcal{S}^2 - \mu_\alpha \mathcal{S} + (Z_\alpha \mu_{\dot{z}} - Z_{\dot{z}} \mu_\alpha + \Gamma \mu_{\dot{z}}))$, which can be reduced to 2) $\mathcal{G}_{\text{rigid}}(\mathcal{S}) = -\mu_{\delta_y} / (\mathcal{S}^2 - \mu_\alpha \mathcal{S})$, when assuming negligible contributions from the directional channel, and finally to 3) $\mathcal{G}_{\text{rigid}}(\mathcal{S}) = -\mu_{\delta_y} / \mathcal{S}^2$ when aerodynamic effects are neglected.

$$\omega_{z_i}^2 = \frac{\Gamma}{\ell_{SM}} \left(\frac{\ell_{\delta} m_{tot} + m_i \ell_{SH}}{\ell_{\delta} m_B} \right),$$

$$2\zeta_{p_i} \omega_{p_i} = 2\hat{\zeta}_{SM} \sqrt{\frac{\Gamma}{\ell_{SM}} \left(\frac{J_{yy_B} + m_i \ell_{SM} (\ell_{SM} - \ell_{SH})}{J_{yy_B}} \right)},$$

$$\omega_{p_i}^2 = \frac{\Gamma}{\ell_{SM}} \left(\frac{J_{yy_B} m_{tot} - m_B m_i \ell_{SH} (\ell_{SM} - \ell_{SH})}{J_{yy_B} m_B} \right).$$

It is informative to examine the two characteristic frequencies that arise in the numerator and denominator of the transfer function, namely the zero frequency ω_{z_i} and the pole frequency ω_{p_i} . According to the findings of Pei [44], which are consistent with classical references in the field, such as Greensite [11], phase stability of the sloshing mode requires that the inequality $\omega_{z_i} < \omega_{p_i}$ is satisfied.

This inequality indicates that the position of the i th sloshing mass must lie below the center of percussion, defined as $\ell_{perc} = J_{yy}/(\ell_{\delta} m_B)$. The center of percussion is the point along the longitudinal axis where the lateral actuation forces produce pure translation without inducing rotational acceleration about the CoM. If the sloshing mass is located between this point and the vehicle's CoM, the dynamic coupling can induce an unfavorable phase lag or phase instability. In this configuration, control inputs from the actuator may inadvertently inject energy into the slosh motion, thereby propagating instability from the sloshing mode to the rigid-body attitude dynamics.

This phenomenon can be interpreted more intuitively using a root-locus perspective: as the control gain increases, the trajectories of the closed-loop poles originating from the open-loop complex-conjugate poles migrate toward the open-loop complex-conjugate zeros. If these trajectories cross into the right half of the complex plane, the mode becomes unstable.

During early design phases, evaluating the position of the sloshing masses relative to the center of percussion provides a conservative and rapid screening method for identifying potentially critical configurations across the flight envelope. As the project progresses, more detailed frequency-domain analyses, including the effects of damping (e.g., via slosh-damping devices) and control bandwidth, become necessary to rigorously assess both the stability and performance of the control system.

4 Application to an RLV mission

4.1 CALLISTO mission

The CALLISTO mission is an RLV demonstrator jointly developed by German Aerospace Center (DLR), Japan Aerospace Exploration Agency (JAXA), and French National

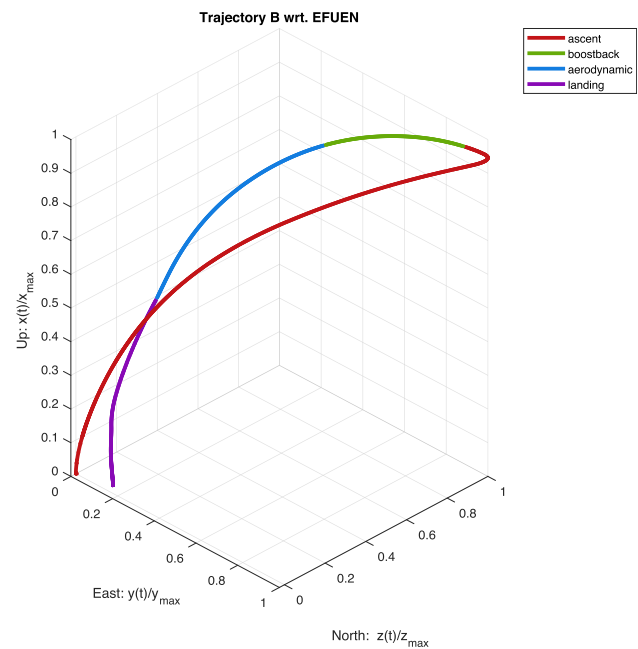


Fig. 6 RTLS reference trajectory and G&C phases

Center for Space Studies (CNES). It is 13 m tall, with a diameter slightly over 1 m. The vehicle uses a 50 kN throttleable engine burning a mixture of LOx and LH₂. The LH₂ tank is developed by DLR [45, 46] and the LOx tank by JAXA [47]. Actuation is provided by (i) two-axis gimbaled TVC for pitch and yaw; (ii) four aerodynamic fins used during reentry (roll, pitch, and yaw); and (iii) a three-axis reaction control system (RCS) fueled by hydrogen peroxide (H₂O₂), used primarily for roll control and for pitch/yaw at low dynamic pressure. All tanks are pressurized with gaseous helium (He).

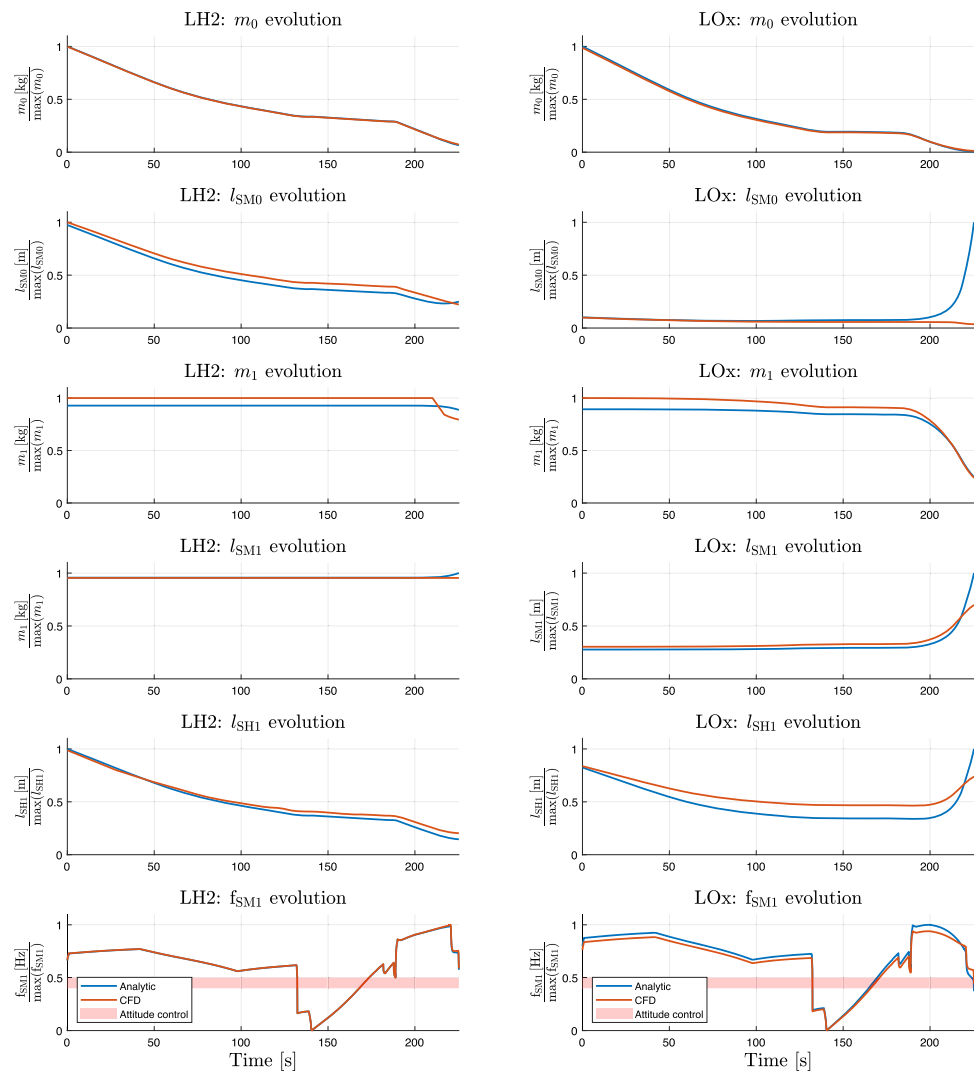
CALLISTO will conduct several test flights (A, B, C, D, and E), ranging from low-energy hover flights (low altitude, low Mach number) to high-energy RTLS scenarios (apogee exceeding 30 km and Mach number exceeding 1). Figure 6 shows the RTLS profile for one of the high-energy flights and its guidance and control (G&C) phases (ascent, boostback, aerodynamic reentry, and landing). Each phase employs distinct guidance and control strategies [24, 35, 48, 49].

For further details on the vehicle architecture, mission aspects, subsystems, objectives of the test flights, and timelines, see [47, 50–53].

4.2 Tank parameters characterization

The behavior of the LOx and LH₂ tanks was characterized within the scope of the models presented in Sect. 2. First, the parameters collected in Table 1 were computed for the baseline vehicle architecture and mission profile (RTLS scenario). Secondly, the initial values were cross-validated against newer datasets from CFD computations [45, 46, 54,

Fig. 7 Comparison between the CFD data and analytical formulas (end to end)



55]. During the early design stage, the analytical estimates were used, whereas the CFD results serve as the current baseline used for the V&V campaign. These results are depicted in Fig. 7, where relative positions, masses, and frequencies for the first mode of both tanks are presented. On the bottom left and right subplots of the figure, it can be observed that the desired closed-loop (CL) attitude bandwidth frequency is quite close to the frequency of both first modes, which poses additional challenges for the control design.

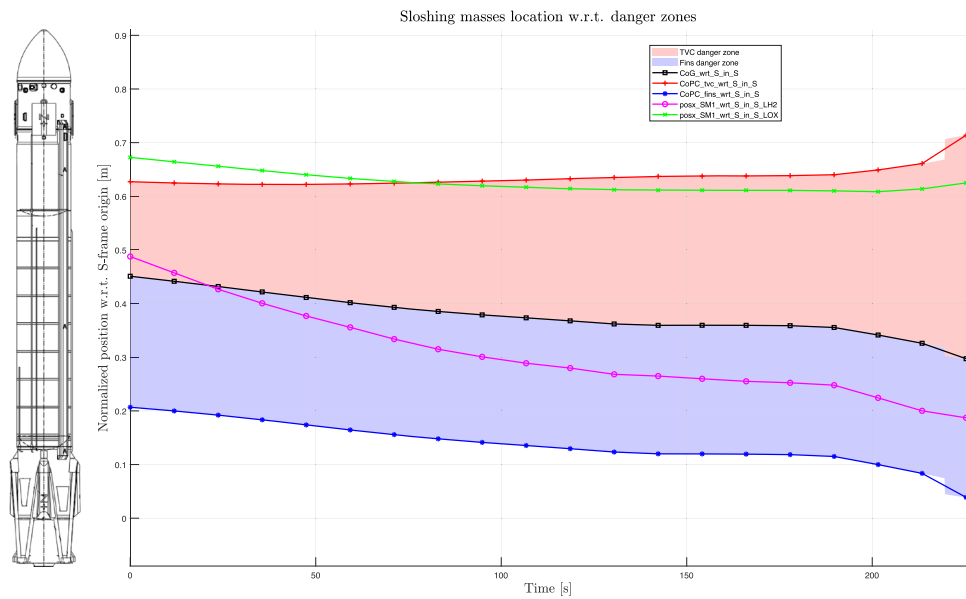
4.3 Stability analysis

Since the CALLISTO vehicle is actuated by commanding the TVC or the fins, there exists a center of percussion associated with each actuator. The evolution of the relative position of the masses throughout the entire mission is shown in Fig. 8, together with the location of the CoM and the two corresponding danger zones. In these investigations, the focus is primarily on the ascent phase of the mission [48]. This phase

employs the FFO (fins folded, legs folded and thrust on) aerodynamic configuration, although the UFO (fins unfolded, legs folded and thrust on) aerodynamic configuration is also an option, in which case the vehicle is purely TVC actuated. It is observed that the first-mode mass for the LOx tank crosses the TVC danger zone for a few seconds at the beginning of the mission. Furthermore, the first-mode mass for the LH₂ tank is always contained within the fins' danger zone. This crossing indicates that further analysis of the stability of the modes was necessary. Since the design of the tanks can only be modified during the early design phases, studies of several configurations for incorporating damping devices were conducted [46, 55] throughout the program.

The current vehicle design has converged to a point where the inclusion of the sloshing modes within the danger zones does not pose any significant risk to the mission. This means that sufficient damping is guaranteed at all times. Furthermore, from the GNC perspective, the control system has been designed so that, even if the modes are stable, the rigid-body

Fig. 8 Positions of the first-mode masses with respect to the danger zones



controller does not actively excite or interact with them. In other words, the attitude control bandwidth is intentionally placed well below the lowest sloshing frequency.

4.4 Control system design

The architecture of the control system used for synthesis during the ascent segment of the trajectory depicted in Fig. 6 is presented in Fig. 9. This system retains the architecture used in prior work [35]. Parametric uncertainties are injected into the system via the Δ_i blocks in the plant and actuator blocks.

The synthesized controller K for CALLISTO is 6-DoF, as shown in Fig. 10a. Although the longitudinal and roll channels are actively controlled, it is assumed that the effect of sloshing on them is small, and consequently is not analyzed here. Accordingly, the pitch-directional and yaw-lateral channels are depicted in Fig. 10b, c.

One of the strengths of using loop-shaping methods for the computation of the control gains of aerospace vehicles is that the CL bandwidth can be directly imposed in the formulation of the control problem via weighting functions. These weighting functions are represented by W_i in Fig. 9. Specifically for the bandwidth, this is achieved by correctly choosing the parameters of the complementary sensitivity weighting function (W_S).

Since CALLISTO has two tanks (LOx and LH₂), one sloshing mode per axis is considered for each tank. The state-space model (83) is extended with two states to account for the other tank. The extended model (represented as G_{plant} in Fig. 9) is evaluated along the red segment in the trajectory depicted in Fig. 6. The perturbations are provided by the lat-

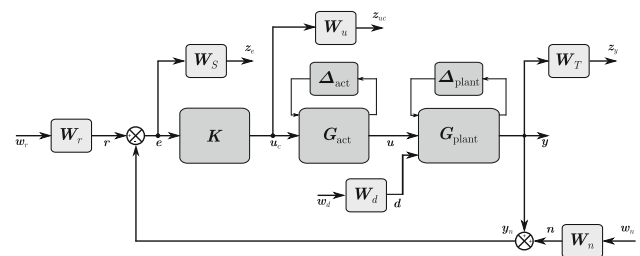


Fig. 9 Control synthesis loop and weighting functions for the ascent phase

est releases of the CFD data.⁸ Block G_{act} models the TVC dynamics, included parametric uncertainties and delays. The objective of the formulation is to compute the value of K that meets the nominal and robust designs.

One of the first design challenges faced during the synthesis of the gains was that direct filtering using anti-resonance filters was not feasible due to their close proximity to the rigid-body bandwidth, as seen in Fig. 7. These filters can also degrade rigid-body control effectiveness. Decreasing the rigid-body bandwidth also degrades performance during trajectory tracking and mitigation of wind disturbances.

The loop-transfer functions (L_θ and L_ψ) for the baseline mission are presented in Fig. 11. Classical gain and phase margins were imposed to account for uncertainties in the plant, sloshing parameters, and wind disturbances (6 dB and 30°). Margins within the design requirements were achieved for both nominal and robust stability of the closed-loop state-space model; however, in Fig. 11, only those corresponding to the decoupled attitude channel are presented for proprietary reasons.

⁸ The CFD data are only available internally within the project.

For a more comprehensive overview of the control system for the mission, the reader is referred to earlier works [35, 48, 49].

4.5 Monte Carlo results

The control design synthesized in Sect. 4.4 was tested in a 6-DoF simulator against nominal and perturbed conditions. The simulator includes high-fidelity models of the CALISTO vehicle. The nonlinear equations derived in Sect. 3 were implemented for the rigid body and the LOx and LH₂ tanks; only the static mass and the first sloshing mode are modeled in each tank.

Other models included in the framework are Mass, Centering, Inertia (MCI) variation, a full aerodynamic model covering all flight configurations, bending effects, advanced actuator models with nonlinearities, environmental models with wind profiles for the launch site in Kourou (French Guiana), Hybrid Navigation System (HNS), and communication delays between subsystems.

All uncertain parameters are randomly sampled according to dedicated probability distributions for n runs. At the end of the campaign, the post-processing framework evaluates several mission requirements for critical states of the vehicle. For instance, in Fig. 12 a subset of 250 runs from a larger MC

campaign is shown for the sloshing angles in both tanks. The top plots show the results for the LOx tank and the bottom plots for the LH₂ tank. The plots on the left side represent the angles in the pitch plane, and the plots on the right depict those in the yaw plane. The red line represents one of the requirements imposed at the system level that limits lateral-acceleration-induced angles.

5 Conclusion

In this paper, a modeling framework for liquid behavior inside the propellant tanks of liquid-propelled space vehicles is presented, tailored to GNC analysis and design. First, the parameters required to approximate sloshing with mechanically equivalent models are identified, and their applicability from early design phases through flight validation is established.

Secondly, the 3D model was derived using the Newton–Euler iterative method for multibody systems for a single mode, and the tools needed for its extension to multiple tanks, multiple modes, and different tank positions relative to the CoM of the vehicle were provided. The process of reducing the 3D equations to a planar simplification was described, followed by the computation of a state-space model for a generic

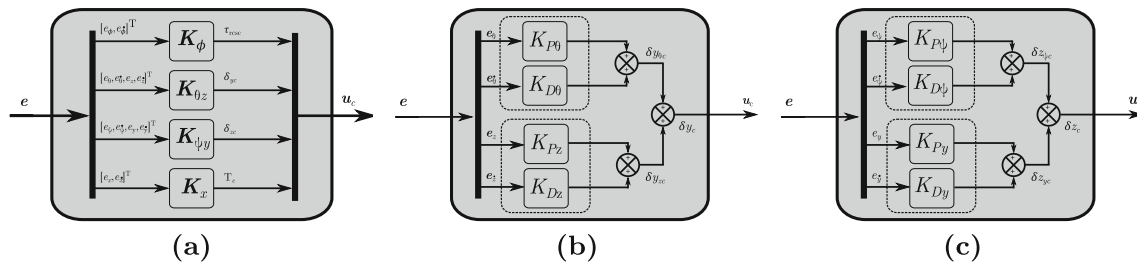


Fig. 10 Flight control system for the ascent phase: **a** 6-DoF controller; **b** pitch-directional controller; **c** yaw-lateral controller

Fig. 11 Nichols plot for the pitch and yaw channels ($L_\theta(\mathcal{S})$ and $L_\psi(\mathcal{S})$), with two tanks (ascent phase)

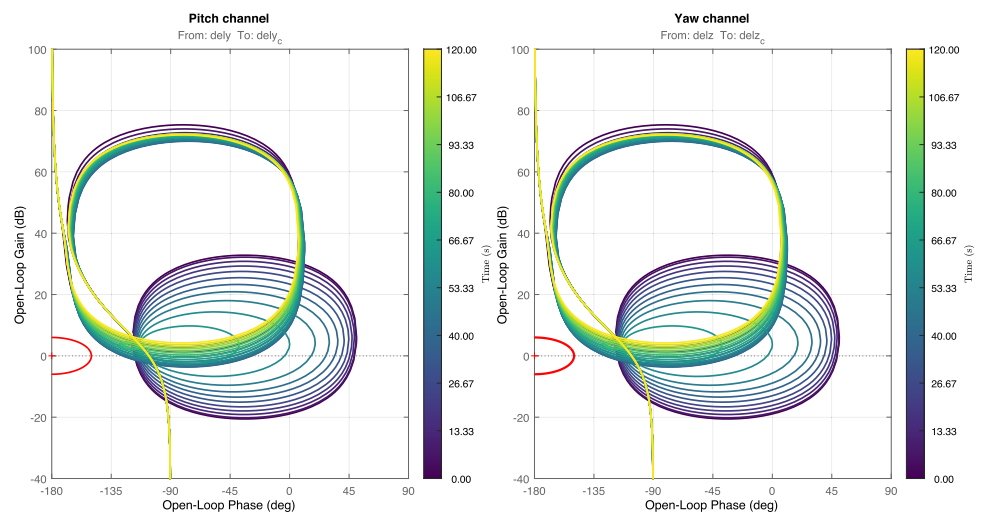
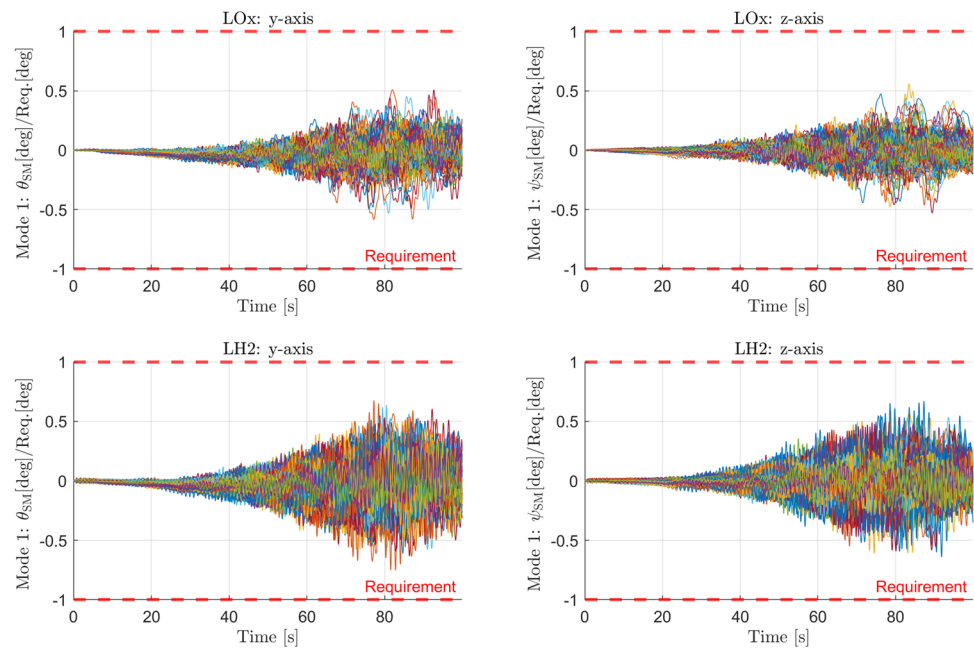


Fig. 12 Monte Carlo results for 250 runs with performance requirement shown



RLV in the pitch plane. Simplifications for the different transfer functions of the attitude channel were also introduced. In this framework, the user has the option to select from the full-order model with aerodynamic effects, the decoupled attitude model with simplified aerodynamics, or the simplest form of decoupled attitude without aerodynamic effects.

In the final part, all the developed models were applied to the CALLISTO mission. The parameters summarized in Table 1 were computed for the vehicle and subsequently validated against newer CFD simulations. The position of the sloshing masses was studied with regard to the danger zones for the baseline mission. As a consequence, the stability of the modes was assessed for the LOx and LH₂ tanks. In the final part, the state-space model was used to extend the $\mathcal{H}_\infty$ -synthesized controllers from earlier investigations, where a proper imposition of the CL bandwidth allowed the identification of a balance between tracking performance and sufficient separation from the sloshing modes' frequencies, while at the same time guaranteeing adequate rejection of wind disturbances and correct requirement compliance.

Appendix A: tables with mechanical parameters

The mechanical parameters for the two tanks of the CALLISTO vehicle were computed based on the summary of parameters presented in Table 1. Table 2 contains the data corresponding to the LOx tank, while Table 3 provides the parameters for the LH₂ tank. The parameters include equiva-

lent masses, heights, lengths, natural frequencies, and damping ratios computed for various filling levels. The reported frequencies were computed assuming a non-gravitational acceleration of $\Gamma = g$. In the case of the evolution along the trajectory, the parameters were computed using the actual Γ and are depicted in Fig. 7.

Appendix B: expanded kinematic quantities

This appendix provides the explicit expressions for the rotation matrices, angular velocities, and angular accelerations used in the main text.

All rotation matrices follow the $\underline{ZYX}$ convention ($\mathcal{R}(\alpha, \beta, \gamma) = \mathcal{R}_1(\alpha)\mathcal{R}_2(\beta)\mathcal{R}_3(\gamma)$).

B.1 Rotation matrices

$$\mathcal{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}. \quad (\text{B1})$$

$$\mathcal{R}_2(\beta) = \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix}. \quad (\text{B2})$$

$$\mathcal{R}_3(\gamma) = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{B3})$$

Table 2 Parameters for the mechanical model using the analytical formulations for the LOx tank

Filling ratio (%)	m_{liq} [kg]	m_0 [kg]	h_{SM_0} [m]	m_1 [kg]	h_{SM_1} [m]	ℓ_{SM_1} [m]	f_{SM_1} [Hz]	ζ_{SM_1} [-]
100	1,872.4	1,609.1	0.92	253.84	1.85	0.3	0.91	1×10^{-3}
94.74	1,773.8	1,510.5	0.88	253.83	1.76	0.3	0.91	1×10^{-3}
89.47	1,675.3	1,412	0.84	253.83	1.66	0.3	0.91	1×10^{-3}
84.21	1,576.7	1,313.5	0.8	253.82	1.56	0.3	0.91	1×10^{-3}
78.95	1,478.2	1,214.9	0.76	253.81	1.47	0.3	0.91	1×10^{-3}
73.68	1,379.7	1,116.4	0.72	253.78	1.38	0.3	0.91	1×10^{-3}
68.42	1,281.1	1,017.9	0.69	253.73	1.28	0.3	0.91	1×10^{-3}
63.16	1,182.6	919.46	0.66	253.63	1.19	0.3	0.91	1×10^{-3}
57.9	1,084	821.1	0.63	253.44	1.1	0.3	0.91	1×10^{-3}
52.63	985.47	722.92	0.61	253.08	1.02	0.3	0.91	1×10^{-3}
47.37	886.92	625.07	0.6	252.38	0.94	0.3	0.91	1×10^{-3}
42.11	788.37	527.85	0.59	251.05	0.87	0.3	0.91	1×10^{-3}
36.84	689.83	431.83	0.6	248.53	0.81	0.31	0.9	1×10^{-3}
31.58	591.28	338.05	0.63	243.76	0.76	0.31	0.89	1×10^{-3}
26.32	492.73	248.38	0.7	234.88	0.73	0.32	0.88	1×10^{-3}
21.05	394.19	165.99	0.85	218.73	0.74	0.35	0.85	1×10^{-3}
15.79	295.64	95.63	1.2	190.6	0.82	0.4	0.79	1×10^{-3}
10.53	197.09	42.88	2.21	145.1	1.05	0.52	0.69	1×10^{-3}
5.26	98.55	11.57	7.06	79.7	1.91	0.95	0.51	1×10^{-3}
0	0	NaN	NaN	NaN	∞	∞	0	1×10^{-3}

Table 3 Parameters for the mechanical model using the analytical formulations for the LH₂ tank

Filling ratio (%)	m_{liq} [kg]	m_0 [kg]	h_{SM_0} [m]	m_1 [kg]	h_{SM_1} [m]	ℓ_{SM_1} [m]	f_{SM_1} [Hz]	ζ_{SM_1} [-]
100	350.91	333.31	2.5	16.97	5.19	0.3	0.91	1×10^{-3}
94.74	332.44	314.84	2.37	16.97	4.91	0.3	0.91	1×10^{-3}
89.47	313.97	296.37	2.23	16.97	4.64	0.3	0.91	1×10^{-3}
84.21	295.5	277.9	2.1	16.97	4.37	0.3	0.91	1×10^{-3}
78.95	277.03	259.43	1.97	16.97	4.09	0.3	0.91	1×10^{-3}
73.68	258.56	240.96	1.84	16.97	3.82	0.3	0.91	1×10^{-3}
68.42	240.1	222.5	1.7	16.97	3.55	0.3	0.91	1×10^{-3}
63.16	221.63	204.03	1.57	16.97	3.28	0.3	0.91	1×10^{-3}
57.9	203.16	185.56	1.44	16.97	3	0.3	0.91	1×10^{-3}
52.63	184.69	167.09	1.31	16.97	2.73	0.3	0.91	1×10^{-3}
47.37	166.22	148.62	1.19	16.97	2.46	0.3	0.91	1×10^{-3}
42.11	147.75	130.15	1.06	16.97	2.18	0.3	0.91	1×10^{-3}
36.84	129.28	111.68	0.94	16.97	1.91	0.3	0.91	1×10^{-3}
31.58	110.81	93.21	0.83	16.97	1.64	0.3	0.91	1×10^{-3}
26.32	92.34	74.75	0.73	16.96	1.38	0.3	0.91	1×10^{-3}
21.05	73.88	56.3	0.64	16.94	1.12	0.3	0.91	1×10^{-3}
15.79	55.41	37.95	0.59	16.82	0.9	0.3	0.91	1×10^{-3}
10.53	36.94	20.2	0.65	16.1	0.74	0.32	0.89	1×10^{-3}
5.26	18.47	5.6	1.32	12.25	0.85	0.42	0.77	1×10^{-3}
0	0	NaN	NaN	NaN	∞	∞	0	1×10^{-3}

$${}^{SH}\mathcal{R}_S = \begin{bmatrix} \cos \theta_{SH} & 0 & \sin \theta_{SH} \\ 0 & 1 & 0 \\ -\sin \theta_{SH} & 0 & \cos \theta_{SH} \end{bmatrix} \begin{bmatrix} \cos \psi_{SH} & -\sin \psi_{SH} & 0 \\ \sin \psi_{SH} & \cos \psi_{SH} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} \cos \theta_{SH} \cos \psi_{SH} & -\cos \theta_{SH} \sin \psi_{SH} & \sin \theta_{SH} \\ \sin \psi_{SH} & \cos \psi_{SH} & 0 \\ -\cos \psi_{SH} \sin \theta_{SH} & \sin \theta_{SH} \sin \psi_{SH} & \cos \theta_{SH} \end{bmatrix}. \quad (20)$$

$${}^{SM}\mathcal{R}_S = \begin{bmatrix} \cos \theta_{SM} & 0 & -\sin \theta_{SM} \\ 0 & 1 & 0 \\ \sin \theta_{SM} & 0 & \cos \theta_{SM} \end{bmatrix} \begin{bmatrix} \cos \psi_{SM} & \sin \psi_{SM} & 0 \\ -\sin \psi_{SM} & \cos \psi_{SM} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} \cos \theta_{SM} \cos(\psi_{SM}) & \cos \theta_{SM} \sin \psi_{SM} & -\sin \theta_{SM} \\ -\sin \psi_{SM} & \cos \psi_{SM} & 0 \\ \cos \psi_{SM} \sin \theta_{SM} & \sin \theta_{SM} \sin \psi_{SM} & \cos \theta_{SM} \end{bmatrix}. \quad (21)$$

B.2 Angular velocities

$${}^{SM}\omega_{B,SM} = \begin{bmatrix} 1 & 0 & -\sin \theta_{SM} \\ 0 & 1 & 0 \\ 0 & 0 & \cos \theta_{SM} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_{SM} \\ \dot{\psi}_{SM} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}_{SM} \sin \theta_{SM} \\ \dot{\theta}_{SM} \\ \dot{\psi}_{SM} \cos \theta_{SM} \end{bmatrix}. \quad (35)$$

B.3 Angular accelerations

$${}^{SM}\alpha_{B,SM} = \begin{bmatrix} 1 & 0 & -\sin \theta_{SM} \\ 0 & 1 & 0 \\ 0 & 0 & \cos \theta_{SM} \end{bmatrix} \begin{bmatrix} 0 \\ \ddot{\theta}_{SM} \\ \ddot{\psi}_{SM} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\dot{\theta}_{SM} \cos \theta_{SM} \\ 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_{SM} \sin \theta_{SM} \end{bmatrix} \\ \begin{bmatrix} 0 \\ \dot{\theta}_{SM} \\ \dot{\psi}_{SM} \end{bmatrix} \\ = \begin{bmatrix} -\ddot{\psi}_{SM} \sin \theta_{SM} - \dot{\theta}_{SM} \dot{\psi}_{SM} \cos \theta_{SM} \\ \ddot{\theta}_{SM} \\ \ddot{\psi}_{SM} \cos \theta_{SM} - \dot{\theta}_{SM} \dot{\psi}_{SM} \sin \theta_{SM} \end{bmatrix}. \quad (36)$$

Appendix C: treatment of the mass variation

Equations (46)–(49) do not fully account for vehicle mass variation. Usually these effects are small, but to accurately model them in nonlinear simulations, they must be included as extra terms. The force due to mass variation ($\star f_{mv}$) and torque due to mass variation ($\star \tau_{mv}$) are introduced. These effects have been extensively studied in recent years by several authors [56–58].

For instance, Eke and Mao [56] state that $\star f_{mv}$ is composed of three components: (1) a component due to thrust (already included in $\star f_{tvc}$ and the T model), (2) a Coriolis component ($\star f_c = \star f_{c1} + \star f_{c2}$), and (3) a component due to linear momentum ($\star f_l$). The authors explain that it is reasonable to assume that, for launch vehicles (LVs), changes occurring in the combustion chamber can be safely neglected; consequently, the Coriolis term depends only on the flow at the nozzle exhaust. Thus, only the second component of the Coriolis term propagates to the mass variation force, i.e.,

$$\star f_{mv} = \star f_{c2}, \quad (C4)$$

with

$$\star f_{c2} = -2 \star \omega_{I,B} \times \int_S \rho \star v_{ex} (\star v_{ex} \cdot \star n) dS, \quad (C5)$$

ρ is the particle density, $\star v_{ex}$ is the exhaust velocity at the nozzle exit, $\star n$ is the vector parallel to the flow, and S is the surface area of the nozzle.

Gäßler et al. [59] show that Eq. (C5) can be further simplified to

$$\star f_{c2} \approx -2 \dot{m} \star \omega_{I,B} \times \star r_{OB,ONE}. \quad (C6)$$

Similarly, the torque due to mass variation $\star \tau_{mv}$ is composed of four components: (1) the component due to thrust, which is already taken into account in $\star \tau_{tvc}$ via T, (2) the component due to the Coriolis term ($\star \tau_c$), (3) the component due to linear momentum ($\star \tau_l$), and (4) the component due to the change in the MoI $\star J_B$, which is usually zero because the rigid body remains unchanged. By assuming that the change happening in the combustion chamber can be neglected and that the exit area is confined to the nozzle exhaust, one can define the torque due to mass variation:

$$\star \tau_{mv} = \star J_B \star \omega_{I,B} + \star \tau_c, \quad (C7)$$

with

$$\star \tau_c = \star \tau_{jet} \\ = - \int_B \rho (\star p \times (\star \omega_{I,B} \times \star p)) (\star v_{ex} \cdot \star n) dS. \quad (C8)$$

Equation (C8) represents the so-called jet-damping torque, which usually has a stabilizing effect on LVs. $\star p = \star r_{OB,ONE} + \star s$ represents the position vector of any particle exiting the vehicle. $\star s$ is the in-plane displacement vector from the center of the nozzle exit to a point on the exit circumference; thus, $\star s \cdot \star n = 0$ and $|\star s| \leq R_N$, where R_N is the

nozzle radius. $\star s$ is usually approximated as the weighted average over the entire exit circumference.

Gäßler et al. [59] provide a simplified expression for Eq. (C8)

$$\star \tau_{\text{jet}} \approx -\dot{m} \star r_{\text{OB, ONE}} \times \left(\star \omega_{\text{I, B}} \times \star r_{\text{OB, ONE}} \right) - \dot{m} \begin{bmatrix} \frac{R_N^2}{2} \\ \frac{R_N^2}{4} \\ \frac{R_N^2}{4} \end{bmatrix} \star \omega_{\text{I, B}}. \quad (\text{C9})$$

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Declarations

Conflict of interest The authors declare no conflict of interest.

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