



POLITECNICO
MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

Optimal Formation Flying for Single-Pass, Multi-Baseline, Across- Track Synthetic Aperture Radar Interferometry

TESI DI LAUREA MAGISTRALE IN
SPACE ENGINEERING - INGEGNERIA SPAZIALE

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Academic Year: 2024-25

To my two best friends —incidentally, sisters

Abstract

Spaceborne Synthetic Aperture Radar (SAR) is at the cutting edge of space technology, providing high-resolution images for applications ranging from Earth observation to planetary exploration, independently of weather and daylight conditions. Among these techniques, SAR Interferometry enables the accurate quantification of a variety of geophysical parameters, such as ground deformation and elevation, fostering the generation of highly accurate Digital Elevation Models (DEMs).

The combination of SAR with formation flying concepts has opened a new frontier in Earth remote sensing, with the TanDEM-X mission by the German Aerospace Centre having generated the most complete and accurate global DEM to date. This mission features a pair of satellites flying in a passively safe helix configuration, thereby reducing the control effort required to maintain the formation in the presence of external disturbances.

However, despite the outstanding achievements, there are intrinsic limitations to this type of concept. Indeed, a formation of only two satellites adopting a non-fixed baseline forces trade-offs between accuracy and robustness to height ambiguities. Conversely, a time-invariant baseline would allow for a neat improvement in the accuracy at the price of a higher control demand. In the case of TanDEM-X, these issues are partially solved by performing repeated acquisitions of the same area with a different baseline after one year. Nevertheless, this solution presents additional problematic aspects, as it loses the benefits of single-pass interferometry in monitoring fast-changing phenomena.

This thesis investigates whether it is possible to overcome such limitations in the context of single-pass, across-track interferometry by employing a multi-baseline approach. More precisely, the purpose is to determine the optimal set of orbital parameters to produce a high-quality DEM robust to height ambiguities while at the same time ensuring the passive safety of the formation.

The proposed approach combines semi-analytical and numerical methods to solve sub-problems with increasing numbers of degrees of freedom. After having determined an analytical solution to the case with two satellites, genetic algorithms are employed to find configurations of three and four satellites compliant with both interferometric and safety-

related requirements. Additionally, a comparative study of the necessary delta-velocity budget leveraging impulsive control strategies is conducted.

The findings suggest that configurations based on nested helix relative trajectories could be designed to meet all the mentioned requirements, thereby providing a starting point for the design of future cost-effective and highly accurate Earth observation missions.

Keywords: Across-track interferometry, digital elevation model (DEM), formation flying, Helix formation, synthetic aperture radar (SAR)

Abstract in lingua italiana

Il radar ad apertura sintetica (SAR) è uno strumento dall'eccezionale potenziale nell'ambito della tecnologia spaziale, data la sua capacità di generare immagini ad alta risoluzione per applicazioni che vanno dall'osservazione della Terra all'esplorazione planetaria, indipendentemente dalle condizioni meteorologiche e di illuminazione. Tra i suoi numerosi possibili impieghi, l'interferometria SAR consente la quantificazione accurata di una varietà di parametri geofisici, come la deformazione del suolo e l'elevazione, consentendo la generazione di modelli digitali di elevazione (DEM) ad alta precisione.

La combinazione di SAR e volo in formazione ha aperto una nuova frontiera nel telerilevamento terrestre: la missione TanDEM-X del Centro Aerospaziale Tedesco ha generato il DEM globale più completo e accurato fino ad oggi. Questa missione prevede una coppia di satelliti che volano in una configurazione a elica passivamente sicura, riducendo così lo sforzo di controllo necessario per mantenere la formazione in presenza di disturbi esterni.

Tuttavia, nonostante gli eccezionali risultati, questa soluzione presenta dei limiti intrinseci. Infatti, una formazione di soli due satelliti che adotta una baseline non fissa impone compromessi tra accuratezza e robustezza alle ambiguità di altitudine. Al contrario, una baseline tempo-invariante consentirebbe un netto miglioramento dell'accuratezza al prezzo di una maggiore richiesta di controllo. Nel caso di TanDEM-X, questi problemi vengono parzialmente risolti eseguendo acquisizioni ripetute della stessa area con una baseline diversa a distanza di un anno. Tale approccio presenta però ulteriori aspetti problematici, in quanto perde i vantaggi dell'interferometria a singolo passaggio nel monitoraggio di fenomeni in rapida evoluzione.

Questa tesi indaga se sia possibile superare tali limitazioni nel contesto dell'interferometria a singolo passaggio trasversale utilizzando un approccio multi-baseline. Più precisamente, lo scopo è determinare l'insieme ottimale di parametri orbitali per produrre un DEM di alta qualità, robusto alle ambiguità di altezza, garantendo al contempo la sicurezza passiva della formazione.

L'approccio proposto combina metodi semi-analitici e numerici per risolvere sottoproblemi con un numero crescente di gradi di libertà. Dopo aver individuato una soluzione

analitica per il caso con due satelliti, vengono impiegati algoritmi genetici per individuare configurazioni di tre e quattro satelliti conformi ai requisiti interferometrici e di sicurezza. Inoltre, viene condotto uno studio comparativo del delta-v budget necessario sfruttando strategie di controllo impulsivo.

I risultati suggeriscono che configurazioni basate su traiettorie relative a eliche annidate potrebbero essere progettate per soddisfare tutti i requisiti menzionati, fornendo così un punto di partenza per la progettazione di future missioni di osservazione della Terra economiche e altamente accurate.

Parole chiave: Interferometria trasversale, modello digitale di elevazione (DEM), volo in formazione, formazione ad elica, radar ad apertura sintetica (SAR)

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1 | Introduction

The exploration of multistatic and distributed concepts for Earth observation and monitoring is a rapidly growing field of research. In recent years, the employment of satellite formations has revolutionised many areas of geoscience; however, despite an increasing number of works in this research area, the design of optimal formation strategies remains an open challenge, especially in terms of performance metrics for specific mission applications. The purpose of this work is hence to contribute to the investigation of such configurations, with a specific focus on Synthetic Aperture Radar Interferometry (InSAR) missions.

This chapter aims to delineate the framework upon which the entire thesis is constructed. In Section 1.1, the background of remote sensing distributed missions is outlined, with a primary focus on InSAR applications. The motivations that drive this work are clarified in section 1.2, together with the formulation of the main research question. Section 1.3 presents an overview of the state-of-the-art Earth observation and formation-flying strategies underlying current multi-satellite remote sensing applications. Section 1.4 presents the novelties and contributions of this thesis to the field. Finally, the structure of the entire work, adopted to meet the proposed objectives, is illustrated in Section 1.5.

1.1. Background

Earth observation from space has come a long way since its beginnings. The dawn of this new means to deliver knowledge —underlying today a variety of scientific fields—dates back to the 1960s, with the advent of the first satellites dedicated to meteorological observation. In 1960, the National Aeronautics and Space Administration (NASA) launched TIROS-1 (Television Infrared Observation Satellite-1), the first operational weather satellite (Dittberner and Vonder Haar [1]). Thanks to its two television cameras, this mission provided the world with its first-ever view of large-scale cloud formations from orbit.

In 1978 the NASA Seasat mission was launched (Born et al. [2]), the first Earth-orbiting satellite specifically designed to carry an instrument that would prove to be a game-

changer for spaceborne remote sensing: the Synthetic Aperture Radar (SAR). Unlike optical systems, SAR can operate day and night, as well as through clouds, providing reliable, high-resolution data under virtually all weather conditions. A particularly relevant application is SAR interferometry (InSAR), which consists of measuring the path length travelled by the radar echo encoded in the phase of the electromagnetic wave. From this information, a wealth of geophysical parameters can be extracted with extreme precision, ranging from surface topography to ice-sheet motion and subsidence processes (Moreira et al. [3]). One of the most relevant applications of InSAR consists of generating highly accurate Digital Elevation Models (DEMs): discrete measurements of the topography of the Earth's surface, extremely valuable to a multitude of branches of geophysics, such as the study of tectonics, volcanism, hydrology and glaciology. Moreover, DEMs are indispensable to support hazard assessment, infrastructure planning, and reliable navigation. Finally, they are also important for the SAR context itself, as they provide the reference for precise geocoding, as well as precise calibration in conventional SAR imaging [3].

A milestone in the demonstration of the potential provided by this technique was set in 2000 by the Shuttle Radar Topography Mission (SRTM), the result of the international cooperation of NASA, the National Imagery and Mapping Agency (NIMA), the German Aerospace Centre (DLR) and the Italian Space Agency (ASI). This single mission produced the first near-global DEM of the Earth, demonstrating the capability of spaceborne InSAR systems to map topography with unprecedented accuracy (Van Zyl [4], and Nikolakopoulos [5]).

In recent years, a new frontier has been opened by the introduction of bi- and multi-static SAR systems, in which two or more spacecraft fly in controlled formation. DLR's TanDEM-X (Krieger et al. [6]) launched in 2010 is, in this regard, the pioneer, having demonstrated for the first time the feasibility of operational single-pass across-track interferometry from space. Building on its heritage, more recent missions such as Lutan-1 (Liu et al. [7]) by the Aerospace Information Research Institute of the Chinese Academy of Science (AIR-CAS), and Hongtu-1 by the China Academy of Space Technology (CAST) (Wang et al. [8]) have progressed further the state of the art, showing the effectiveness of multi-satellite interferometry.

Such outstanding achievements are paving the way for future Earth observation architectures. Among others, ESA's Harmony mission (Lopez-Dekker [9]) envisions two small satellites flying alongside Sentinel-1 (Potin et al. [10]). Similar concepts have been investigated by the DLR, such as the High Resolution Wide Swath (HRWS) mission (Spiridonova and Kahle [11]) and the MirrorSAR extension (Mittermayer et al. [12]). Although not realised, these studies embody the current trend towards modular, distributed architec-

tures: a potential leap forward in SAR formation-flying, resulting from their capability to synthesise performance and cost efficiency.

1.2. Motivation

To ensure that the imaging requirements are consistently met across all observed regions, an ideal formation would entail satellites in fixed relative positions. The quality of the DEM, in fact, is determined by the acquisition geometry, and adopting fixed baselines would always guarantee a perfect positioning of the radars. The drawback, however, is the formation maintenance cost: holding satellites in rigid configurations goes against their natural orbital dynamics, leading to a higher control effort. The alternative is to take advantage of the natural relative motion of satellites in orbit, thereby reducing the required Δv budget but accepting a time-varying geometry. In this case, the path lengths travelled by radar echoes would change throughout the orbit and, with them, the resulting interferograms.

All the said considerations naturally lead to the research question that motivates this thesis:

"What is the optimal formation-flying configuration that guarantees a uniform Digital Elevation Model quality on a global scale?"

In particular, this question is decomposed into the following three investigation directions:

1. The optimisation problem is addressed first in its unconstrained form, in order to identify possible ideal formations without external limitations.
2. A second fundamental aspect to consider is the operational safety of the space segment, as any proposed configuration must indeed prevent the risk of mutual collisions. The second aim is therefore to determine the conditions under which the identified formations can be guaranteed to remain safe (*i.e.*, collision free). By extension, to recognise which optimal configurations are inherently compliant with safety requirements.
3. Even a safe and optimal formation is only viable if it can be sustained operationally. This raises the third and final point of discussion, which is to provide an estimate of the expected costs of maintaining such a formation in the harsh Low Earth Orbit (LEO) environment.

1.3. Literature Review

This section presents the state-of-the-art techniques, methods, and technologies upon which this thesis is built. The following discussion is divided into three parts, each focusing on a subject fundamental to this work: SAR, satellite formation-flying, and the synthesis of the two: distributed SAR concepts. Finally, a review of the most relevant missions employing formation flying in LEO is provided.

1.3.1. Synthetic Aperture Radar

Firstly introduced by Carl Wiley [13] in 1954, SAR is nowadays a widely employed remote sensing technology that provides highly accurate images for purposes ranging from climate monitoring to planetary exploration. As explained by Moreira et al. [3], its independence from weather conditions and daylight makes it an ideal solution for continuous and reliable Earth monitoring. Of particular interest for this thesis is the Interferometric SAR (InSAR) technique, based on the comparison of the phases of two complex radar images (Bamler [14]). There are two possible ways this technology may be operated: across-track and along-track interferometry. The former necessitates two separated antennas moving on parallel tracks to measure the elevation angles of the elemental scatterers, thus providing information about the imagined objects' height. This strategy was employed, for example, on the TanDEM-X mission, as detailed by Krieger et al. [15]. For the latter, instead, the two antennas are aligned with the velocity direction, allowing them to capture motion accurately [14]. Finally, Differential InSAR (DInSAR) is a well-established technique in which two SAR images taken with a time separation are superimposed to produce an interferogram [14]. The result is the detection of deformations in large areas with an accuracy of fractions of a wavelength over long time scales.

Another relevant dichotomy is that between monostatic and bi-/multistatic, which is well explained by D'Errico [16]. In the first scenario, the transmitter and receiver antennas coincide, and the locations of the antennas define the synthetic aperture during the illumination of a target. In contrast, two (bistatic) or more (multistatic) transmitting and receiving antennas are used, forming a distributed system. In the last case, there are two possibilities for the transmitting system: cooperating, in which the transmitter and receiver jointly fulfil a common task; and non-cooperating, in which an independent source is leveraged, typically indicated with the name of *illuminator of opportunity* [16].

1.3.2. Formation flying dynamics

Multiple studies in the literature address the problem of an analytical description of the relative motion between two or more satellites. The most well-known model was proposed by Clohessy and Wiltshire [17] and was built upon a set of linearised differential equations formulated by Hill [18] to describe the lunar motion: the so-called Hill-Clohessy-Wiltshire equations (HCW). Its simplicity, combined with a low computational burden, has led to the widespread adoption of this model, especially for rendezvous scenarios. However, its applicability is limited by stringent hypotheses (*i.e.*, quasi-circular orbits and absence of perturbations), making it less usable when considering long-duration formation-flying phases. In fact, as explained by Wang [19], it accumulates significant errors over long-term scenarios. To address these limitations, several works have managed to extend the validity of the HCW equations, firstly removing the hypothesis of quasi-circular orbit, as done by Tschauner and Hempel [20]. On the opposite side, Kechichian [21] developed an exact formulation of the relative dynamics problem, incorporating J_2 and air drag perturbing effect. This solution is based on a set of twelve non-linear differential equations that must be integrated. Hence, despite its accuracy, due to its numerical complexity and computational cost, it is often impractical for on-board applications. A more straightforward strategy to only include perturbation induced by the first zonal harmonic of the geopotential (J_2) was proposed by Schweigart and Sedwick [22]: first, the motion with respect to a reference circular orbit is solved in closed-form; then, more notably, a set of constant-coefficient, linearised differential equations is derived. The combined effects of J_2 and air drag have been considered by Sabatini and Palmerini [23]: the equations of motion (EoM) are linearised around a reference circular or slightly eccentric orbit. This paper compares the derived method with other linear strategies, such as the one mentioned in [22], highlighting that the inclusion of drag enhances accuracy only when the ballistic coefficients of the satellites are sufficiently diverse. In this case, the drift is higher at lower altitudes and with larger baselines. The computational overload, although slight, may provide a reasonable argument for preferring J_2 -only models when considering satellites of (almost) identical design and a slight relative separation.

HCW equations-based methods are formulated by describing the state in Cartesian coordinates, in the so-called Hill frame. However, formulations based on a set of Relative Orbital Elements (ROEs) have been extensively investigated, as they provide an immediate physical interpretation, an easier definition of the geometry, and a more straightforward inclusion of perturbations, making them often better suited for describing formation flying and proximity operations. Starting from Gauss' variational equations (GVE), which express the EoM in terms of variations of the orbital element, D'Amico [24] demonstrated

the appropriateness of the definition of a set of relative orbital elements matching the integration constants of the HCW equations. More precisely, it is shown that the integration of Gauss' equations (which are typically used to describe the *absolute* state of the spacecraft) makes it possible to interpret the result as a map from a change of the initial conditions of the relative motion into a change of the orbital elements of the participating spacecraft. Therefore, the ROEs are conveniently defined as the HCW integration constants. Additionally, it is possible to derive an analytic map from Hill's frame into ROEs at a generic mean argument of latitude.

This concept was extended by Gaias [25], developing an analytical framework based on the representation of the STM in mean ROEs, capable of accounting for the geopotential effect up to an arbitrary order of the gravitational harmonics. The primary focus of the proposed method is the derivation of the transformation between mean and osculating orbital elements. This is achieved by employing Hamiltonian perturbation theory through Lie transforms. A first canonical transformation is used to remove the parallax, followed by a Delaunay normalisation that eliminates the lower orders of the mean anomaly, and a final transformation eliminates the long-period terms. The result is an analytical solution free from singularities related to equatorial and circular orbits, even though it is not viable close to the critical inclination. Periodic corrections based on the Kaula first-order method are then applied to the semi-major axis to compensate for zonal, sectorial, and tesseral harmonics of the geopotential of order higher than two. The STM is thus directly derived in ROE and accounts for an arbitrary order of the geopotential. To account for cases in which a first-order expansion in the ROEs (i.e. STM-based) is not sufficient, the second-order State Transition Tensor is derived as well. The accuracy provided by the proposed approach is finally analysed through numerical simulations, comparing it against the algorithm employed by the AVANTI flight software developed by Gaias et al. [26].

The ROE parametrisation has been extensively explored to develop guidance strategies: D'Amico [27] adopted it to the formation-flying problem, creating an impulsive guidance solution. Remarkable results in this direction have been obtained by Gaias and Lovera [28] in the context of close-range proximity operations (where larger manoeuvres are typically needed than simple formation keeping). Starting from the GVE, the impulsive control actions are modelled in the ROE frame: by solving a two-boundary problem, the pulses required at each step on a time grid are computed. The impulsive guidance algorithm is then used as the basis to derive a piecewise constant thrust profile; to do so, a modification of the grid scheme is introduced, aiming to reduce the Δv cost. The discussed methodology is validated using a fly-around scenario as a test bed: the results show that

the impulsive solution tends towards the continuous one as the number of nodes increases. For formation keeping and reconfigurations that use small corrections, the impulsive control strategy is often acceptable, since the errors introduced in the desired relative eccentricity, inclination, and mean argument of longitude are within a prescribed control accuracy threshold.

A critical problem when discussing formation flying is collision avoidance. A widely used analytical method in this regard is based on the *eccentricity/inclination vector separation* concept. Although initially developed for the geostationary framework, it was extended to LEO formations by D’Amico and Montenbruck [29]. This work shows that an (anti-)parallel configuration of the relative eccentricity and inclination vectors avoids any possible collision in an unperturbed scenario. However, it also demonstrates that a control action would be necessary to maintain the desired configuration, showing that a suitable choice of the nominal configuration could partially mitigate the effect caused by Earth’s oblateness. The relative orbit control is effectively addressed with a couple of opposite, symmetric thrusts in the tangential direction, while numerical simulations using the TanDEM-X formation as a test bed confirm the correctness of the analytical derivation.

The proposed strategy has been applied to a longitude swap manoeuvre with along-track separation higher than 200 km, such as the GRACE [30] formation (the two spacecraft nominally maintain a 220 km separation). Such a manoeuvre is described in depth by Montenbruck et al. [31]. The eccentricity/inclination vector separation method was chosen as the best one compared with traditional alternatives, such as the *circularisation* method. In fact, the former is shown to be intrinsically safe, since a careful choice of the argument of latitude maximises the radial separation at the moment of the manoeuvre, while at the same time reducing design complexity and fuel cost. Concerning formations of more than two satellites, Koenig and D’Amico [32] have formulated a constraint that the ROEs must comply with, to ensure a given minimum separation between each possible pair of satellites in the radial across-track plane. A further development in this regard was provided by Scala et al. [33], who introduced an extended version of the ROEs: the eROEs, fostering the systematic modelling of the relative dynamics for all possible satellite pairs.

1.3.3. Multistatic and distributed SAR systems

Performing SAR imaging with satellite formations is one of the most promising advancements in Earth observation technologies that has recently been developed. A comprehensive analysis of possible solutions for the design of formation flying SAR has been

conducted by Scala [34]. In this work, the relative motion is described through the modelling of the STM in both Cartesian and ROE parametrisations, deriving harmonic (helix, cross-track pendulum, and cartwheel) and closed-form (general circular orbit, projected circular orbit, and leader-follower) analytical solutions for the unperturbed case. Subsequently, building upon the said results in [25], the perturbing effects of air drag and Earth's oblateness, including full gravitational harmonic terms, are included. The results are validated through numerical simulations, assuming a two-spacecraft helix configuration with a separation of about 40 km. On the one hand, the findings of Sabatini and Palmerini [23] suggest that air drag is relevant only for spacecraft with diverse ballistic coefficients. Furthermore, they extend them by modelling the gravitational perturbation up to the 6th order, nevertheless showing that only the J_2 term is relevant for the case study.

Two main categories of configurations are possible depending on the baseline evolution over time: time-varying or fixed. The former has been implemented practically in the TanDEM-X mission, adopting a helix-shaped configuration as described by Krieger et al. [6]. The design of a fixed across-track baseline for SAR observation purposes is addressed by Scala et al. [35]: a configuration composed of a chief spacecraft and an arbitrary number of deputies is considered, and the relative dynamics is represented through the STM in the ROE framework. Since the maintenance of a fixed relative geometry necessarily requires a continuous control strategy, a piecewise constant control action is assumed with the aim of maintaining a chosen nominal geometry. An analytical solution is thus derived for the unperturbed case, providing an initial estimation of the required Δv budget. A subsequent parametric analysis, including external perturbations, refines the estimate, providing a relationship between baseline, altitude, and cost.

Scala et al. [36] present a comparative analysis of time-varying and fixed baseline strategies. Two scenarios are considered: a first one with nested helix relative trajectories and time-varying baselines; and a second one featuring fixed parallel relative tracks. The two cases are investigated through a parametric analysis, which shows that for the first one, the control effort is low, primarily due to the requirement for absolute orbit maintenance. On the other hand, the second one is less cost-effective, but with potentially better observation performance. This work can be considered a valuable starting point in the development of mixed strategies, based on alternating phases of fixed and time-varying baselines, to merge the benefits from the two approaches.

1.3.4. Relevant formation-flying missions in LEO

The following paragraphs provide a brief overview of relevant past and ongoing formation-flying missions in Low Earth Orbit (LEO). A condensed summary is also reported in Table 1.1.

Name	Operator	N° of satellites	Separation	Year	Main scientific purpose	References
GRACE	NASA/DLR	2	≈ 220 km	2002	gravitational field mapping	[30, 37, 38]
PRISMA	SSC	2	50 m - 10 km	2010	technology demonstrator	[38–40]
TanDEM-X	DLR	2	250–500 m	2010	global DEM in X band	[6, 15, 38]
CanX-4/5	CSA	2	50 m – 1 km	2014	technology demonstrator	[41]
AVANTI	DLR	2	50 m - 10s km	2016	technology demonstrator	[38, 42, 43]
LuTan-1	AIR-CAS	2	700–7000 m	2022	DEM generation in L band, deforestation monitoring	[7]
Starling	NASA	4	10s–100s km	2022	technology demonstrator	[44, 45]
Hongtu-1	CAST	4	< 1 km	2023	DEM generation in X band, multistatic SAR demonstration	[8]

Table 1.1: Examples of relevant state-of-the-art formation-flying missions in LEO; missions that encompass a SAR are highlighted in orange.

Gravity Recovery and Climate Experiment (GRACE)

GRACE (replaced from 2018 by its nearly identical follow-on version GRACE-FO) consisted of two satellites flying in tandem at an altitude between 300 and 500 km (Tapley and Reigber [37]) with separations of ~ 200 km in along-track direction (Ardaens et al. [38]). It demonstrated the use of precise inter-satellite ranging (K/Ka-band microwave) to recover Earth’s gravity field. Its relevance for formation flying missions is particularly notable for the first implementation of eccentricity-inclination vector separation in LEO, used to plan the swap manoeuvre of the two satellites in 2005. It demonstrated, therefore, the viability of a simple yet effective method for ensuring passive safety, thereby setting a precedent for scientific tandem missions. Additionally, it has demonstrated the capability of the satellite-to-satellite (SST) tracking link, allowing for correcting changes induced by atmospheric drag with a precision of approximately one micrometre (Davis et al. [30]).

Prototype Research Instruments and Space Mission technology Advancement (PRISMA)

PRISMA was a technology demonstration mission consisting of a two-satellite formation operating in LEO in a small eccentricity orbit, with perigee and apogee radii of 730

km and 790 km altitude. It was conducted by the Swedish Space Corporation (SSC) (now OHB-Sweden), in cooperation with the French Space Agency (CNES), the German Aerospace Centre (DLR), and the Technical University of Denmark (DTU). It involved two satellites, Mango, fully manoeuvrable with a mass of 150 kg, and Tango, passive, weighing 40 kg (D’Amico and Larsson [40]), launched on June 15 2010, with the objective of testing formation flying solutions (Persson et al. [39]). Specifically, it tested several navigation and control techniques, including GPS-based relative positioning, radiofrequency metrology, and optical sensors, to validate homing and rendezvous solutions. Particularly notable was the implementation of the along-track control leveraging its coupling with a slight difference in the semi-major axis, caused by the asphericity of Earth’s geopotential [40]).

TanDEM-X

TanDEM-X (TerraSAR-X add-on for Digital Elevation Measurement), developed by the DLR and the European Aeronautic Defence and Space Company (EADS) Astrium (now Airbus Defence and Space), is the first single-pass radar interferometer utilising a formation of two SAR satellites. It consists of two twin TerraSAR-X spacecraft orbiting in LEO. The first of the two was launched in 2007, while the second one was launched in 2010, effectively starting the TanDEM-X mission. Each TerraSAR-X [46] satellite has a mass at launch of roughly 1350 kg [47], a cylindrical shape of length 5.0 m and diameter 2.4 m (Wermuth et al. [48]) and flies inside an orbital tube of radius 250 m (Kahle et al. [49]). It is still operational and has generated the most accurate global DEM to date in the X frequency band. More precisely, it delivered the first two DEMs in 2013: a first one acquired with smaller inter-satellite distances and a second one, a year after, with longer baselines (Buckreuss et al. [50]). The two satellites fly in a specific kind of formation known as Helix (Moreira et al. [51]), which pairs the safety requirements with robustness to J_2 perturbations. This is achieved by employing two circular Sun-Synchronous Orbits at ~ 514 km altitude with identical inclination. A revisit time of 11 days enables the mapping of the entire surface of the Earth in slightly more than one year [50]. Moreover, the formation allows for reconfigurations with baselines varying from hundreds of meters to kilometres (Krieger et al. [15]), enabling the modification of the relative geometry to satisfy specific imaging requirements. Such flexibility grants, at the two extremes, the capability of acquiring strips 100 km wide at 16 m resolution, or 5-10x10 km size tiles at 1 m resolution [15], depending on the acquisition mode.

Canadian Advanced Nanospace eXperiment-4 & 5 (CanX-4&5)

CanX-4&5 was a two-satellite formation of the University of Toronto (UTIAS) and of the Institute for Aerospace Studies within the Space Flight Laboratory (SFL), sponsored by the Canadian Space Agency (CSA). The goal was to prove cube-sat formation-flying with reduced control effort in LEO. The mission flew at an altitude of 650 km, where different baseline configurations were tested, a pure along-track separation of 500 metres and a projected circular orbit of 50 metres (Bonin et al. [52]). Using GPS and inter-satellite communication, it achieved meter-level precision control with satellites of only 7 kg mass, paving the way for future distributed EO systems at a small scale. (Roth et al. [41]).

Autonomous Vision Approach Navigation and Target Identification (AVANTI)

AVANTI was a formation flying experiment conducted by the DLR involving two satellites: the non-cooperative target picosatellite BeeSat-4 and the main spacecraft BIROS. The mission demonstrated the capability to control formation flying using angles-only optical measurements, without GPS or RF links: an approach to a distance as close as 50 metres was achieved by employing a monocular camera (Gaias and Ardaens [42, 43] and Gaias et al. [53]), demonstrating lightweight, low-cost relative navigation technologies. Additionally, because the two satellites were substantially different in shape and mass, the differential acceleration induced by atmospheric drag had to be accurately modelled. This was achieved by Gaias et al. [26] with the introduction of a semi-empirical model, which refined the understanding of the ROE-based representation of the relative dynamics.

Lutan-1

Lutan-1 is a Chinese bistatic SAR mission launched at the beginning of 2022, consisting of two L-band SAR satellites flying in a Sun-Synchronous Orbit (SSO) at 607 km altitude. Two principal phases are envisioned: in the first, the satellites fly in a Helix formation with a time-varying baseline that can range from 0.7 km to 7.0 km. In the second, started roughly one year after launch, they share the same orbit on which they are located, with an angular separation of 180 degrees. The repeat cycle is eight days in phase one and four days in phase two. Among the objectives are the measurement of forest height, the estimation of biomass, and deformation monitoring (Liu et al. [7]). After TanDEM-X, it confirmed the operational viability of formation-flying SAR in a different frequency band, the L-band.

Starling

Starling is a NASA mission of four CubeSats in LEO to test autonomous swarm technologies, such as relative navigation, inter-satellite communication, and cooperative decision-making. The goal is to demonstrate that an entire constellation can be operated as a single unit, without the need to control each satellite separately. As such, it envisions a greater degree of autonomy than previous analogous missions (Grashuis and Cannon [45]). Notably, it has delivered the first demonstration of angles-only navigation for satellite swarms, within the Starling Formation-Flying Optical Experiment (StarFOX).

Hongtu-1

Hongtu-1, also referred to as PIESAT-1, is a formation flying mission by the China Academy of Space Technology (CAST). It is an X-band interferometric SAR composed of four satellites flying in a cartwheel formation. More precisely, it consists of a primary 320 kg spacecraft acting as a transmitter and three 270 kg receivers orbiting around it (Wang et al. [8]), which reach maximum baselines of approximately 500 km. Notably, the satellites maintain a relative formation known as an interferometric cartwheel, introduced first by Massonnet [54]. The underlying concept requires satellites flying in the same orbital plane in slightly elliptical orbits (D’Errico [16]). Remarkably, it represents the first single-pass multi-baseline SAR formation in LEO. The constellation orbits at an altitude of approximately 528 km, with a revisit period of 15 days [8]. Launched in 2023, it is intended to provide a global DEM in a 1:50000 scale at a resolution of 3 metres in its nominal acquisition mode, where strips of 20-30 km are acquired at a time, pushing forward the standard set with TanDEM-X.

1.4. Thesis contributions

The contributions of this thesis can be summarised in five main points.

1. An analytical mapping linking the ROE-based representation of the relative dynamics to the HoA has been derived, providing a better understanding of how a formation-flying solution influences interferometric performance.
2. The categories of satellite formations that allow for a homogeneous HoA on a global level have been identified. Specifically, a purely analytical solution has been derived for the case with two satellites, subsequently extended to larger constellations.
3. The eccentricity/inclination vector separation has been extended to constellations with an arbitrary number of satellites, recognising two categories of formations

capable of maximising mutual passive safety constraints.

4. It has been demonstrated that interferometric-optimal formations can also meet passive safety requirements. Within the subset of compliant configurations, two remarkable cases have been identified. A first one, that pushes the safety to the limit, yielding an optimal formation that maximises the minimum distance between any pair of deputies. A second one, that maximises robustness against perturbations from the Earth's geopotential, thereby leading to a minimisation of the formation maintenance costs.
5. Finally, it has been shown how, given a predetermined satellite formation, the acquisition geometry can be pushed to optimality by a suitable selection of the angle of incidence: a possible option for specific, TanDEM-X-like systems.

Overall, this thesis demonstrates that multi-baseline SAR formations can overcome the limitations of current single-baseline missions, enabling the generation of highly accurate, robust DEMs with reduced formation maintenance costs.

1.5. Thesis Structure

The contents of this thesis are organised as follows:

Chapter 2 provides an overview of the background of SAR techniques, with a specific focus on interferometry. Therefore, the key figures of merit considered in the optimisation are rigorously defined.

Chapter 3 introduces the absolute and local systems in which the relative dynamics of the satellite formation are formulated. Moreover, it describes the effects of central disturbances on close formations in the LEO environment, providing motivation for the control strategies proposed in the following chapters.

In Chapter 4, the mathematical formulation of the optimisation problem is presented, which is to be solved in the following chapters. A rigorous definition of the objectives and constraints. Specifically, the requirements arising from the generalisation of the eccentricity/inclination vector separation method are derived.

Chapter 5 illustrates the methodologies adopted and the solutions found to the HoA optimisation problem. More precisely, three approaches are considered: a first, purely analytical approach for the two-satellite case, and a heuristic and a numerical approach for larger constellations.

Chapter 6 examines the behaviour of the identified optimal formations in the hypothesis

of a perturbed relative motion. A consequent estimate of the costs for the formation maintenance is subsequently provided. In particular, this is done for various possible control strategies and different degrees of disturbance severity.

Chapter 7 proposes a possible extension of the methodology employed in the previous chapters. The idea is primarily aimed at systems that, similarly to TanDEM-X, have the potential to exploit an additional degree of freedom in the acquisition geometry to meet interferometric requirements.

Finally, the conclusions are drawn in Chapter 8, followed by a discussion of possible future developments.

2 | Synthetic Aperture Radar Fundamentals

This chapter discusses the fundamental principles of synthetic aperture radar, with particular focus on interferometry. It therefore aims to address the need for a more rigorous introduction to the challenges faced by Earth observation missions tailored to digital elevation model generation. In section 2.1, an overview is provided concerning the acquisition geometry, key steps in the signal processing chain, and the trade-offs intrinsic to a synthetic aperture radar. Building upon this, section 2.2 introduces the foundations of interferometric image generation, acquisition modes and relevant performance parameters such as the height of ambiguity and the coherence. Finally, in section 2.3, the basics of synthetic aperture radar tomography are briefly illustrated.

2.1. Synthetic Aperture Radar

This section presents an introduction to the basic principles of Synthetic Aperture Radar (SAR). The typical acquisition geometry is presented first, followed by image resolution and basic processing steps. The constitutional trade-off any SAR system is subject to is then addressed. Finally, relevant characteristics of SAR images are discussed.

First introduced in 1951 by the American engineer Carl Wiley [13], Synthetic Aperture Radar (SAR) is nowadays a well-established tool for remote sensing. Unlike passive sensors, which leverage the electromagnetic field spontaneously emitted or reflected by the observed bodies, SAR is an active instrument: it produces an electromagnetic pulse to then measure the backscattered echoes. SAR operates in the microwave region of the electromagnetic spectrum, with wavelengths ranging from 0.75 to 120 cm (see Table 2.1). The selection of the frequency band provides different insights into the imaged areas. Specifically, the longer wavelengths of P and L bands, capable of penetrating volumetric structures such as tree canopies, are employed to estimate biomass. Higher frequencies allow –in different capacities –to study topography, oceans, and ice sheets with higher resolution. Finally, as an active instrument working in the microwave domain, SAR

Frequency Band	Ka	Ku	X	C	S	L	P
Frequency [GHz]	40-25	17.6-12	12-7.5	7.5-3.75	3.75-2	2-1	0.5-0.25
Wavelength [cm]	0.75-12	1.7-2.5	2.5-4	4-8	8-15	15-30	60-120

Table 2.1: Frequency bands in the microwave spectrum, from Moreira et al. [3]

imaging capabilities remain consistent during both day and night, regardless of weather conditions.

A SAR consists of a moving antenna transmitting pulses at a rate given by the Pulse Repetition Frequency (PRF). As it moves, the radar synthesises a large effective aperture by integrating all the echoes received from targets at different positions along the flight path (Curlander and McDonough [55]). Conventional radar systems such as the Side-Looking Airborne Radar (SLAR) technology (Mastin et al. [56]) –the state of the art until the 50s –are indeed limited in the achievable resolution by the physical size of the antenna (Moreira et al. [3]). With a SAR, each ground target is illuminated for a significantly longer time, providing a much finer azimuth resolution, which is also independent of the position of the scattering object across the swath (Curlander and McDonough [55]). However, this comes at the cost of a higher power demand and a more convoluted signal processing to form a meaningful image from raw data, a procedure commonly referred to as *focusing* [3].

2.1.1. SAR acquisition geometry

In Figure 2.1, the typical geometry of a SAR acquisition is illustrated. Specifically, it shows the simplest and most commonly employed acquisition configuration: the *stripmap*, in which the radar acquires a single, continuous swath [3]. The (x, y, z) reference frame is fixed to the ground, with the x -axis (azimuth) pointing in the direction of flight (for this reason, it is also called *along-track*), the y -axis (ground range) across the swath, and the z -axis (vertical). The altitude at which the radar flies is indicated with h . The radar-target distance is referred to as *slant range*, or simply as *range*. Considering the relative position between the platform and the imaged objects, it is often convenient to introduce another reference system, with unchanged azimuth direction, but having the slant range (r) and the elevation (ζ) as the two other principal components. This alternative frame is reported in Figure 2.4.

Regarding the angular coordinates, $\Delta\psi$ and $\Delta\theta$ are the antenna beamwidth in the azimuth and elevation directions, respectively; the look-angle θ_l and the incidence angle (also

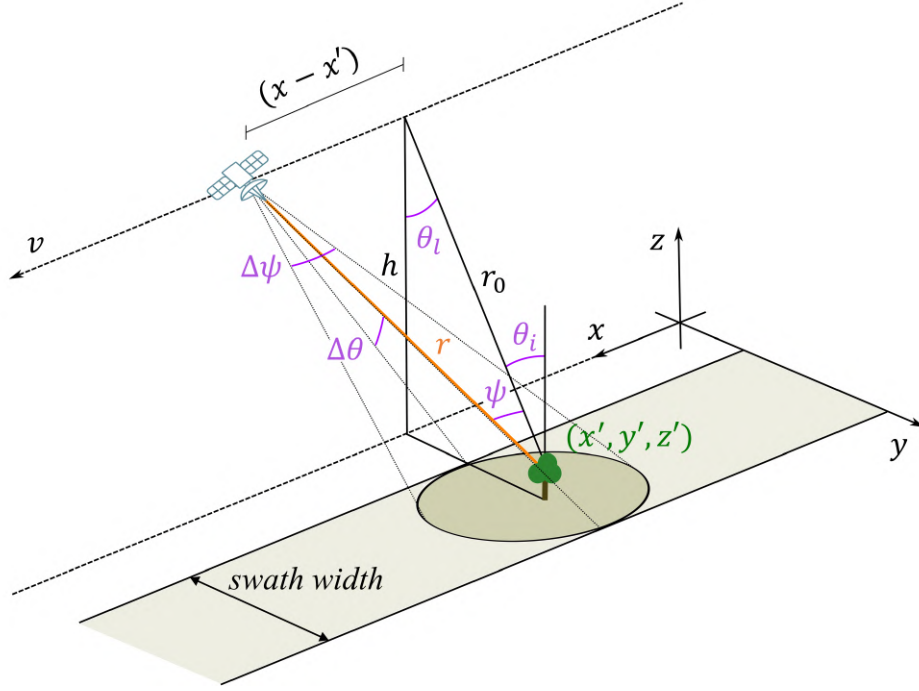


Figure 2.1: SAR imaging geometry, adapted from Bamler [14]

referred to as angle of incidence) θ_i define the angular separation between the radar and the target in the y - z plane. The former is relative to the local vertical at the radar position, the latter is measured with respect to the vertical at the target location. Note that, because of the curvature of the Earth, the two are different. Finally, ψ is the squint angle, which defines the relative position in the x - r plane.

In the following, it is assumed that the satellite follows a straight-line trajectory with constant velocity v , allowing for a neat derivation of fundamental equations in SAR theory. Indeed, if the trajectory were curved or irregular, the Doppler history (*i.e.* the time-dependent variation of the Doppler frequency for a given target) would be more complex, and the concept of synthetic aperture length would be less cleanly defined. This is a perfectly reasonable assumption, even for satellites flying in circular orbits. In fact, over the few seconds of aperture time, their motion is well approximated as linear. Finally, possible deviations from the ideal trajectory due to orbital curvature or attitude errors can be treated as perturbations and corrected via motion compensation (Manzoni [57]).

2.1.2. Synthetic aperture and resolution

SAR produces 2D images that, as such, need to be resolved in two directions: range and azimuth. Concerning range, the resolution is dependent on the system bandwidth,

analogously to conventional radars [56]. It is defined as follows [3]:

$$\delta r = \frac{c}{2B_w}, \quad (2.1)$$

where c is the speed of light and B_w the bandwidth. Regarding the azimuth, in a SLAR, the resolution changes across the swath according to: $\delta x = r_0 \Delta \theta$ [55], given $\Delta \theta$, the antenna beamwidth, narrower for a large antenna, and r_0 the slant range. This means that targets closer to the radar are acquired with a higher resolution. In a SAR, the azimuth resolution can be computed starting from the angular resolution $\delta \psi$, which reads [55]:

$$\delta \psi = \frac{\lambda}{2A_s}, \quad (2.2)$$

where A_s is the length of the synthetic aperture and λ the wavelength. Eq. 2.2 is a well-known result in antenna array imaging (Carr [58]). It is indeed possible to assimilate a SAR to an antenna array by leveraging the start-stop approximation: according to it, the radar transmits a pulse, waits until the echo is received, and only then moves to the next along-track position [55]. The underlying assumption is that the radar velocity is negligible with respect to that at which the impulses travel. A target remains in the radar sight as long as it is within the angular sector of width $\Delta \psi$, which is determined by the antenna size. Consequently, the synthetic aperture can be approximated as follows [55]:

$$A_s \approx r_0 \Delta \psi \approx \frac{\lambda}{L_x} r_0, \quad (2.3)$$

where L_x is the antenna length in the azimuth direction. The azimuth resolution δx , therefore, reads [55]:

$$\delta x \approx r_0 \delta \psi = r_0 \frac{\lambda}{2A_s} \approx \frac{L_x}{2}. \quad (2.4)$$

Notably, the azimuth resolution is given by half of the physical antenna length, regardless of the target's position on the ground. Eq. 2.4 demonstrates that, in a SAR, a shorter antenna leads to a better resolution: a seemingly paradoxical result according to classical antenna theory, where a higher resolution entails a larger antenna [58]. The explanation, however, is simple and lies in the fact that the larger antenna allows for capturing a wider area while the radar flies, effectively increasing the time during which the target is illuminated, and hence the synthetic aperture length.

2.1.3. Image processing fundamentals

Data are focused along a fixed direction defined by the squint angle ψ . The conventional choice is to focus at boresight, *i.e.*, with a null squint angle, a configuration known as *zero-Doppler* (Bamler [14]). Specifically, the position of the target relative to the flight path –ahead or behind– determines the Doppler shift (the change in the apparent frequency caused by the relative motion between the radar and the target). The region where the Doppler shift is zero is called the *Doppler centroid zero line* or *zero-Doppler plane* and corresponds to the imaginary plane passing through the position of the satellite, perpendicular to its direction of flight (Krieger et al. [59], and Rodrigues Silva et al. [60]).

To understand how images are formed, it is first necessary to derive the path travelled by the signal, which, in turn, depends on the relative positions of the target and the radar. Given a target identified by the (x', y', z') set of coordinates in the azimuth (x) - ground range (y) - altitude (z) frame depicted in Figure 2.1, its relative position at the time t_x with respect to a radar moving along the x -direction at constant velocity v can be computed through Pythagoras' theorem [3]:

$$r = \sqrt{r_0^2 + (x - x')^2} \approx r_0 + \frac{(v t_x)^2}{2r_0}, \quad (2.5)$$

where r is the so-called *range history* of the point scatterer in the position (x', y', z') . The quadratic expression on the rightmost term of eq. 2.5 has been obtained from a Taylor expansion up to the second order, assuming $v t_x \ll r_0$ [3]. Eq. 2.5. Again, employing Pythagoras' theorem, the slant range r_0 reads [14]:

$$r_0 = \sqrt{(h - z')^2 + y'^2}, \quad (2.6)$$

where h is the altitude at which the radar flies.

The signal transmitted by the radar is in the form [14]:

$$s_{tx}(t_r) = g(t_r) \cdot \exp(j 2 \pi f_0 t_r), \quad (2.7)$$

where g is the signal envelope as a function of the time t_r , whereas f_0 is the carrier frequency. Regarding the signal envelope $g(t_r)$, in SAR *chirps* are typically used, *i.e.*, constant amplitude signals with fixed duration and frequency varying linearly with time [55]. After its interaction with the point target, the signal gets reflected to the radar, delayed by an interval $\Delta t = 2r/c$, with c the speed of light. Neglecting the carrier contribution (removed through demodulation [55]), and recalling that for an electromagnetic wave

$c = f_0\lambda$, the reflected signal measured by the radar is [14]:

$$s_{rx}(t, r) = g(t_r - \Delta t) \cdot \exp\left(-j\frac{4\pi}{\lambda}r + j(\varphi_{scat} + \varphi_{prop})\right), \quad (2.8)$$

where φ_{scat} is the scattering phase introduced by the target itself, and φ_{prop} accounts for any phase variation due to the propagation medium. Finally, by substituting eq. 2.5 into eq. 2.8, the 2D raw data response of a point scatterer is obtained:

$$s_{rx}(t_r, t_x) = g\left(t_r - \frac{2}{c}\left(r_0 + \frac{(v t_x)^2}{2 r_0}\right)\right) \cdot \exp\left(-j\frac{4\pi}{\lambda}\left(r_0 + \frac{(v t_x)^2}{2 r_0}\right) + j(\varphi_{scat} + \varphi_{prop})\right), \quad (2.9)$$

where the slant range r_0 is given by eq. 2.6. The backscattered signals are organised in the raw SAR data matrix, indexed by fast time t_r (related to range) and slow time t_x (related to platform position) [55]. More precisely, t_r corresponds to the travel time of the pulse along the radar line-of-sight; it thus embeds information about how far the radar is from the target at the instant of transmission. On the other hand, t_x corresponds to the motion of the radar along its flight path; as such, it measures how the signal evolves from pulse to pulse as the radar moves. Consequently, by collecting the measurements from multiple pulses transmitted along the flight direction, the bidimensional raw data array is constructed.

After having been collected, the raw data must undergo a sequence of processing steps to be transformed into an interpretable image: most notably, range and azimuth compression.

Range compression: In this process, the received echo is convolved with a time-reversed and conjugated copy of the transmitted chirp, a procedure known as *match filtering* (Turin [61]). The result is a sharp peak in the fast-time direction centred at the time delay corresponding to the target's range [3].

Azimuth compression: This process is the counterpart of range compression, but performed along the slow-time axis. However, since the phase history of a point target depends on the relative position between radar and target, the reference function used in the matched filter must be adapted to the geometry. More precisely, different range bins require slightly different azimuth matched filters, because the chirp rate in azimuth varies with range. By correlating the received signal with the appropriate range-dependent reference function, the target's energy is hence coherently focused into a single resolution cell [3, 55].

The result of the focusing procedure is the transformation of the raw data into the now interpretable *Single Look Complex* (SLC) [55].

Finally, a particularly delicate step in the generation of a SAR image is the *calibration*. It is typically addressed with a two-fold process involving both internal and external calibration, where, for the latter, targets of known reflectivity are employed. A detailed review of calibration methods and related science requirements is provided by Freeman [62].

2.1.4. SAR image characteristics

Radar images are formed in the range (r) - elevation ζ plane. Consequently, these results are distorted with respect to the true geographical frame given by azimuth and ground range directions. An important step in the signal processing chain is thus the *geocoding*, with which the images are rotated back to the x - y frame (see Figure 2.1). The map of the relative coordinates of each point with respect to it is given as follows [14]:

$$\begin{bmatrix} \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sin \theta_l & \cos \theta_i \\ -\cos \theta_l & \sin \theta_i \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \zeta \end{bmatrix} \quad (2.10)$$

In addition to the distortion due to imaging in the radar frame rather than in geographic coordinates, the side-looking geometry and the interaction with distributed targets can cause the effects described below.

Layover: Since the acquired targets are not aligned with the nadir direction, but viewed under the look-angle, the ground range resolution becomes dependent on the local slope of the terrain. This can be formulated as follows (Rossi and Eineder [63]):

$$\delta y = \frac{\delta r}{\sin(\theta_i - \alpha)}, \quad (2.11)$$

where α is the local inclination with respect to a flat surface, θ_i is the incidence angle (see Figure 2.1), δr the slant range resolution defined in eq. 2.1 and δy is the ground range resolution. From eq. 2.11, the source of the layover effect is readily interpreted. It consists of the shrinking of the fore-slopes, for which $-\pi < \alpha < -\pi + \theta_i$, and the expansion of the back-slopes, for which $-\pi + \theta_i < \alpha < 0$.

Radar shadow: Taken to the extreme, the effect of the terrain slope may cause some areas to become completely inaccessible by the radar (e.g., behind high buildings or mountains). No data can be retrieved from these regions, which therefore appear dark in the SLC [63].

Speckle: Distributed targets are typically modelled as a sum of randomly distributed elementary scatterers. As their relative positions with respect to the radar change, the individual echoes may vary the way they combine - constructively or destructively - resulting in potentially abrupt variations of intensity between neighbouring cells. The coherent sum of the backscattered signals from all the stochastic targets within a single resolution cell gives rise to a noise-like pattern known as speckle. Its visual impact can be reduced through the multi-looking technique, consisting of the incoherent averaging of multiple images of the same scene, thereby reducing the variance at the cost of spatial resolution. A deeper investigation on the physical foundation of speckle is given by Goodman [64], while Argenti et al. have provided a thorough comparison of several methods for speckle reduction [65].

2.2. SAR Interferometry

This section aims to provide an overview of the fundamentals of Interferometric SAR (InSAR), with particular focus on the key figures of merit considered in the formation flying design. Initially, the mathematical basics of InSAR are introduced, followed by a description of the most widely used acquisition strategies. The two main purposes of InSAR are presented next: this technique can in fact be employed to either generate digital elevation models (DEMs), or measure variations in the observed scene (Bambler [14]). The former is achieved by *across-track interferometry*, while the latter by *along-track interferometry* or by *differential interferometry*, depending on the time scale over which changes are measured. The remainder of this section covers the fundamentals of interferometric coherence and the compromise single-baseline systems face between accuracy and height ambiguity avoidance.

2.2.1. InSAR principles

A decisive advantage of SAR lies in its ability to provide a coherent measurement of the signal phase. Consequently, it allows for the coherent processing of multiple images acquired at different times or from slightly different positions. This concept is at the basis of SAR interferometry (InSAR), where the measurement of phase differences allows for quantifying a wide variety of geophysical parameters with sub-wavelength accuracy, such as topography, ground deformation, subsidence, glaciers and ice sheets movements [3].

InSAR has brought outstanding results in a multitude of missions: here are recalled the European Space Agency (ESA)'s ERS-1/ERS-2 Tandem mission (D'Elia and Jutz [66]), Shuttle Radar Topography Mission (SRTM) by the National Aeronautics and Space Ad-

ministration (NASA), Department of Defence (DoD), German Aerospace Center (DLR) and Italian Space Agency (ASI) (Farr et al. [67]), and the still operational TanDEM-X mission by DLR (Krieger et al. [6]) for DEM generation. Regarding subsidence and terrain movement monitoring, remarkable examples are ESA's Sentinel-1 (Attema et al. [68]) and Japanese Aerospace Exploration Agency (JAXA)'s Advanced Land Observing Satellite-2 (ALOS-2) (Kankaku et al. [69]), among others.

In the following, it is shown how relevant geophysical information can be derived from the phase difference between two images.

Recalling the formula for the backscattered signal of a point target given by eq. 2.8, the i -th SAR image acquired reads:

$$I_i(r, x) = A(r, x) \exp \left(-j \frac{2m\pi}{\lambda} r + j\varphi_{scat} + j\varphi_{prop} \right), \quad (2.12)$$

where A is the amplitude, which depends on the position of the target with respect to the radar, m is a constant factor taking into account the difference in the paths taken by the signal; it is equal to 1 for bi-static and 2 for monostatic acquisitions (the difference is better detailed later in this section). Moreover, φ_{scat} represents the variation induced by the scattering object, which can be either deterministic, with a change in the dielectric constant, or stochastic [14]. Finally, φ_{prop} brings information regarding the phase perturbations due to noise and propagation medium.

The interferogram is then obtained by taking the complex product of the first image with the complex conjugate of the second, each of them defined accordingly with eq. 2.12 [14]:

$$I_2(r, x)I_1^*(r, x) = |A_1(r, x)||A_2(r, x)|e^{j\Delta\varphi}, \quad (2.13)$$

where $(\cdot)^*$ denotes the complex conjugate, and $\Delta\varphi$ is the interferometric phase defined as follows:

$$\Delta\varphi = \frac{2m\pi}{\lambda}\Delta r + \Delta\varphi_{scat} + \Delta\varphi_{prop}. \quad (2.14)$$

Here Δr is the separation between the range positions of the same target measured in the two images, whereas $\Delta\varphi_{scat}$ and $\Delta\varphi_{prop}$ indicate the differences between the phase variations due to scattering and to perturbations, respectively.

Eq. 2.14 represents the very essence of InSAR: it provides indeed a direct link between the interferometric phase $\Delta\varphi$ and the range difference Δr due to either a different acquisition geometry or a movement of the target in the time interval separating the acquisitions. Consistent with Bamler [14], the assumption $\Delta\varphi_{scat} = 0$ is retained hereafter for a more

neat derivation of the quantities of interest for this work. Specifically, this holds for small incidence angle differences between the two acquisitions. Finally, the term φ_{prop} is also neglected. It is mainly due to two contributions: an imperfect knowledge of the radar trajectory, and random variations of the range history caused by the atmosphere. Concerning the former, current state-of-the-art methods allow isolating and compensating for it, as explained by Manzoni [57]. Regarding the latter, it can be (partially) corrected employing turbulence theories (see Zhang et al. [70] and Bekaert et al. [71]). Therefore, with the additional assumption of a linear movement of the target in between the acquisitions, eq. 2.14 is reduced to:

$$\Delta\varphi = \frac{2m\pi}{\lambda}\Delta r \approx k_z \Delta z + \frac{4\pi}{\lambda}v(t_2 - t_1), \quad (2.15)$$

where t_1 and t_2 are time instants at which the two images are acquired, Δz is the measured altitude variation, whereas k_z is a coefficient linking the variations of altitude and phase; its complete expression is explicitly derived later in this section. Eq.2.15 is particularly instructive, as it allows for highlighting the two different contributions of the altitude Δz and the displacement $v(t_2 - t_1)$ on the phase difference.

An explicative representation of the interferometric phase is provided in Figure 2.2 taken from Bamler [14]. It is characterised by a typical fringe pattern, where the iso-phase curves follow the coloured bands. Here, a more rapid change in hue indicates a steeper altitude variation; the phase ranges from 0 to 2π as the colour shifts from red to blue. In the left picture, the effect of topography is almost completely masked out by the contribution of an ideally flat Earth, which is removed in the right picture, explaining the closer resemblance to topographic contour lines.

2.2.2. Acquisition strategies

SAR systems can be designed to perform acquisitions employing different modes (Mora et al. [3], Krieger et al. [72]). Some of the most widely employed are illustrated in Figure 2.3. The left picture shows the so-called *pursuit monostatic* mode, in which both satellites transmit a pulse and record its echo [3]. In this configuration, the radars do not acquire the same area simultaneously, hence an along-track separation larger than in other modes is allowed. The central picture shows the *bistatic* mode, in which one of the two satellites acts as an illuminator and both receive the reflected signal [3]. In this case, the same area is mapped simultaneously, eliminating temporal decorrelation. A combination of these two modes is achieved in the *alternating bistatic* mode, also known as *ping-pong* mode, shown in the right picture. In this case, the two radars act as trans-

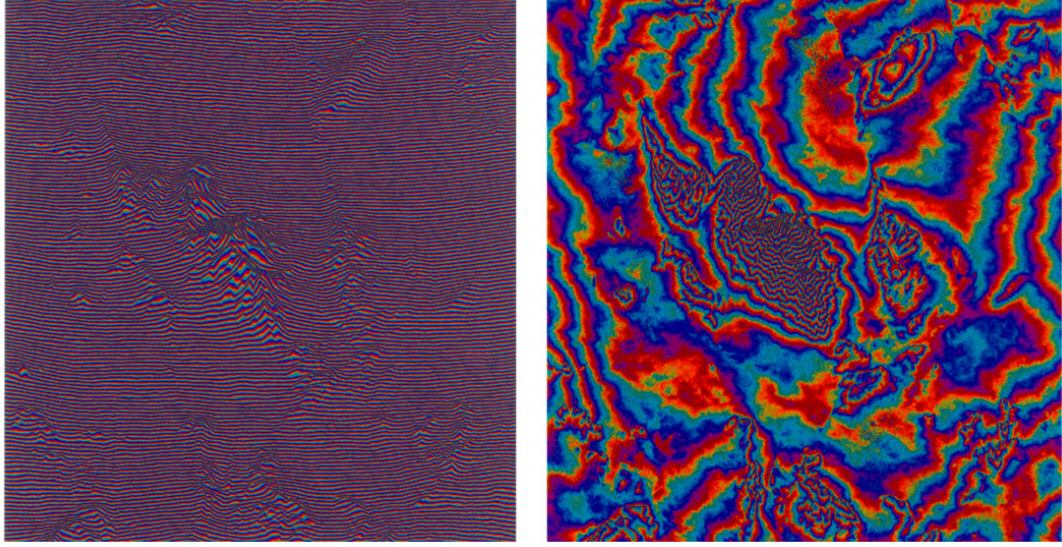


Figure 2.2: SAR interferogram, from Bamler [14]. It shows a $\sim 25\text{km} \times 25\text{km}$ area in Mojave Desert, CA, taken with ERS-1/2 ©ESA sensor. Left: with flat earth; Right: after flat earth removal

mitters in an alternate fashion (the coloured icons), while the scattered signal is recorded by both (coloured and grey icons) [3]. This allows for the generation of two monostatic and two bistatic images in a single pass. The aforementioned acquisition modes are all implemented in TanDEM-X. As detailed by Krieger et al. [72], each of them is indeed employed in different phases of the mission. In particular, the bistatic mode is the nominal one, adopted for the DEM generation; the pursuit monostatic mode instead is used as a backup in the event of formation flying or synchronisation issues; finally, the alternating bistatic mode is employed for calibration and validation purposes.

A further dichotomy concerning the acquisition strategy arises between single- and multi-pass InSAR. This work is particularly aimed at systems adopting a *single-pass* configu-

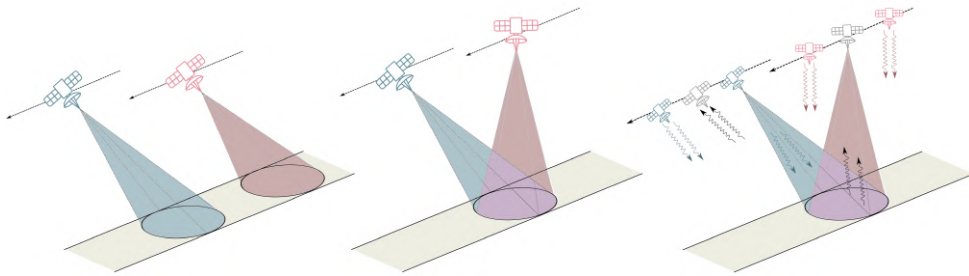


Figure 2.3: InSAR data acquisition modes, adapted from Krieger et al. [6]. Left: pursuit monostatic mode; Centre: bistatic mode; Right: alternating bistatic mode.

ration. This technique entails two or more satellites acquiring the same scene simultaneously, providing immunity to temporal decorrelation (explained later in this section) and atmospheric variability, fostering the generation of highly accurate DEMs (see Massonnet and Feigl [73], Curlander and McDonough [55]). Conversely, *multi-pass* InSAR rely on repeat observations of the same area from nearly identical orbital tracks. It allows indeed for monitoring surface deformation over time at the price of introducing decorrelation. Additionally, it suffers from a higher sensitivity to changes in scattering properties and to atmospheric path delays [55, 73].

2.2.3. Across-track interferometry

The main purpose of across-track interferometry is the generation of a DEM, *i.e.* a gridded digital representation of the Earth's surface topography, where each grid cell corresponds to an elevation value. Its functioning mechanism consists of acquiring two images of the same scene from two slightly different positions. The two acquisitions may be simultaneous, as in the bistatic case, or performed during different passes, as in the monostatic mode.

In Figure 2.4 is reported the typical acquisition geometry for across-track interferometry. The two satellites are at distances r_c and r_d from the target (the subscripts are used to denote the chief and deputy satellite, respectively), with direction identified by the look angles θ_l , and the incidence angle θ_i . Their relative distance is given by the baseline B , whereas its component perpendicular to the look direction is the *orthogonal* or *effective baseline* $B_\perp$. In the following, it is considered the position of a reference target whose distance with respect to the chief and deputies' satellites is $r_{0,c}$ and $r_{0,d}$, respectively. Every other point can then be expressed in range and elevation coordinates (r', ζ) with respect to the latter.

Once the interferometric phase has been computed, it is necessary to convert it to an altitude measure. Considering a target at: (y, z) , and the position of the satellite: (y_s, z_s) , their relative distance is [55]:

$$r(y, z) = \sqrt{(y - y_s)^2 + (z - z_s)^2}. \quad (2.16)$$

Consider now a reference point target at (y_0, z_0) , the origin of the r - ζ frame in Figure 2.4. By taking a first-order Taylor expansion centred at (y_0, z_0) , eq. 2.16 yields, for the chief

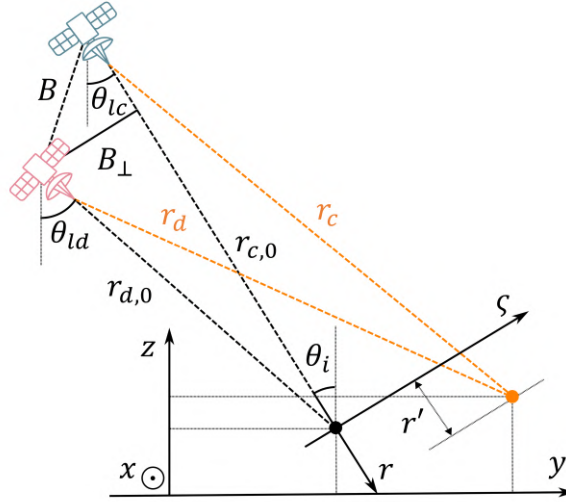


Figure 2.4: InSAR imaging geometry.

and deputy satellites, respectively:

$$\begin{cases} r_c(y, z) \approx r_{c,0} + \Delta y \sin \theta_{i,c} - \Delta z \cos \theta_{i,c} \\ r_d(y, z) \approx r_{d,0} + \Delta y \sin \theta_{i,d} - \Delta z \cos \theta_{i,d} \end{cases} \quad (2.17)$$

where $\Delta y := y - y_0$ and $\Delta z := z - z_0$ are the components of the separation between the target and the reference, r_0 is the separation between the radar and the reference target at (y_0, z_0) , and the subscripts $(\cdot)_c$ and $(\cdot)_d$ are used to indicate quantities related to the chief and to the deputy satellites, respectively. Moreover, the eqs. 2.17 are obtained considering that the derivatives in the Taylor expansion are:

$$\frac{\partial r}{\partial y} \approx \sin \theta_i \quad \frac{\partial r}{\partial z} \approx -\cos \theta_i. \quad (2.18)$$

By taking another Taylor series expansion truncated to first-order of the difference between the two eqs. 2.17 entered on the angle $\theta_{i,c}$, the one-way path difference Δr reads:

$$\Delta r := (r_d - r_{d,0}) - (r_c - r_{c,0}) \approx (\theta_{i,d} - \theta_{i,c}) \cdot (\cos \theta_{i,c} \Delta y + \sin \theta_{i,c} \Delta z) = (\theta_{i,d} - \theta_{i,c}) \zeta, \quad (2.19)$$

where the rightmost term is obtained from the map in eq. 2.10; the dependency on x, y is omitted for readability. The angular difference $\theta_{i,d} - \theta_{i,c}$ can be written in terms of the effective baseline as follows:

$$B_{\perp} \approx (\theta_{i,d} - \theta_{i,c}) r_{c,0}. \quad (2.20)$$

By substituting it into eq. 2.19, and recalling that the acquisitions take place in the same

time instant ($t_1 = t_2$), the phase difference defined in eq. 2.15 yields:

$$\Delta\varphi = \frac{B_{\perp}}{r_{c,0}} \frac{2m\pi}{\lambda} \zeta. \quad (2.21)$$

To determine the sensitivity of the interferometer to an altitude change, it is necessary to differentiate the phase difference with respect to the height at a fixed image pixel, *i.e.*, at fixed slant range. This is done as follows:

$$\left. \frac{\partial \Delta\varphi}{\partial \Delta z} \right|_r = \frac{d \Delta\varphi}{d \zeta} \cdot \left. \frac{\partial \zeta}{\partial \Delta z} \right|_r. \quad (2.22)$$

From the map in eq. 2.10, the elevation is expressed in terms of altitude and range:

$$\zeta = \frac{\Delta z + r \cos \theta_{i,c}}{\sin \theta_{i,c}}, \quad (2.23)$$

from which its derivative with fixed range reads:

$$\left. \frac{\partial \zeta}{\partial \Delta z} \right|_r = \frac{1}{\sin \theta_{i,c}}. \quad (2.24)$$

Finally, the height sensitivity measured for a resolution cell at slant range $r_{c,0}$ yields:

$$\left. \frac{\partial \Delta\varphi}{\partial \Delta z} \right|_{r=r_{c,0}} = \frac{B_{\perp}}{r_{c,0}} \frac{2m\pi}{\lambda \sin \theta_i}. \quad (2.25)$$

A crucial parameter for evaluating interferogram quality is the *height of ambiguity* (HoA), *i.e.*, the height difference corresponding to a phase variation of 2π , *i.e.*, *i.e.* one fringe in the interferogram. This can be readily obtained from eq. 2.25 as:

$$\text{HoA} = \frac{\lambda r_{c,0} \sin \theta_i}{m B_{\perp}}. \quad (2.26)$$

where m is 1 or 2 for a bistatic and monostatic acquisition mode, respectively. The HoA defines the interferometer's sensitivity to topographic variations and, as such, it is the scaling factor between the uncertainty on the phase and on the height:

$$\sigma_z = \frac{\text{HoA}}{2\pi} \sigma_{\varphi}. \quad (2.27)$$

Considering its pivotal role in determining the interferometric performance, the HoA is regarded as the principal figure of merit throughout this thesis. Specifically, the capability

to maintain it close to the nominal value is considered an indicator of the quality of the satellite formation under consideration.

The phase difference has been extracted from the interferogram, eq. 2.25 can be leveraged to obtain the corresponding altitude variations and hence reconstruct the topography. However, the phase is a quantity defined modulo 2π . To retrieve the height information, it is therefore necessary to unwrap it through a procedure known as *phase unwrapping* [14].

A large variety of phase unwrapping methods have been developed: Ghiglia and Pritt [74] have provided a comparison between different algorithms. Such methods are based on the assumption that the numerical gradient of the true phase always falls in the interval $(-\pi, \pi]$. As a second step, they try to find the best estimate of the unwrapped phase from its numerical gradient $\nabla\phi$. The latter can be computed from the variation of the phase along the x , y coordinates as follows [14]:

$$\nabla\varphi(x, y) = \begin{bmatrix} \Delta_x\varphi(x, y) \\ \Delta_y\varphi(x, y) \end{bmatrix} = \begin{bmatrix} \varphi(x+1, y) - \varphi(x, y) \\ \varphi(x, y+1) - \varphi(x, y) \end{bmatrix} \quad (2.28)$$

where Δ_i is the phase difference measure between nearby pixels in the i direction, whereas x and y are the azimuth and ground range coordinates, respectively. Being defined in a bi-dimensional data matrix, the gradient can be integrated along the smoothest path, improving the robustness of the algorithm.

However, a problem arises if for adjacent samples the true phase difference does not belong to the $(-\pi, \pi]$ interval. In this case, the height information contained in those pixels would be wrongly computed by integer multiples of 2π . Such a mistaken attribution may have two causes: an abrupt change in the phase due to a particularly steep height difference –generally arising when imaging rough terrains –or a loss of coherence, a concept further clarified later in this section. In order to prevent errors in the phase unwrapping procedure, a higher HoA is required. In fact, this would cause a large change in altitude to be mapped into a small variation in phase, consistently with eqs. 2.25 and 2.26.

The last considerations make clear the critical trade-off to which the choice of HoA is forced. According to eq. 2.27, a low HoA would have the advantage of reducing the uncertainty on the measured altitude, thereby producing more accurate interferograms. On the other hand, however, it would lead to incorrect attributions due to errors in the phase unwrapping procedure. One possible strategy to circumvent the ambiguity problem is to adopt time-varying baselines, as done for TanDEM-X. In this mission, two DEMs were acquired approximately one year apart: in the first, short baselines were used to avoid phase unwrapping issues; in the second, the accuracy was increased by using longer

baselines (Krieger et al. [15]).

In Figure 2.5 is reported the flow of the main signal processing operations typically employed for across-track InSAR. Note that the phase unwrapping is performed almost at the end: this reduces the chance of introducing errors by working on data with reduced phase variance thanks to the previous multi-looking operation. Two steps have not yet been discussed: the *co-registration* and the *spectral filtering*.

Co-registration: before combining the images to form the interferogram, it is also necessary to ensure a coincidence of the pixels corresponding to the same points on the ground. This is typically done by maximising the cross-correlation between the two reflected signals to determine the correcting shift to apply [3].

Spectral filtering: After interferogram generation, the phase contains both the desired topographic or deformation signal and noise arising from decorrelation, thermal noise (these two noise sources are better detailed later in this section), and co-registration errors. To improve the interferogram quality before multi-looking and phase unwrapping, a spectral filter is applied, allowing for suppressing high-frequency components dominated by noise [55].

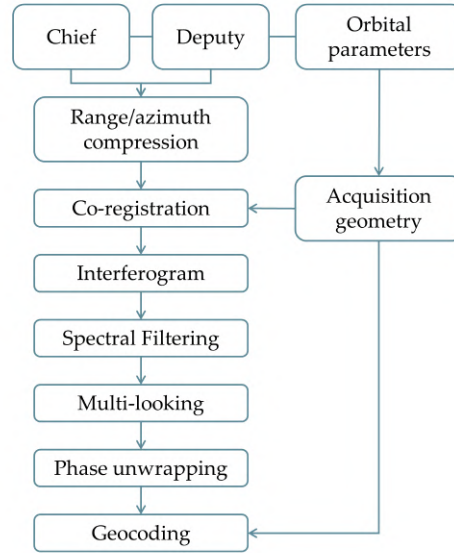


Figure 2.5: Main across-track InSAR signal processing operations.

2.2.4. Along-track and differential interferometry

While in across-track interferometry, the measured phase difference comes from a different looking geometry, to gather information about moving targets, the acquisitions must

be separated in time. This is achieved in along-track interferometry by leveraging two antennas separated in the along-track coordinate, as the name suggests. Typically, one of the two transmits a pulse and both receive the backscattered echoes, thus enabling the measurement of a movement in the range direction. The measured phase variation is proportional to the target's speed, which can be retrieved given a known velocity of the platform. This allows the monitoring of fast-changing phenomena, such as cars and ship movements or surface oceanic currents measurements (Bamler [14]). A similar approach is applied in differential InSAR (D-InSAR), with the aim of measuring changes on much longer time scales, from days to years. By leveraging wavelengths in the order of cm, it allows for gathering information about geophysical parameters such as subsidence, glaciers' evolution, as well as seismic and volcanic movements, potentially with sub-wavelength level accuracy. From eq. 2.15, it would be impossible to isolate the time varying term $v(t_1 - t_2)$ from that due to topography by sole measurement of the interferometric phase. Therefore, D-InSAR must also rely on the a priori knowledge of a DEM in order to cancel out the altitude difference Δz . This operation bounds the achievable accuracy, which can never exceed that of the employed DEM. Additionally, since the acquisitions must take place over a particularly long period of time, the consequent temporal decorrelation represents the main limitation to the D-InSAR performance (Moreira et al. [3]).

2.2.5. Interferometric coherence

The interferometric phase is generally affected by noise. This could be inherent to the system itself, as in the case of thermal noise, or being caused by a random variation of the reflected signal as a result of targets' changes over time or under different looking geometries. The coherence factor brings information about both these noise sources by measuring the correlation coefficient between two SAR images. It is defined as follows [14]:

$$\gamma(r, x) = \frac{E[I_2(r, x)I_1^*(r, x)]}{\sqrt{E[|I_1(r, x)|^2]E[|I_2(r, x)|^2]}}, \quad (2.29)$$

where $E[\bullet]$ denotes the expected value operator, whereas the subscript $(\cdot)_i$ indicates the i -th image combined to form the interferogram. What is relevant to the evaluation of the phase noise is the absolute value of the coherence factor, as it can be related to the phase variance through the Cramer-Rao Bound (CRB) (Rosen et al. [75]):

$$\sigma_\phi^2 \geq \frac{1 - |\gamma|^2}{2N|\gamma|^2}, \quad (2.30)$$

where N is the *independent number of looks*. These are statistically independent measurements of the same pixel, averaged to improve phase variance. Specifically, the bandwidth is divided into sub-bands to form separate looks of the same ground area that are subsequently averaged. Each look has an independent speckle noise, and therefore their averaging effectively abates the phase noise by a factor $1/N$. On the other hand, because each look makes use of a portion of the total bandwidth, the overall range accuracy decreases, consistently with eq. 2.1. This forces a trade-off between range resolution and phase accuracy. As the bandwidth is limited, the maximum number of statistically independent looks is given by the total number of independent resolution cells available in the averaging window. For this reason, real missions typically employ tens of independent looks (Krieger et al. [15]).

Eq. 2.30 highlights how the coherence is linked to the phase accuracy, explaining how this parameter can provide insights about the expected quality of the interferogram. Moreover, the coherence can also be leveraged ex-post to derive information about the observed targets by comparing their correlation across different acquisitions [3].

In the following, an overview of possible sources of decorrelation is provided, with specific focus on those affected by the geometry of the satellite formation.

Noise decorrelation

Given two images I_1 and I_2 affected by the white noises w_1 and w_2 with identical variance σ_w^2 , by employing the amplitude-phase formulation of the backscattered signal in eq. 2.12 each of them yield [14]:

$$I_1 = Ae^{-j\frac{4\pi}{\lambda}r_1} + w_1; \quad I_2 = Ae^{-j\frac{4\pi}{\lambda}r_2} + w_2, \quad (2.31)$$

where A are the amplitudes, r_1 and r_2 are the distances of the target from the radar in the two acquisitions. From eq. 2.29, considering that for a point target $N = 1$, the contribution to the coherence γ_{noise} of a white noise is:

$$|\gamma_{noise}| := \frac{A^2}{A^2 + \sigma_w^2} = \frac{SNR}{1 + SNR}, \quad (2.32)$$

where SNR stands for the signal-to-noise ratio. Note that, as one would expect, $|\gamma_{noise}| \rightarrow 1$ if $SNR \rightarrow \infty$.

Temporal decorrelation

Changes in the relative positions and in the scattering behaviour of the observed elementary targets can occur during the time elapsed between different acquisitions. These changes cause a loss of coherence in the resulting interferogram, which affects along-track interferometry, D-InSAR, and monostatic across-track interferometry. Several models have been proposed to accurately describe this phenomenon. One of the most relevant interpretations has been provided by Zebker and Villasenor [76], according to which the decorrelation is due to the Brownian motion of the elementary scatterers within the resolution cell. The movement of each target is hence random and completely independent of the others. According to this model, the contribution to the coherence given by temporal decorrelation γ_{temp} reads:

$$\gamma_{temp} := \exp \left(-T \frac{(4\pi\sigma_d)^2}{2\lambda^2} \right), \quad (2.33)$$

where T is the time interval separating the two acquisitions, σ_d the rate of change of the variance of the elementary displacement, and λ the wavelength.

Spatial decorrelation

When the images forming the interferogram are acquired from two different positions - the natural scenario in across-track interferometry - the optical paths followed by the pulses can differ significantly because of the diverse distribution of the elementary scatterers with respect to the radars. The result is a different speckle pattern that leads to a loss of coherence [14].

As one of the objectives of this work is to understand how the formation geometry may influence the interferometry performance, more refined mathematical details are provided regarding the impact of the acquisition geometry on the coherence. In the following, reliance is placed once more on the geometry in Figure 2.4. The signal $I_{C,d}$ from a single resolution cell C of the SLC measured by the deputy in the range (r'), azimuth (x'), and elevation (ζ) coordinates is given by:

$$I_{C,d} = \int_C o(x', r', \zeta) \exp \left(-j \frac{4\pi}{\lambda} r_d \right) dx' dr' d\zeta, \quad (2.34)$$

where r_d is the distance between the target and the deputy satellite, whereas $o(x', r', \zeta)$ is the *complex backscatter*: it incorporates information about amplitude and phase variations caused by the interaction of the pulse with the elementary target. It is modelled as white noise in all directions.

The deputy-target distance r_d is given in eq. 2.19. By combining it with the definition of the effective baseline in eq. 2.20, it reads:

$$r_d = r' + r_{d,0} + (\theta_{i,d} - \theta_{i,c}) \cdot (\cos \theta_{i,c} \Delta y + \sin \theta_{i,c} \Delta z) = \frac{B_{\perp}}{r_{c,0}} \zeta + r' + r_{d,0}, \quad (2.35)$$

where the subscript $(\cdot)_0$ indicates the reference target in correspondence with the origin of the r - ζ reference frame. Moreover, $\theta_{i,c}$ and $\theta_{i,d}$ are the look angles relative to the chief and deputy satellite, respectively, whereas $r' := (r_c(y, z) - r_{c,0})$ is the position of the scatterer with respect to the reference target in the slant range direction (see eq. 2.19 and Figure 2.4). The contribution given by $r_{d,0}$ is known from the acquisition geometry and can be promptly removed. Therefore, by inserting the expression in eq. 2.35 into that in eq. 2.34 after having removed the term $r_{d,0}$, it yields:

$$I_{C,d} = \int_C o(x', r', \zeta) \exp \left(-j \frac{4\pi}{\lambda} r' \right) \exp \left(-j \frac{4\pi}{\lambda} \frac{B_{\perp}}{r_{c,0}} \zeta \right) dx' dr' d\zeta, \quad (2.36)$$

From eq. 2.36, the projection of the complex backscatter along the elevation $s(\zeta)$ can be extracted; it is defined as follows:

$$s(\zeta) := \int o(x', r', \zeta) \exp \left(-j \frac{4\pi}{\lambda} r' \right) dx' dr'. \quad (2.37)$$

Since $o(x', r', \zeta)$ is a white noise, s inherits this property; as such, its autocorrelation is proportional to a Dirac delta in elevation:

$$E[s(\zeta)s^*(\zeta + \zeta')] = \sigma_s^2(\zeta)\delta(\zeta'),$$

where σ_s^2 is the variance of s and $(\cdot)^*$ the complex conjugate. Therefore, the terms defining the coherence in eq. 2.29 are:

$$\begin{cases} E[|I_1(r, x)|^2] = E[|I_2(r, x)|^2] = \int \sigma_s^2(\zeta) d\zeta \\ E[|I_2(r, x)I_1^*(r, x)|] = \int \sigma_s^2(\zeta) \exp \left(-j \frac{4\pi}{\lambda} \frac{B_{\perp}}{r_{c,0}} \zeta \right) d\zeta \end{cases} \quad (2.38)$$

Finally, the contribution to the coherence of the spatial decorrelation $\gamma_{spatial}$ is (Rosen et al. [75]):

$$\gamma_{spatial} := \frac{\int \sigma_s^2(\zeta) \exp \left(-j \frac{4\pi}{\lambda} \frac{B_{\perp}}{r_{c,0}} \zeta \right) d\zeta}{\int \sigma_s^2(\zeta) d\zeta}. \quad (2.39)$$

The variance σ_s^2 depends on the observed target. Therefore, the spatial decorrelation is different, for instance, between flat and forested areas. As a special case, note that the response of a point target located at ζ_0 is a Dirac delta: $\sigma_s^2(\zeta) = \delta(\zeta - \zeta_0)$, and therefore its coherence γ_{point} reads:

$$|\gamma_{point}| = 1. \quad (2.40)$$

This result is consistent with the fact that a point scatterer does not create a speckle pattern; in other words, it lacks the source of spatial decorrelation itself.

The spatial decorrelation makes the coherence drop the farther apart the two acquisitions are. Specifically, γ becomes null when the effective baseline reaches a critical value $B_{\perp,crit}$. Zebker and Villasenor [76] as well as Rodriguez and Martin [77] derive this limit by assuming a bi-dimensional cardinal sine in the azimuth and range coordinates and directly substituting it into eq. 2.39. The same result can be obtained with the approach by Ferretti et al. [78], which is shown in the following. The idea is to compute the variation in the incidence angle $\Delta\theta_i$ such that the paths from the nearest and farthest points of a resolution cell differ by a wavelength λ [78]:

$$m \delta y \sin(\theta_i + \Delta\theta_i) - m \delta y \sin \theta_i = \lambda, \quad (2.41)$$

where δy and δr are the resolutions in the ground range and slant range directions, respectively, θ_i is the angle of incidence (see Figure 2.1), and m is 1 or 2 for a bistatic or a monostatic acquisition, respectively. By virtue of elementary geometric considerations relying on the sketch in Figure 2.4, the ground range resolution is linked to the slant range by a factor $\sin \theta_i$. Therefore, expanding the first term on the left-hand side of eq. 2.41 around θ_i , it yields:

$$\Delta\theta_i \approx \frac{\lambda}{m \delta y \cos \theta_i} = \frac{\lambda \sin \theta_i}{m \delta r \cos \theta_i}. \quad (2.42)$$

In conclusion, the critical baseline $B_{\perp,crit}$ corresponds to the variation of the incidence angle computed in eq. 2.42 and is given by the following expression:

$$B_{\perp,crit} \approx r_{0,c} \Delta\theta_i = \frac{\lambda r_{0,c}}{m \delta r} \tan \theta_i. \quad (2.43)$$

Eq. 2.43 effectively imposes a compelling requirement on the relative geometry of the satellite formation. In fact, to avoid the complete loss of coherence, the effective baseline shall never exceed the limit imposed by the critical baseline, in correspondence of which $\gamma = 0$.

In general, the longer the effective baseline, the lower the decrease in coherence. Consid-

ering the acquisition of a terrain without volumetric scatterers, the variance $\sigma_s^2(\zeta)$ can be modelled as a cardinal sine centred at the position ζ' :

$$\sigma_s^2(\zeta) \approx \left(\text{sinc} \left(\frac{\zeta - \zeta'}{\delta r} \tan \theta_i \right) \right)^2. \quad (2.44)$$

Substituting it into eq.2.39, the following formula expressing the coherence loss depending on the effective baseline $\gamma_{baseline}$ is obtained in agreement with Bamler [14]:

$$\gamma_{baseline} := \left(1 - \frac{B_{\perp}}{B_{\perp crit}} \right), \quad (2.45)$$

where $B_{\perp}$ is recalled to be the effective baseline. The contribution of the factor $\gamma_{baseline}$ to the coherence is often denoted as *baseline decorrelation* (see Moreira et al. [3]) to distinguish it from the *volumetric decorrelation* γ_{vol} , arising in the presence of volumetric scatterers. In any case, these could be analogously treated by employing an appropriate model for the function $\sigma_s(\zeta)$, allowing for considering both in the wider context of spatial decorrelation [75].

According to eq. 2.45, the coherence decreases linearly with the effective baseline. Moreover, considering the relationship between coherence and phase variance given by the CRB in eq. 2.30, a lower coherence leads to a higher phase uncertainty. The latter determines the height variance σ_z by means of eq. 2.27, where the scale factor linking the two is given by the HoA, which, in turn, is inversely related to the effective baseline. The overall effect on the height uncertainty is given by the following expression:

$$\sigma_z = \frac{\text{HoA}(B_{\perp})}{2\pi} \frac{1}{|\gamma_{baseline}|} \sqrt{\frac{1 - |\gamma_{baseline}|^2}{2N}} \quad (2.46)$$

where N is the number of independent looks (see). The behaviour of σ_z as a function of the effective baseline and the HoA is shown in Figure 2.6, where the SAR relevant parameters have been taken accordingly to those of TanDEM-X, as summarised in Table 2.2. The curve shows a clear minimum which corresponds to an effective baseline (see Appendix A):

$$B_{\perp} = \frac{3 - \sqrt{5}}{2} B_{\perp crit} \approx 0.382 B_{\perp crit}.$$

The peculiar trend is due to the two-folded role played by $B_{\perp}$: on the one hand, a longer baseline leads to the said decrease in the coherence and the consequent worsening of the phase uncertainty. On the other, it decreases the scale factor linking the latter to the height variance. Theoretically, the best possible interferometric performance is attained

for the $B_{\perp}$ (or the HoA) that leads to a minimum of σ_z . However, issues concerning the phase unwrapping procedure described previously in this section (see eq. 2.28) force SAR systems to operate at shorter baselines (and higher HoAs) [77].

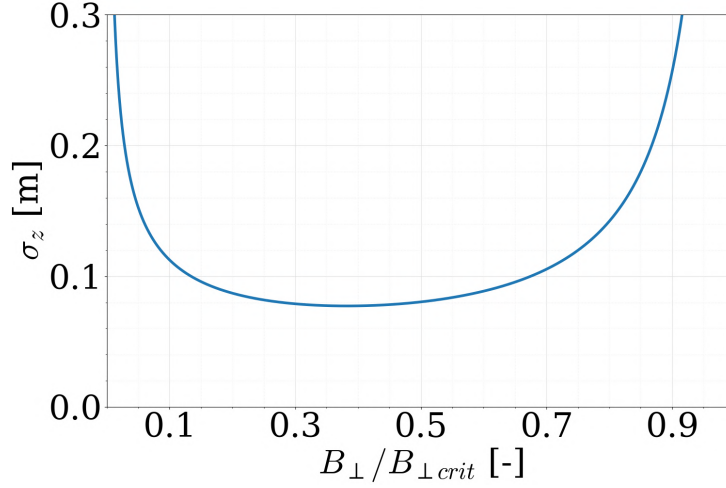


Figure 2.6: Standard deviation of the altitude σ_z relative to the effective baseline and the HoA; relevant SAR parameters from Table 2.2.

Wavelength	Altitude	Incidence angle	Look-angle	Bandwidth	N° independent looks	Critical baseline
3.10 cm	512.0 km	36.9°	33.8°	150 MHz	15	136.2 km

Table 2.2: Nominal parameters of TanDEM-X mission used as a reference [72].

In conclusion, the absolute value of the total coherence factor is obtained as the product of all the contributions from the aforementioned decorrelation sources [3] in eq. 2.32, 2.33, and 2.39:

$$|\gamma| = |\gamma_{noise}| |\gamma_{temp}| |\gamma_{spatial}|. \quad (2.47)$$

2.2.6. Accuracy and phase unwrapping robustness compromise

Figure 2.6 makes it evident the existence of an optimal value of the effective baseline to minimise the height uncertainty and therefore maximise the accuracy. However, section 2.2.3 explained that, to extract topographic information from the interferometric phase, it is necessary to *unwrap* it. Consequently, any mistake in the phase unwrapping would lead to an incorrect height measurement, corresponding to a phase shift multiple of 2π . Specifically, any change in the true altitude bigger than the HoA would lead to an error in the phase unwrapping procedure (Moreira et al. [3], Bamler [14]). The higher the HoA, the less error-prone the height reconstruction would be, because a considerable

altitude variation would be mapped into a slight difference in the phase. On the other hand, this would also mean that the interferometer is less sensitive to small changes in the topography. In other words, InSAR must compromise between low and high HoA, between accuracy and robustness in the phase unwrapping. An example of such a trade-off is shown in Figure 2.7. It displays two interferograms obtained by the Hongtu-1 mission and the corresponding height error map following phase unwrapping, here adapted from Wang et al. [8].

A possible solution to overcome this limitation resides in the availability of multiple images of the same area, which can be combined to form interferograms with different HoAs. In this way, a principal, low-HoA interferogram could be used to achieve high accuracy; the other, with a higher HoA, would provide support in correcting the phase unwrapping. For instance, this is achieved in TanDEM-X by performing two global acquisitions in succession separated by a time interval of approximately one year (Krieger et al. [6]). In the first one, the satellites fly at a closer distance, delivering an interferogram with a higher HoA. The accuracy is then enhanced in the second acquisition, where the inter-satellite

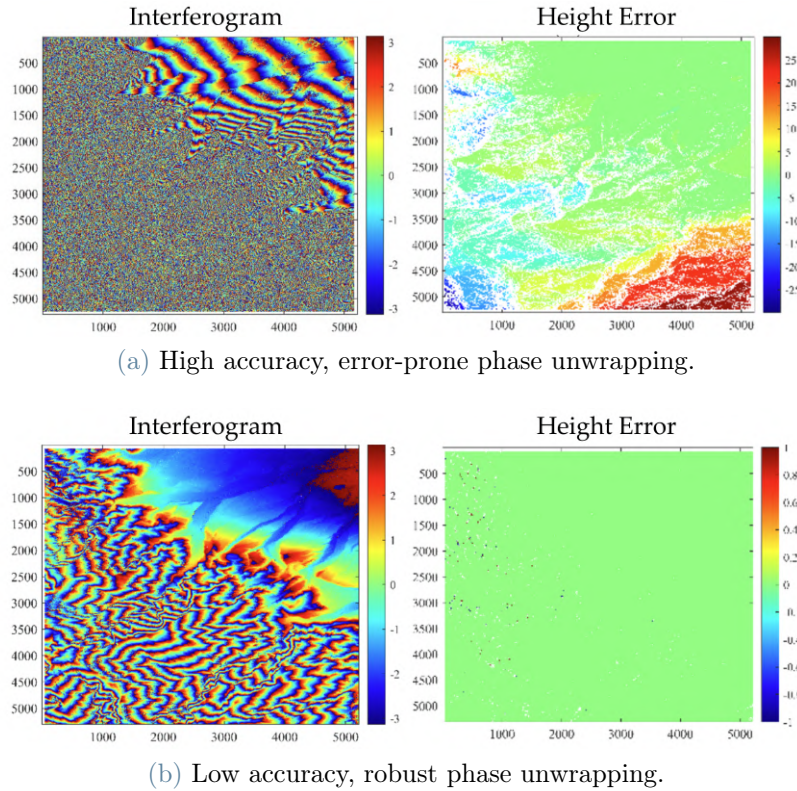


Figure 2.7: Examples of phase unwrapping error for interferograms with different accuracy –the denser the fringes, the greater the height sensitivity –from the Hongtu-1 mission, adapted from Wang et al. [8].

distance is increased, leading to a lower HoA.

However, while effective in solving the accuracy-robustness compromise, this solution does not come without problems. The main issue is precisely the time elapsed in between the acquisitions: changes in the scenes that happened in the meantime would cause a loss of coherence between the images, affecting the overall quality of the DEM.

There is only one possible way of having more than one interferogram at disposal without losing coherence, and this requires constellations of more than two satellites. In such a setup, every pair of satellites would produce an interferogram and, by carefully controlling their mutual distances, two interferograms with different HoA could be obtained at the same time. The HoA is, in fact, determined by the relative distance between the radars and, thus, an ideal configuration would entail satellites in fixed relative positions (or fixed *baselines*). Such a formation would ensure that imaging requirements are consistently met across all observed regions, as the radars would always be perfectly positioned. The drawback, however, is the cost: holding satellites in rigid configurations goes against their natural orbital dynamics, thus requiring a higher control effort (Scala et al. [35]).

The alternative is to take advantage of the natural relative motion of satellites in orbit, which reduces the cost but requires accepting a time-varying geometry. The result is that the HoA is not constant throughout the orbit, meaning that the interferograms would lose homogeneity.

All these considerations are of fundamental importance for the purpose of this work, as they allow for a more rigorous formalisation of the research question it aims to answer. By employing formations of more than two satellites, it could be possible indeed to dedicate one effective baseline to reach maximum accuracy (so, minimum σ_z), while at the same time having another, shorter one, to solve phase unwrapping. Hence, the need for investigating suitable formation flying solutions.

2.3. SAR Tomography

Although this work is primarily aimed at applications in the field of interferometry, other SAR techniques could also benefit from the use of satellite formations. Among these, SAR tomography is certainly a prominent one; a brief introduction to this technology is therefore provided in this section.

The theoretical foundations of tomographic SAR are due to Knael and Cardillo [79], while Pasquali et al. [80], Fortuny-Guasch [81] gave the first experimental confirmations. The first airborne demonstration was performed by Reigber et al. [82]. Chen et al. [83]

experimentally validated a bistatic tomographic SAR concept in 2024. In 2025, the ESA P-band mission BIOMASS [84] was launched: it is expected to provide the first global tomographic coverage of forested areas

In a SAR, the transmitted pulse is reflected by structures of dimensions comparable to those of the incident wavelength. Consequently, by selecting an appropriate frequency, it is possible to penetrate composite volumetric structures, such as forests. More precisely, longer wavelengths are transmitted through the canopies, composed of small elements —leaves and branches — and are instead reflected by the underlying bodies: trunks and soil. This allows the retrieval of information regarding the forest height, structure, and concealed topography. Recent results by Tebaldini et al. [85] have also demonstrated the possibility of measuring above-ground biomass by correlating it with radar intensity. A similar reasoning can also be applied to the study of glaciers and snow, for which SAR tomography can measure depth, optical properties, as well as internal structure and stratification (Tebaldini et al. [86]).

The acquisition of a single SAR image does not allow for distinguishing targets located at the same radial distance, which would all be mapped into the same resolution cell. In InSAR, two images are used indeed. However, this is still not sufficient for the study of volumetric structures, as it would only provide an intermediate value between terrain and canopy height corresponding to the phase centre of all scatterers within each resolution cell (Reigber et al. [82]). SAR tomography is therefore based on this principle: acquiring a stack of images from slightly different points of view, obtained during multiple passes of the radar over the area of interest. Specifically, the need for multiple overflights refers to the monostatic case, but — as for InSAR — the use of a satellite constellation would allow acquisitions to be performed in a multi-static configuration, simultaneously. This would have the undoubted advantage of making tomography virtually immune to temporal decorrelation.

As in classical SAR theory, the resolution can be calculated similarly to an array antenna, in which the length is replaced by the synthetic aperture. In the case of tomography, this is given by the maximum distance between all pairs of radar acquisitions, which are obtained throughout multiple orbits. The accuracy in elevation $\delta\zeta$ can hence be computed from the angular resolution multiplied by the slant range, obtaining [3]:

$$\delta\zeta = \frac{\lambda r_0}{2b_\zeta}, \quad (2.48)$$

where λ is the wavelength, r_0 the slant range, and b_ζ is the baseline aperture in elevation. Rather than producing a bidimensional image, the result is a cubic data structure in the

range, azimuth, and elevation coordinates. The range and azimuth resolution are given from classic SAR theory (see eq. 2.1, 2.4), while eq. 2.48 gives the resolution in elevation. Another relevant result for the purpose of this thesis has been derived by Reigber et al. [82] and reads:

$$d < \frac{\lambda r_0}{m z} \sin \theta_i. \quad (2.49)$$

where d is the minimum admissible separation between two acquisitions, r_0 the slant range, z the vertical extent of the structure under observation, θ_i the incidence angle, and m a factor that takes the value 1 or 2 depending on whether the acquisition geometry is monostatic or bistatic (see section 2.2). Eq. 2.49 expresses the minimum required distance between the acquisitions, depending on the scene height, to avoid ambiguities. In this sense, it can be regarded as the tomographic extension of the HoA concept initially defined in the context of interferometry.

In conclusion, it has been demonstrated the critical role of the HoA for both InSAR and tomographic SAR. Keeping it close to its nominal value is remarkably important and, as such, provides the underlying motivation for this work. In the next chapter the instrument intended to address this purpose –the formation flying –is introduced.

3 | Formation Flying Fundamentals

This chapter outlines the formation flying background, providing the foundations of the rest of the thesis. Initially, the absolute and local reference systems are defined, introducing the ROE-based parameterisation of the kinematics in section 3.1. A description of the unperturbed dynamics is then provided in section 3.2, paying particular attention to the linear models derived for quasi-circular orbits. Section 3.3 is dedicated to a characterisation of the main orbital disturbances affecting satellite formations in Low Earth Orbit, namely the first geopotential harmonics and the atmospheric drag. Finally, section 3.4 introduces the perturbed form of the state transition matrix that is adopted throughout the entire work.

3.1. Reference Systems Definition

This section provides the definitions of the systems adopted to model satellite absolute and mutual dynamics, considering both the Cartesian and the orbital elements-based representations. It is divided into two parts: in the first one, the absolute reference systems, centred on the Earth, are illustrated; in the second, the local systems employed in the formulation of the relative dynamics are described.

3.1.1. Absolute systems

In the following are defined the absolute systems employed throughout this work. Specifically, the Cartesian systems are presented first, followed by the representation through the Keplerian elements. The classical definition of the reference systems, prior to the 2010 International Earth Rotation and Reference Systems Service (IERS) Convention [87], is employed. This is justified since, considering the purpose of this thesis, those are indistinguishable from their more modern version, as better detailed in the following.

Earth Centred Inertial

The Earth Centred Inertial (ECI) are a family of reference frames with the property of not rotating with respect to the position of the fixed stars. They are usually defined with respect to the plane of the ecliptic or the Earth's equator, with origin at the Earth's centre of gravity. The classically employed realisation of an ECI is the Earth's Mean Equator and Mean Equinox at the reference January 1, 2000, 12:00 Terrestrial Time (J2000 or EME2000). It is defined as follows:

- The x-axis points towards the direction of the vernal equinox, *i.e.*, the intersection of the equatorial and ecliptic planes at the reference epoch.
- The z-axis is directed towards the Celestial North Pole (CNP), given by the direction of the Earth's rotation axis at the reference epoch.
- The y-axis is defined to complete a right-handed system.

The term ECI is often used synonymously with J2000. However, other realisations of an ECI are possible, like *e.g.*, the Geocentric Celestial Reference Frame (GCRF), which is recognised as the current IAU standard [88]. Rather than being aligned with the CNP at a specific epoch, its pointing direction is defined accordingly to the International Celestial Reference System (ICRS). As such, it is fixed with respect to the positions of distant quasars and accounts for the relativistic effects. On a practical level, the difference between J2000 and GCRF becomes meaningful only for high-precision astrometry or geodesy, as the measured deviations are in the order of tens of milliarcseconds ($\sim$ meters at Earth's surface). For the purpose of this work, the difference between the two is negligible, and therefore J2000 is adopted hereafter. It will be equivalently referred to as ECI, and its principal direction will be denoted as: $[X_{\text{ECI}}, Y_{\text{ECI}}, Z_{\text{ECI}}]$.

True-of-Date frame

One of the limitations of the ECI frames/systems is the misrepresentation of the non-constant direction of the vernal axis and the CNP over time. This variation is caused by the Earth's precession (*i.e.* a slow ~ 26000 -year motion) and nutation (shorter-term periodic oscillation), which alter the direction of the rotation axis. To properly take the change of the CNP and vernal axis into account, the true-of-date (ToD) reference frame is introduced. It differs from the J2000 because it is referred to the *current* epoch. Given the time distance t between the reference and the current epoch, the map between the

J2000 and the ToD reference frames can be defined consistently with Vallado [89]:

$$\mathbf{r}_{\text{ToD}} = \mathbf{R}_{\text{ToD} \leftarrow \text{J2000}}(t) \cdot \mathbf{r}_{\text{J2000}}, \quad \mathbf{R}_{\text{ToD} \leftarrow \text{J2000}}(t) = N(t) \cdot P(t), \quad (3.1)$$

where $\mathbf{r}_{\text{J2000}}$ and $\mathbf{r}_{\text{ToD}}$ define the position of a same point in the J2000 and ToD frames, respectively, whereas the two matrices $N(t)$ and $P(t)$ define the rotations due to the nutation and precession motions, respectively [87]. In particular, the precession matrix is [88]:

$$P(t) = R_z(-\zeta_A) \cdot R_y(\theta_A) \cdot R_z(-z_A), \quad (3.2)$$

where $R_j(\phi)$ represents the rotation about the axis j of an angle ϕ , whereas ζ_A, θ_A, z_A are the precession angles defined by the IAU [88]. Regarding the nutation, the matrix $N(t)$ is defined as [88]:

$$N(t) = R_x(-\epsilon) \cdot R_z(-\Delta\psi) \cdot R_x(\epsilon + \Delta\epsilon), \quad (3.3)$$

where ϵ is the mean obliquity of the ecliptic, $\Delta\psi$ and $\Delta\epsilon$ are the nutation angles defined by the IAU [88].

Earth-Centred-Earth-Fixed

In contrast to inertial reference frames, Earth-Centred-Earth-Fixed Frames (ECEF) remain fixed with respect to the Earth's surface during its rotation. The IERS has provided a set of guidelines for the theoretical definition of an ECEF: the International Terrestrial Reference System (ITRS). Practical realisations of the ITRS are defined, namely the International Terrestrial Reference Frames (ITRFs). Throughout this work, the term ECEF will be used as a synonym of ITRF to indicate the reference system with the main directions given below (Vallado [89]).

- The x-axis is aligned with the Greenwich meridian.
- The z-axis points towards the CNP at the current epoch.
- the y-axis completes the right-handed system.

The ECEF can be computed from the ToD by adding the contributions of Earth's rotation and polar motion. The first consists of a rotation of the Greenwich Apparent Sidereal Time (GAST) about the z-axis: $R_z(\text{GAST}(t))$ [89]. The GAST is defined as [89]:

$$\text{GAST} := \text{GMST} + \Delta\psi \cos \epsilon, \quad (3.4)$$

where $\Delta\psi$ and ϵ are the same angles defining the nutation matrix in eq. 3.3. whereas

GMST is the Greenwich Mean Sidereal Time, *i.e.*, the rotation angle relative to the mean equinox of date. The latter is a function of the universal time UT1 defined by the IERS Convention 2010 [87].

Concerning the polar motion, it consists of the z-axis misalignment due to Chandler wobble [90] (~ 14 -month period), annual wobble (due to seasonal mass shifts), and other irregularities. It is defined through the polar motion matrix $W(t)$, which, in turn, depends on the coordinates of the pole in arcseconds: (x_p, y_p) . According to the IAU Resolution B1.8, since x_p and y_p are in the order of 10^{-6} rad, a first-order expansion is usually sufficient to model it [88]:

$$W(t) = \begin{bmatrix} 1 & 0 & -x_p \\ 0 & 1 & -y_p \\ x_p & y_p & 1 \end{bmatrix} + \mathcal{O}(x_p^2, y_p^2). \quad (3.5)$$

Finally, the complete transformations from the ECI to the ECEF reads [87]:

$$\mathbf{r}_{\text{ECEF}} = W(t) \cdot R_z(\text{GAST}(t)) \cdot N(t) \cdot P(t) \cdot \mathbf{r}_{J2000}, \quad (3.6)$$

where $\mathbf{r}_{J2000}$ and $\mathbf{r}_{\text{ECEF}}$ denote the position of a same point in the J2000 and ECEF systems, respectively.

Keplerian elements

In addition to the Cartesian coordinates, another possible system can be formulated from a set of orbital elements. The most commonly used are the Keplerian elements (Vallado [89]):

$$\boldsymbol{\alpha} := [a, e, i, \Omega, \omega, f]^T. \quad (3.7)$$

Respectively, they are: the semi-major axis (a), the eccentricity (e), the right ascension of the ascending node (Ω), the argument of perigee (ω), and the true anomaly (f). Another useful parameter often referred to in the rest of this work is the mean motion [89]:

$$n := \sqrt{\frac{\mu_{\oplus}}{a^3}}, \quad (3.8)$$

where $\mu_{\oplus} \approx 3.986 \cdot 10^{14} \text{ m}^3 \text{ s}^{-2}$ is the standard gravitational parameter of the Earth [89]. Under the hypothesis of unperturbed, two-body, Newtonian orbits, they are all constant except for f . However, the presence of orbital perturbations makes all elements time-

dependent (Brouwer and Clemence [91]). For this reason, two different terms are used: *osculating elements*, which indicate the actual trajectory, and *mean* or *secular elements*, referred to a mean orbit. Specifically, the mean elements are obtained by averaging the osculating elements over short- and long-periodic oscillations caused by perturbations (typically conservative). The resulting averaged, or secular, elements are therefore free from periodic variations.

The spacecraft's position along the orbit can be parametrised using different angular variables. Besides the true anomaly, one can use the mean anomaly M , defined as [89]:

$$M = n(t - t_0), \quad (3.9)$$

where n is the mean motion, t is the current time, and t_0 is the time of passage at pericentre. Another common choice is the mean argument of latitude, defined as:

$$u = \omega + M. \quad (3.10)$$

While the osculating elements provide the current position of a satellite, the mean ones are used to model its long-term behaviour. Typically, Guidance, Navigation, and Control (GNC) systems must handle information in both sets; thus, the availability of an efficient reciprocal mapping is crucial. As an example, Gaia et al. [25] have addressed the problem with an analytical solution based on Lie transform canonical perturbation theory. Specifically, their approach uses three consecutive transformations: 1) parallax elimination, 2) Delaunay normalisation, and 3) perigee elimination, yielding secular and periodic terms up to third and second order in J_2 , respectively (see section 3.3):

$$\tilde{\alpha} = \alpha_{osc} - f_{parallax}^{(1)}(\alpha_{osc}) + \frac{1}{2}g_{parallax}^{(2)}(\alpha_{osc}) \quad (3.11)$$

$$\hat{\alpha} = \tilde{\alpha} - f_{delaunay}^{(1)}(\tilde{\alpha}) + \frac{1}{2}g_{delaunay}^{(2)}(\tilde{\alpha}) \quad (3.12)$$

$$\bar{\alpha} = \hat{\alpha} - f_{perigee}^{(1)}(\hat{\alpha}) + \frac{1}{2}g_{perigee}^{(2)}(\hat{\alpha}) \quad (3.13)$$

where $\bar{\alpha}$ are the mean elements, α_{osc} the osculating ones, $f^{(1)}$ and $f^{(2)}$ are the first and second order corrections whose expressions are explicitly given in the supplemental appendix of [25], the subscripts refer to the computed correction (parallax, delaunay, perigee), $g^{(1)}$ and $g^{(2)}$ are the inverse corrections of first and second order respectively, also reported in the supplemental appendix of [25]. Additionally, in the proposed model, the effect of higher-degree harmonics on the semi-major axis is added with Kaula's first-order theory.

The corresponding complete expressions are reported in [25].

3.1.2. Local systems

In the following are introduced the local systems. Consistent with what has been done for the absolute ones in the first part of this section, the local systems based on Cartesian coordinates are presented first, followed by the representation based on the Relative Orbital Elements.

Hill frame:

Considering a formation of N satellites, the one with respect to which the positions of the others are defined is the *chief*, whereas the others are the *deputies*. The Hill frame has its origin at the chief's centre of mass and its principal axes defined as follows (Vallado [89]):

- The x-axis points from the centre of the Earth to the chief satellite's position.
- The z-axis in the direction of the angular momentum of the chief's orbit.
- The y-axis completes the right-handed system.

From the definition, the y-axis is oriented in the direction of the velocity but not necessarily parallel to it, unless the chief's orbit is perfectly circular. Denoting with $\mathbf{r}_{\text{ECI}}$ the position of the chief in the ECI frame, and with $\dot{\mathbf{r}}_{\text{ECI}}$ its velocity in the same frame, the coordinates in the Hill frame are expressed as follows [89]:

$$\begin{cases} \mathbf{x}_{\text{Hill}} = \frac{\mathbf{r}_{\text{ECI}}}{\|\mathbf{r}_{\text{ECI}}\|} \\ \mathbf{z}_{\text{Hill}} = \frac{\mathbf{r}_{\text{ECI}} \times \dot{\mathbf{r}}_{\text{ECI}}}{\|\mathbf{r}_{\text{ECI}} \times \dot{\mathbf{r}}_{\text{ECI}}\|} \\ \mathbf{y}_{\text{Hill}} = \mathbf{z}_{\text{Hill}} \times \mathbf{x}_{\text{Hill}} \end{cases} \quad (3.14)$$

Depending on the field of application, it is also referred to by other names, such as: Radial - Tangential - Normal (RTN), Radial - Along-track - Cross-track (RSW), Radial - In-track - Cross-track (RIC), Local-Vertical-Local-Horizontal (LVLH) or Gaussian frame (Vallado [89], Curtis [92], Kaplan [93], Schaub and Junkins [94]).

The nomenclature RTN will be employed synonymously with the Hill frame hereafter. Its main directions are denoted as:

$$[\hat{\mathbf{R}}, \hat{\mathbf{T}}, \hat{\mathbf{N}}] = [\mathbf{x}, \mathbf{y}, \mathbf{z}]_{\text{Hill}} = [\mathbf{x}, \mathbf{y}, \mathbf{z}]_{\text{RTN}} \quad (3.15)$$

Figure 3.1 shows a sketch of the Hill/RTN frame with respect to ECI [89]. The position of the chief is defined by its mean argument of latitude u_c , in purple. The RTN coordinates are depicted in red, the RTN ones in black. Useful quantities for the definition of the orbit of the chief are also reported in blue. Those are the eccentricity vector $\mathbf{e}$ and the angular momentum $\mathbf{h}$, computed from the semi-major axis of the chief a_c and its eccentricity e_c as:

$$\mathbf{h} = \sqrt{\mu_{\oplus} a_c (1 - e_c^2)}. \quad (3.16)$$

Moreover, it is indicated the location of the ascending node *i.e.*, the intersection between the equatorial and orbital planes when the spacecrafts pass from the southern to the northern hemisphere.

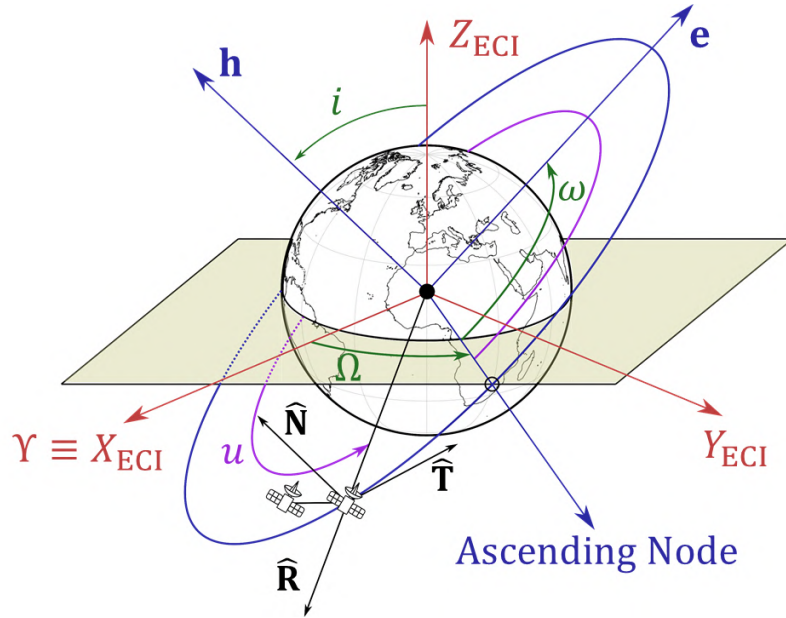


Figure 3.1: Graphical representation of absolute/local frames. ECI (red) - RTN (black)

Zero-Doppler frame:

From the remote-sensing standpoint, the position relative to the surface of the rotating Earth is an important aspect, and the Hill frame does not account for it. For this reason, a different reference frame is considered: the Zero-Doppler reference frame (ZD) introduced by Rodrigues Silva et al. [60]. It is defined so as to have one of the principal axes aligned with the satellite's velocity expressed in the ECEF frame. In this way, the other two identify the zero-Doppler plane (see Chapter 2). The ZD is hence formally defined as

follows [60]:

$$\begin{cases} \mathbf{y}_{\text{ZD}} = \frac{\mathbf{v}_{\text{ECEF}}}{\|\mathbf{v}_{\text{ECEF}}\|} \\ \mathbf{z}_{\text{ZD}} = \frac{\mathbf{r}_{\text{ECEF}} \times \mathbf{v}_{\text{ECEF}}}{\|\mathbf{r}_{\text{ECEF}} \times \mathbf{v}_{\text{ECEF}}\|} \\ \mathbf{x}_{\text{ZD}} = \mathbf{y}_{\text{ZD}} \times \mathbf{z}_{\text{ZD}} \end{cases} \quad (3.17)$$

where $\mathbf{r}$ is the chief's position with respect to the Earth's centre of mass, whereas $\mathbf{v}$ is the chief's velocity; the subscripts $(\cdot)_{\text{ECI}}$ and $(\cdot)_{\text{ECEF}}$ indicate which reference frame the quantities are defined in. Specifically, the velocity $\mathbf{v}_{\text{ECEF}}$ can be computed from the transport theorem applied to the ECEF-ECI systems [95]:

$$\mathbf{v}_{\text{ECEF}} = \mathbf{v}_{\text{ECI}} - \boldsymbol{\omega}_{\oplus} \times \mathbf{r}, \quad (3.18)$$

where $\boldsymbol{\omega}_{\oplus}$ is the Earth's angular velocity. The authors of [60] derive the complete transformation from the ZD to the Hill frame as a sequence of two rotations. In the case of near circular orbits, this could be approximated by a rotation about the x-axis of the ZD of the angle $\beta_{\text{ZD} \rightarrow \text{Hill}}$ defined as [60]:

$$\beta_{\text{ZD} \rightarrow \text{Hill}} := \arctan \left(\frac{\boldsymbol{\omega}_{\oplus} \sin(i) \cos(u)}{n - \boldsymbol{\omega}_{\oplus} \cos(i)} \right), \quad (3.19)$$

where n is recalled to be the mean motion defined in eq. 3.8.

In Figure 3.2 is reported an example of how the relative position between satellites in closed periodic orbits may change when seen from the ZD instead of the Hill frame. The specific type of relative orbit depicted (the xz-Projected Circular Orbit (PCO)) is described in detail in section 3.2.

Relative Orbital Elements:

While the Keplerian elements provide a simple interpretation of motion in an absolute reference frame, the Relative Orbital Elements (ROEs) address the same task in describing relative dynamics. Their systematic parametrisation and geometric interpretation for LEO formation design is due to D'Amico [27]. In their dimensionless form, *i.e.* scaled by

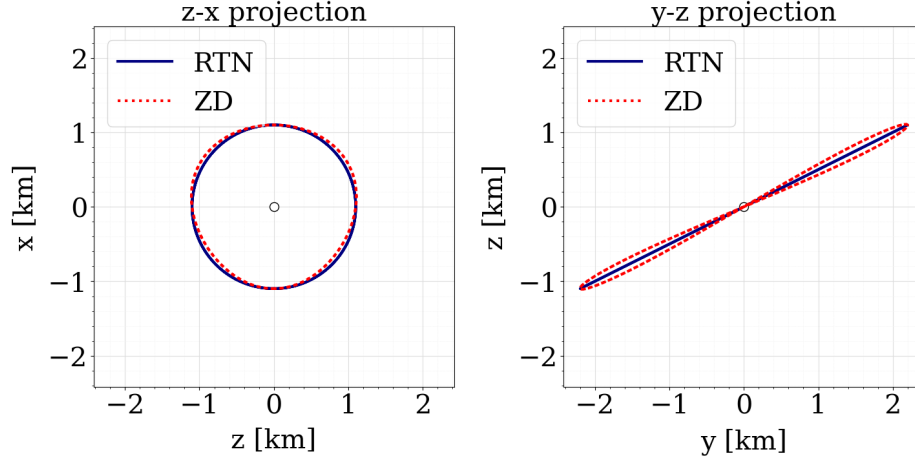


Figure 3.2: Projections of a xz-PCO (see Table 3.1) viewed from Hill frame (blue), and ZD frame (red) (see section 3.1.2).

the chief's semi-major axis a_c , they are defined as follows:

$$\delta\alpha = \begin{bmatrix} \delta a \\ \delta\lambda \\ \delta e_x \\ \delta e_y \\ \delta i_x \\ \delta i_y \end{bmatrix} = \begin{bmatrix} (a_d - a_c)/a_c \\ (u_d - u_c) + (\Omega_d - \Omega_c) \cos i_c \\ e_d \cos \omega_d - e_c \cos \omega_c \\ e_d \sin \omega_d - e_c \sin \omega_c \\ i_d - i_c \\ (\Omega_d - \Omega_c) \sin i_c \end{bmatrix} \quad (3.20)$$

where the subscripts c and d indicate the chief and the deputy satellite, respectively. Notably, as with Keplerian elements, ROEs can also be defined as either osculating or mean.

3.2. Unperturbed Relative Equations of Motion

This section presents the equations of relative motion in both Cartesian coordinates and with ROEs, with particular focus on the specific cases in which they can be solved in closed form.

3.2.1. Cartesian representation

For a Cartesian two-body problem, the inertial equations of motion yield [89]:

$$\ddot{\mathbf{r}}_{c,d} = -\frac{\mu_{\oplus}}{r_{c,d}^3} \mathbf{r}_{c,d}. \quad (3.21)$$

where $\mathbf{r}_{c,d}$ is the relative position of the deputy with respect to the chief and $r_{c,d}$ its norm. Given the position of the deputy relative to the chief $\boldsymbol{\rho} := \mathbf{r}_d - \mathbf{r}_c$, the relative cinematic is defined by the relation [95]:

$$\ddot{\boldsymbol{\rho}}_{\text{ECI}} = \ddot{\boldsymbol{\rho}}_{\text{Hill}} + 2\boldsymbol{\omega}_{\text{rel}} \times \dot{\boldsymbol{\rho}}_{\text{Hill}} + \dot{\boldsymbol{\omega}}_{\text{rel}} \times \boldsymbol{\rho}_{\text{Hill}} + \boldsymbol{\omega}_{\text{rel}} \times (\boldsymbol{\omega}_{\text{rel}} \times \boldsymbol{\rho}). \quad (3.22)$$

If the chief's orbit is purely Keplerian, the angular velocity of the Hill frame relative to ECI is simply given by the true anomaly of the chief: $\boldsymbol{\omega}_{\text{rel}} = [0, 0, \dot{f}_c]_{\text{Hill}}$. From eq. 3.21 and 3.22, the complete set of equations of motion giving the deputy's position in Hill frame therefore reads (Alfriend et al. [96]):

$$\begin{cases} \ddot{x} - 2\dot{f}_c\dot{y} - \dot{f}_c^2 x = -\frac{\mu_{\oplus}(r_c + x)}{((r_c + x)^2 + y^2 + z^2)^{3/2}} + \frac{\mu_{\oplus}}{r_c^2} \\ \ddot{y} + 2\dot{f}_c\dot{x} + \dot{f}_c^2 y = -\frac{\mu_{\oplus}y}{((r_c + x)^2 + y^2 + z^2)^{3/2}} \\ \ddot{z} = -\frac{\mu_{\oplus}z}{((r_c + x)^2 + y^2 + z^2)^{3/2}} \\ \ddot{r}_c = r_c\dot{f}_c^2 - \frac{\mu_{\oplus}}{r_c^2} \\ \ddot{f}_c = -\frac{2\dot{r}_c\dot{f}_c}{r_c} \end{cases} \quad (3.23)$$

The system of eqs 3.23 consists therefore of 10 non-linear equations in the variables:

$$\{x, \dot{x}, y, \dot{y}, z, \dot{z}, f_c, \dot{f}_c, r_c, \dot{r}_c\}.$$

Assuming that the chief's orbit is circular, taking a first-order approximation of the relative position for a circular reference orbit yields the well-known Hill-Clohessy-Wiltshire equations (Clohessy and Wiltshire [17]) are obtained as follows:

$$\begin{cases} \ddot{x} = 2n\dot{y} + 3n^2x \\ \ddot{y} = -2n\dot{x} \\ \ddot{z} = -n^2z \end{cases} \quad (3.24)$$

They were derived in 1960 by Clohessy and Wiltshire [17], building upon the findings of Hill regarding the motion of the Moon relative to the Earth [18]. The general solution of the HCW is given by a combination of in-plane harmonics in y coupled with secular and constant terms in x , and an out-of-plane simple harmonic oscillation in z [96]:

$$\begin{cases} x(t) = A + B \cos(nt) + C \sin(nt) \\ y(t) = D - 3/2 A nt - 2B \sin(nt) + 2C \cos(nt) \\ z(t) = E \cos(nt) + F \sin(nt) \end{cases} \quad (3.25)$$

where the constant terms A, B, C, D, E and F are determined by the initial conditions [96]. The solution can be equivalently represented in its state-space form. Thus, given the state vector $\mathbf{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$, it yields:

$$\mathbf{x}(t) = \Phi_{\text{HCW}}(t, t_0) \cdot \mathbf{x}(t_0), \quad (3.26)$$

where the State Transition Matrix (STM) Φ_{HCW} is given by [96]:

$$\Phi_{\text{HCW}}(t, t_0) = \begin{bmatrix} 4 - 3c_u & 0 & 0 & \frac{s_u}{n} & \frac{2}{n}(1 - c_u) & 0 \\ 6(s_u - u) & 1 & 0 & -\frac{2}{n}(1 - c_u) & \frac{1}{n}(4s_u - 3nu) & 0 \\ 0 & 0 & c_u & 0 & 0 & \frac{s_u}{n} \\ 3ns_u & 0 & 0 & c_u & 2s_u & 0 \\ 6n(c_u - 1) & 0 & 0 & -2s_u & 4c_u - 3 & 0 \\ 0 & 0 & -ns_u & 0 & 0 & c_u \end{bmatrix} \quad (3.27)$$

indicating with $c_u = \cos u$, $s_u = \sin u$. Note that the mean argument of latitude u embeds the time information. It reads: $u = nt$, since the argument of the pericentre is by convention set equal to zero for a circular orbit.

3.2.2. ROE-based representation

As shown by D'Amico [27], the parameterisation through the ROE is particularly convenient because they match the integration constants in eq. 3.25 at their initial state, indicated with the subscript $(\cdot)_0$:

$$[A, B, C, D, E, F] = [+ \delta a_0, - \delta e_{x0}, - \delta e_{y0}, + \delta \lambda_0, - \delta i_{y0}, + \delta i_{x0}].$$

Leveraging this result, the solution of the HCW can be formulated in terms of the ROEs as follows:

$$\begin{cases} x/a_c = +\delta a_0 & -c_u \delta e_{x0} & -s_u \delta e_{y0} \\ y/a_c = -\frac{3}{2} \delta a_0 (u - u_0) & +\delta \lambda_0 + 2s_u \delta e_{x0} & -2c_u \delta e_{y0} \\ z/a_c = & & +s_u \delta i_{x0} & -c_u \delta i_{y0} \\ \dot{x}/v_c = & +s_u \delta e_{x0} & -c_u \delta e_{y0} \\ \dot{y}/v_c = -\frac{3}{2} \delta a_0 & +2c_u \delta e_{x0} & +2s_u \delta e_{y0} \\ \dot{z}/v_c = & & +c_u \delta i_{x0} & +s_u \delta i_{y0} \end{cases} \quad (3.28)$$

where $v_c = n a_c$ and $\delta \lambda = -\frac{3}{2} \delta a_0 (u - u_0) + \delta \lambda_0$. Hence, the system of eqs. 3.28 provides an instantaneous mapping from the state expressed in RTN to a ROE-based formulation. This transformation can be condensed in the form: $\mathbf{x}(t) = a_c T(t) \cdot \delta \boldsymbol{\alpha}(t)$. As also reported by Gaias and Lovera [28], the matrix T is always invertible, having a constant determinant equal to $n^3/2$. Consequently, the conversion is a *Lyapunov transform*, *i.e.* a time-varying linear change of coordinates relating two equivalent descriptions of the same dynamics. The inverse transformation $a_c \delta \boldsymbol{\alpha}(t) = T^{-1}(t) \cdot \mathbf{x}(t)$ is therefore equally always possible; recalling that $u = nt$ it yields:

$$\mathbf{T}^{-1}(t) = \begin{bmatrix} 4 & 0 & 0 & 0 & \frac{2}{n} & 0 \\ 0 & 1 & 0 & -\frac{2}{n} & 0 & 0 \\ 3 \cos u & 0 & 0 & \frac{\sin u}{n} & \frac{2 \cos u}{n} & 0 \\ 3 \sin u & 0 & 0 & -\frac{\cos u}{n} & \frac{2 \sin u}{n} & 0 \\ 0 & 0 & \sin u & 0 & 0 & \frac{\cos u}{n} \\ 0 & 0 & -\cos u & 0 & 0 & \frac{\sin u}{n} \end{bmatrix} \quad (3.29)$$

3.2.3. Harmonic solutions to the relative motion

The only solution of the HCW in the system of eqs. 6.33 relevant for formation-flying requires all satellites to have orbital periods in a 1:1 ratio. Otherwise, the relative motion would be unbounded in the local reference frame. This condition translates into the necessary equality of the semi-major axes of the satellites' orbits, *i.e.* $\delta a = 0$, which is equivalent to cancelling the A -dependent secular term in the second equation of 3.25. As a consequence of this simplification, the solution becomes harmonic with a possible along-

track shift. Specifically, the deputy's trajectory would be centred with respect to the chief in the x-z plane, while there would be an offset $\delta\lambda_0$ in the y direction. At this point, a convenient way to formalise the relative dynamics is to introduce the amplitude/phase form of the relative eccentric and inclination vectors [29]:

$$\Delta\mathbf{e} = \delta\mathbf{e} = \begin{bmatrix} \delta e_x \\ \delta e_y \end{bmatrix} = \delta e \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}; \quad \Delta\mathbf{i} = \delta\mathbf{i} = \begin{bmatrix} \delta i_x \\ \delta i_y \end{bmatrix} = \delta i \begin{bmatrix} \cos \vartheta \\ \sin \vartheta \end{bmatrix} \quad (3.30)$$

From this, the harmonic solutions of the HCW can be formulated within the ROE framework as:

$$\begin{cases} x(u) = -a_c \delta e \cos(u - \varphi) \\ y(u) = a_c \delta \lambda_0 + 2 a_c \delta e \sin(u - \varphi) \\ z(u) = a_c \delta i \sin(u - \vartheta) \end{cases} \quad (3.31)$$

Depending on how the ROEs are initialised, particularly remarkable harmonic solutions arise. The mathematical conditions for obtaining them are listed in Table 3.1.

- **Helix:** Initially introduced by Moreira et al. [51] for the DLR TanDEM-X mission, the Helix concept has the (anti-)parallel orientation of the relative eccentricity and inclination vectors as its peculiar characteristic. This is achieved by setting a phase difference of a multiple of π : $\vartheta = \varphi + k\pi, k \in \mathbb{Z}$ together with a non-zero magnitude of the two vectors, *i.e.* $\delta e, \delta i > 0$. This configuration offers the great advantage of an intrinsic passive safety. In fact, the separation between the satellites in the x-z plane is always guaranteed, so any along-track drift would not pose a risk of collision (D'Errico [16]). This is evident from the equations 3.31, from which it yields:

$$\begin{cases} x(u) = -a_c \delta e \cos(u - \varphi) \\ z(u) = \pm a_c \delta i \sin(u - \varphi) \end{cases} \quad (3.32)$$

and clearly no u exists for which both terms nullify. This is the essence of the e/i -vector separation method introduced by D'Amico [29], of which the Helix configuration represents the practical implementation. As such, it was successfully implemented in the TanDEM-X - for which it was originally conceived - with the addition of the condition $\varphi = \pi/2$. The merit of this additional requirement is that it increases the stability of the formation, which thus becomes less sensitive to the perturbation induced by J_2 (see section 3.4.2).

- **Cartwheel:** [16] This configuration requires that the relative distances along the z direction are zero for all satellites. This is obtained by initialising the magnitude

of the relative inclination and the along-track shift λ_0 to zero and imposing the same magnitude of the relative eccentricity for all deputies. Consequently, all the satellites belong to the same orbital plane, while collisions are avoided by different arguments of the perigee, leading to different φ .

- **Cross-track pendulum:** [16] This configuration is, in a sense, the dual case of the cartwheel, resulting in a pendulum-like oscillation in the z direction. The separation along z is introduced by collocating the satellites on different orbital planes, achieved with either a difference in the inclinations or the right ascensions of the ascending node. On the other hand, the magnitudes of the relative eccentricities are all null. Note that this requires setting $\delta\lambda_0 \neq 0$, otherwise z would be the only non-degenerate component, leading to a collision as soon as $u = \vartheta (+\pi)$.
- **General Circular Orbit (GCO):** The GCO can also be formalised starting from eq. 3.31. It consists of a circle in the 3D space centred on the chief. Its systematic formulation is due to the works of Sengupta [97] and Sengupta and Vadali [98]. If more than a deputy is present, all relative orbits must have equal radius; thus, the relative eccentricity and inclination magnitudes must be equal for all deputies. While D’Errico[16] and Scala [34] derive the corresponding ROE initialisation directly from the HCW, it can be readily obtained by imposing the coordinates from eq. 3.31 to satisfy $x^2 + y^2 + z^2 = R^2 = \text{constant}$. The complete proof is given in Derivation 3.2.1.
- **Projected Circular Orbit (PCO):** The requirement to obtain a PCO is that the projection of the orbit onto one of the principal planes is a circle. Clearly, the $x - y$ plane is excluded, as the result would always be an ellipse whose axes differ by a factor of 2. The initialisation can be obtained similarly to the GCO. However, accordingly to Sengupta [97], there is no strict requirement about the formation being centred around the chief: this gets reflected on the λ_0 , which is no longer necessarily 0. Notably, the x - z PCO can also be regarded as a special case of the Helix with $\delta e = \delta i$, with the application of the eccentricity/inclination vector separation method (see D’Amico and Montenbruck [29]). An example of an xz -PCO is shown in Figure 3.2 as seen from the Hill/RTN frame (perfectly circular on the x - z plane) and the ZD frame (see section 3.1.2).
- **Leader-Follower:** [16]. In the LF configuration, the oscillatory terms are eliminated and the satellites are separated only by an along-track offset. This is fixed by the choice of λ_0 , while the relative eccentricity and inclinations are null.

	δa_0	$\delta \lambda_0$	δe_0	φ_0	δi_0	ϑ_0
Helix	0	–	> 0	–	> 0	$\varphi (+\pi)$
Cartwheel	0	–	$\delta e_k = \delta e$	$\varphi_k \neq \varphi_j$	0	–
XT Pendulum	0	$\neq 0$	0	–	> 0	–
GCO	0	0	$\delta e_k = \delta e$	$\varphi_k \neq \varphi_j$	$\sqrt{3}\delta e$	$\varphi \pm \pi/2$
y-z PCO	0	–	> 0	$\varphi_k \neq \varphi_j$	$2\delta e_0$	$\varphi \pm \pi/2$
x-z PCO	0	–	> 0	$\varphi_k \neq \varphi_j$	δe_0	$\varphi (+\pi)$
LF	0	$\neq 0$	0	–	0	–

Table 3.1: ROE initialisation for notable harmonic solutions, $(\cdot)_k$ indicates the k -th deputy.

Derivation 3.2.1: ROE Initialisation for a GCO

Starting from eq. 3.31, the condition $\delta \lambda_0 = 0$ is imposed because the chief must be in the centre. By calling $\beta = u - \varphi$ and $\Delta\beta = \vartheta - \varphi$, it yields:

$$\begin{aligned}
 x^2 + y^2 &= \delta e^2 (\cos^2 \beta + 4 \sin^2 \beta) = \delta e^2 \frac{5 - 3 \cos(2\beta)}{2} \\
 z^2 &= \delta i^2 \sin^2(\beta - \Delta\beta) = \frac{\delta i^2}{2} - \frac{\delta i^2}{2} (\cos(2\beta) \cos(2\Delta\beta) + \sin(2\beta) \sin(2\Delta\beta)).
 \end{aligned} \tag{3.33}$$

Since $R^2 = x^2 + y^2 + z^2$ must be constant - so, independent from β - both coefficients of $\cos(2\beta)$ and $\sin(2\beta)$ must vanish. The coefficient of $\sin(2\beta)$ is $\sin(2\Delta\beta)$, so $\Delta\beta = k\frac{\pi}{2}$. The coefficient of $\cos(\beta)$ is $-\frac{3}{2}\delta e^2 - \frac{\delta i^2}{2} \cos(2\Delta\beta)$, from which: $\delta i^2 \cos(2\Delta\beta) = -3\delta e^2$. Now combining the two conditions: if $\Delta\beta = \pm\pi/2$, then $\delta i^2 = 3\delta e^2$; whereas if $\Delta\beta = 0 (+\pi)$ then $\delta i^2 = -3\delta e^2$, which is not acceptable. In conclusion, the ROEs to initialise a GCO must satisfy:

$$\delta i = \sqrt{3}\delta e, \quad \vartheta - \varphi = \pm\pi. \tag{3.34}$$

3.3. Orbital Perturbations

The equations of motion presented in section ref 2sec:perturbedMotion are obtained in an ideal scenario, in which the only force acting on the satellites is the gravitational pull of the central body, assumed to be a point. This section presents a brief discussion of the main perturbations that can influence the motion of satellites. The main effect of such perturbations is to introduce secular or periodic variations in the orbital elements,

making the difference between osculating and mean elements manifest.

3.3.1. Asphericity of the Earth

The Earth is not perfectly spherical. Its mass distribution is not symmetrical about its centre, resulting in an anisotropy of its gravitational field. The consequent acceleration is canonically calculated as the gradient of the gravitational potential, defined with the following series of spherical harmonics [89]:

$$U = -\frac{\mu_{\oplus}}{r} \left(1 + \sum_{l=2}^{\infty} \left(\frac{R_{\oplus}}{r} \right)^l J_l P_l^0(\sin \tilde{\lambda}) + \sum_{l=2}^{\infty} \sum_{m=1}^l \left(\frac{R_{\oplus}}{r} \right)^l P_l^m(\sin \tilde{\lambda}) (C_l^m \cos(m\phi) + S_l^m \sin(m\phi)) \right) \quad (3.35)$$

In the previous equation, P_l^m are the associated Lagrange polynomials of degree l and order m , $R_{\oplus}$ is the Earth's radius, $\mu_{\oplus}$ its standard gravitational parameter, $\tilde{\lambda}$ is the geocentric co-latitude ($\tilde{\lambda} = \pi/2 - \lambda$, for latitude λ), ϕ is the longitude, the coefficients J_l , C_l^m , S_l^m tune the spherical harmonics. By writing it in this form, the different contributions to the geopotential are highlighted: the term $-\frac{\mu_{\oplus}}{r}$ accounts for the spherical symmetry, the series with J_l contains the zonal terms with rotational north-south symmetry; in the second series, the sectorial terms ($l = m$) give the variation across m meridian planes, the tesseral terms ($m \neq l$) combine the last two in a chequered pattern. Figure 3.3 shows a chart with terms of the geopotential up to the fourth order, omitting the trivial, spherical one. The values of the coefficients in eq. 3.35 come necessarily from the empirical knowledge of the Earth's gravitational field, which has seen a significant contribution from missions such as NASA's and DRL's GRACE [99], and ESA's GOCE [100]. These have

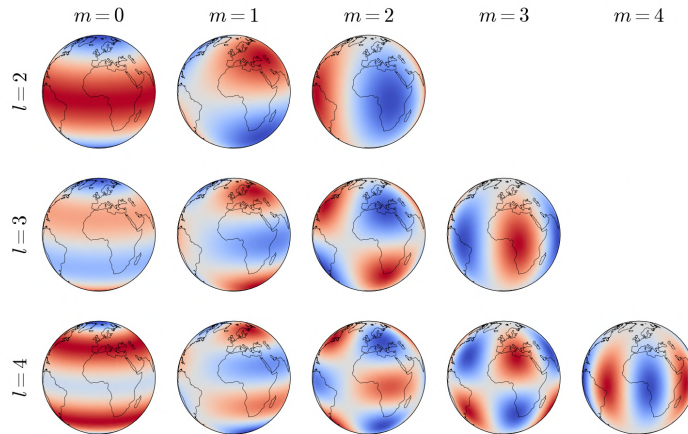


Figure 3.3: Tabulation illustrating the terms of the geopotential for $l \in \{2, 3, 4\}$.

produced the GGM (GRACE-Derived Models) series [101], which combines data from both missions together with surface ones. To date, the most accurate model available is the EGM2008, reaching degree and order up to 2159 [102]. Nevertheless, the most relevant term is by far that due to the ellipsoidal shape of the Earth, associated with the coefficient J_2 . It introduces secular trends and long- and short-term oscillations in the relative motion. Specifically, the secular effect consists of a linear variation of the right ascension of the ascending node, argument of perigee, and mean anomaly. Their rates have been computed by Brouwer [103] and yield:

$$\begin{cases} \dot{\Omega} = -\frac{3}{2} \frac{J_2 R_\oplus^2 n_\oplus}{a^2 \eta^4} \cos i \\ \dot{\omega} = \frac{3}{4} \frac{J_2 R_\oplus^2 n_\oplus}{a^2 \eta^4} (5 \cos^2 i - 1) \\ \dot{M} = \frac{3}{4} \frac{J_2 R_\oplus^2 n_\oplus}{a^2 \eta^4} (3 \cos^2 i - 1) \end{cases} \quad (3.36)$$

where $\eta = \sqrt{1 - e^2}$ and, for $e \rightarrow 0, \eta \rightarrow 1$. Eqs. 3.36 are particularly relevant for the specific context of remote sensing. Often, among the scientific requirements, there is consistency in the illumination of the acquired scenes, which entails the use of a Sun-synchronous orbit (SSO). This involves the satellites flying over a certain point at the same solar time at each pass. Such an orbit can be obtained by leveraging the secular variation of Ω due to J_2 ; in particular, by requiring Ω to complete a full rotation of 2π in the course of a solar year. Consequently, from the first equation in 3.36 one can derive the necessary inclination angle from the following equation:

$$\cos i = -\frac{2}{3} \frac{\dot{\Omega}_\odot a^2}{J_2 n R_\oplus^2}, \quad (3.37)$$

where $\dot{\Omega}_\odot \approx \frac{2\pi}{365.24 \text{ days}}$ is the angular velocity of the Sun seen from the Earth, whereas $R_\oplus$ is the radius of the Earth. Figure 3.4 shows the value of i for a circular orbit depending on the altitude: as it is slightly higher than 90° , SSOs are quasi-polar and retrograde.

3.3.2. Atmospheric drag

Although rarefied, Earth's atmosphere is still significantly present at LEO altitude. The friction generated by satellites' interaction with it produces a force that opposes their motion: the atmospheric drag. This can be computed with the formula [89]:

$$\mathbf{a}_{\text{drag,ECI}} = -\frac{1}{2} \frac{C_d S}{m} \rho v_{\text{rel}}^2 \hat{\mathbf{v}}_{\text{rel}}, \quad (3.38)$$

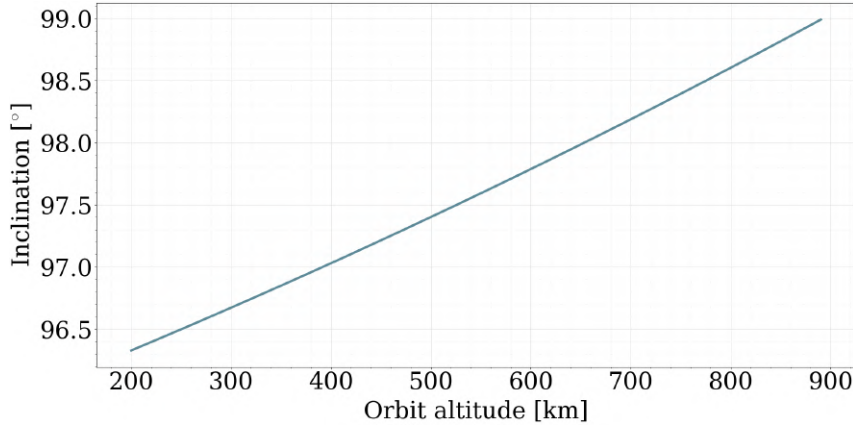


Figure 3.4: Inclination angle required for a circular SSO at given altitude

where C_d is the drag coefficient, S the cross-sectional area of the satellite, m its mass, ρ the air density, and $\mathbf{v}_{rel}$ the velocity relative to the atmosphere measured in ECI. The quantity $\frac{C_d S}{m}$ is also known as *ballistic coefficient* C_B . Regarding the velocity, it is typically assumed that the atmosphere rotates in solidarity with the Earth, so its velocity $\mathbf{v}_{rel}$ can be computed thanks to the transport theorem [95]:

$$\mathbf{v}_{rel} = \dot{\mathbf{r}}_{\text{ECI}} - \boldsymbol{\omega}_{\oplus} \times \mathbf{r}. \quad (3.39)$$

The critical term in eq. 3.38 is the air density, as a large variety of models exist in the literature, some more accurate than others. A common choice in spacecraft modelling is the Jacchia model [104, 105], valid for altitudes between 90 and 2500 km. A more modern option is instead the NRLMSISE-00 (Naval Research Laboratory - Mass Spectrometer and Incoherent Scatter radar) empirical model developed by Picone et al. [106], which is considered to be the current standard for space research activities. However, as stated by Kumar et al. [107], their reliability is not always easy to evaluate and compare. In modelling the effect of drag on relative motion, the semi-empirical model proposed by Gaias et al. [26], flight-proven by the AVANTI mission (see section 3.4.2), is employed.

The two main effects of atmospheric drag are a lowering of altitude due to a secular decay of the semi-major axis and a decrease in eccentricity, which causes the circularisation of the orbit. From eqs. 3.38 and 3.6, it is possible to transform the acceleration into the RTN frame. At this point, one could use Gauss' Variational Equations (GVE) to calculate the effect of drag on the Keplerian elements (Battin [108]). However, since the main interest of this work is on circular orbits, this would not be ideal as the GVEs become singular in this case. For this reason, the approach of Khalil Ben Larbi and Stoll [109] is followed, deriving the GVEs in terms of the non-singular elements and computing their limit for

$e \rightarrow 0$. This results in the following set of equations:

$$\begin{cases} na \dot{a} = & + 2 a a_T \\ na \dot{u} = n^2 a & - 2 a_R & - \frac{\sin(u)}{\tan(i)} a_N \\ na \dot{e}_x = & + \sin(u) a_R + 2 \cos(u) a_T \\ na \dot{e}_y = & - \cos(u) a_R + 2 \sin(u) a_T \\ na \dot{i} = & + \cos(u) a_N \\ na \dot{\Omega} = & + \frac{\sin(u)}{\sin(i)} a_N \end{cases} \quad (3.40)$$

where a_R, a_T, a_N denote the components of the acceleration in the radial, transversal, and normal directions of the RTN frame. If the orbit is perfectly circular, one can consider the relative velocity vector given by eq. 3.39 to lie in the TN plane, with no components in the $\hat{\mathbf{R}}$ direction. Moreover, assuming that the properties of the atmosphere are constant with respect to the position, *i.e.* with respect to u , the only term in the system of eqs. 3.40 that presents a secular behaviour is a , which decreases linearly.

3.3.3. Other perturbations

The effects characterised above are by far the most influential on satellite formations in LEO. As such, they are also the only ones considered in this thesis. However, there are other orbital disturbances that should be included in any more detailed analyses. They are therefore briefly described in this section for completeness.

Solar Radiation Pressure: The SRP is determined by the impact of solar radiation on the spacecraft. Therefore, its effect depends on the position relative to the Sun and becomes evident for long-duration and interplanetary missions. It causes an acceleration which reads (Vallado [89]):

$$\mathbf{a}_{\text{SRP}} = \frac{\text{SC}}{c} \frac{A_{\perp\odot}}{m} \varrho \hat{\mathbf{r}}_{\odot}, \quad (3.41)$$

where SC is the solar constant, *i.e.* the average solar irradiance per unit area at 1AU (equal to $\sim 1367 \text{ W/m}^2$), c the speed of light, ϱ is the spacecraft's reflection coefficient, m its mass, $\hat{\mathbf{r}}_{\odot}$ is the one-unit vector directed from the Sun to the satellite, $A_{\perp\odot}$ the extension of the illuminated surface projected onto the plane perpendicular to $\hat{\mathbf{r}}_{\odot}$.

Third-Body Perturbations: This category includes gravitational disturbances generated by bodies other than the Earth, mainly the Moon and the Sun. The resulting acceleration is given by the sum of the contributions from all the relevant bodies (Val-

lado [89]):

$$\mathbf{a}_{3\text{body}} = \sum_{b=1}^{N^{\circ}\text{bodies}} \mu_b \left(\frac{\mathbf{r}_b - \mathbf{r}}{\|\mathbf{r}_b - \mathbf{r}\|^3} - \frac{\mathbf{r}_b}{\|\mathbf{r}_b\|^3} \right), \quad (3.42)$$

where $\mathbf{r}_b$ and $\mathbf{r}$ are the position vectors of the third body and of the spacecraft relative to the primary, respectively. The relevance of this perturbation on the satellite's relative dynamics has been investigated by Roscoe et al. [110], who have derived the effect of the Moon's gravitational pull on differential orbital elements within the circular restricted-three-body-problem framework.

3.4. Perturbed Relative Motion

This section discusses how the perturbations described above influence the relative dynamics. Since the focus of this work is on the LEO environment, only the perturbations due to J_2 and atmospheric drag are considered. Additionally, the relative motion is described by means of the ROEs since, as shown by Scala [34], it facilitates the design of guidance and control algorithms. Nevertheless, extensions of the HCW that partially take perturbations into account are also briefly mentioned.

3.4.1. Cartesian representation

Soon after their introduction, many models were formulated to extend the validity of the HCW equations. One of the first examples is the Tschauner-Hempel model of 1965 [20], which considers the linearised relative dynamics around an eccentric orbit. In 2002 Schweighart and Sedwick [22] formulated a linear model that accounts for the J_2 perturbation. Their approach is based on the linearisation of the satellite motion with respect to a reference circular orbit and the subsequent derivation of the chief-deputy dynamics. The resulting set of equations resembles the HCW, except for the presence of J_2 -dependent coefficients $c(J_2), q(J_2), \phi(J_2)$ which introduce the averaged J_2 effect on the dynamics [22]:

$$\begin{cases} \ddot{x} = 2n c(J_2) \dot{y} + (5c(J_2)^2 - 2)n^2 x \\ \ddot{y} = -2n c(J_2) \dot{x} \\ \ddot{z} = -q(J_2)^2 z + p(J_2) \cos(q(J_2)t + \phi(J_2)) \end{cases} \quad (3.43)$$

3.4.2. ROE-based representation

The representation of the relative dynamics through ROEs is particularly convenient for the purpose of this thesis. In the following are therefore presented the modifications to the

unperturbed model illustrated in section 3.2 to account for geopotential and differential atmospheric drag.

Geopotential

The first formulation of a linear model based on ROEs that took into account J_2 was introduced by D'Amico [27]. It is based on the substitution of the secular rates defined in eq. 3.36 directly into the definition of ROE, obtaining, after setting $e = 0$ and $\eta = 1$:

$$\begin{cases} \delta a = \delta a_0 \\ \delta \lambda = \delta \lambda_0 - \frac{21}{2} \left(\gamma \sin(2i) \delta i_x + \frac{1}{7} \delta a_0 \right) (u - u_0) \\ \delta e_x = \delta e_0 \cos(\varphi_0 + \dot{\varphi}(u - u_0)) \\ \delta e_y = \delta e_0 \sin(\varphi_0 + \dot{\varphi}(u - u_0)) \\ \delta i_x = \delta i_{x0} \\ \delta i_y = \delta i_{y0} + 3\gamma \sin^2 i_c \delta i_{x0} (u - u_0) \end{cases} \quad (3.44)$$

where $\gamma = \frac{J_2}{2\eta^4} \left(\frac{R_\oplus}{a_c} \right)^2$, and $\dot{\varphi} = \frac{3}{2}\gamma(5 \cos^2 i_c - 1)$.

Notably, the terms δa and δi_x are not affected in any way. Conversely, the two components of the eccentricity vector have an oscillatory motion that leaves its magnitude unchanged but makes its phase vary linearly. The effect is that the relative eccentricity vector travels a circle in the perifocal frame completing it at $u = u_0 + \frac{2\pi}{\dot{\varphi}}$. Finally, the terms $\delta \lambda$ and δi_y exhibit a secular behaviour that, if not adequately compensated, would cause the motion to be unbounded. This could be achieved through periodic manoeuvres; however, as shown by D'Amico and Montenbruck [29], a proper initialisation can eliminate the secular effect of δi_y . The idea is to make the satellites share the same inclination, which would imply $\delta i_x = 0$. The separation along the z-axis would then be achieved by the sole difference in the right ascensions of the ascending node. This approach is the basis of the Helix orbit used by TanDEM-X [15], in which the phase terms are defined respectively as: $\vartheta = 3/2\pi$ (to cancel δi_x), and $\varphi = \vartheta + \pi$ (respecting Helix definition, see Table 3.1).

A step forward from the formulation in 3.44 was made by Gaias et al. [26], who have shown the existence of other secular terms depending on δa . In their approach, they start from the secular rates, but impose the simplifications associated with circular orbits only *after* mapping from absolute element rates to relative ones. In doing so, the $\delta \lambda$ and δi_y

expressions are enriched with new components:

$$\begin{cases} \delta\lambda = (\text{DA}_{\delta\lambda}) - \frac{21}{2}\gamma (3\cos^2(i_c) - 1) \delta a (u - u_0) \\ \delta i_y = (\text{DA}_{\delta i_y}) + \frac{21}{4}\gamma \sin(2i_c) \delta a (u - u_0) \end{cases} \quad (3.45)$$

where $(\text{DA}_{\delta\alpha})$ denotes the expression for $\delta\alpha$ obtained by D'Amico on the right side of eq. 3.44. To be precise, also the relative eccentricity vector rates contain terms proportional to δa and δi_x in addition to the main coupling, but these are of order e , therefore negligible in the near-circular orbit assumption. From the last formulation, one can refine the condition for bounded relative orbits subject to J_2 of [27], which yields:

$$\gamma \sin(2i_c) \delta i_x + \left(\frac{1}{7} + \gamma (3\cos^2(i) - 1) \right) \delta a = 0 \quad (3.46)$$

The resulting model was applied in the AVANTI experiment [43], with the only difference that a linear approximation was retained for the relative eccentricity vector. This was fully justified by the short duration of the experiment. However, it is clear that the oscillatory behaviour is of considerable importance for a remote-sensing mission with a potential lifetime of years. Therefore, the full expression of these terms is included in the following. Figure 3.5 illustrates the time evolution of the relative orbit obtained using this last formulation.

A further improvement to the model –which can be considered state of the art –is provided by Gaias et al. [25]. The authors derive an STM accounting for the secular effects of even zonal terms of the geopotential up to an arbitrary order. In this regard, note that the odd terms do not produce secular effects either on the absolute or on the relative motion: they are only responsible for periodic contributions. The significant accuracy provided by this model, however, goes beyond the scope of this thesis; thus, the analysis is limited to considering only the first-order zonal gravitational harmonic.

Differential drag

As anticipated in section 3.3.2, the semi-empirical model initially derived by Colombo [111] and used for the AVANTI experiment (Gaias and Ardaens [43], Gaias et al. [26]), which is not dependent on atmospheric density models, is adopted. It assumes that, for sufficiently close formations, the satellites do not experience a difference in air density or absolute velocity. Therefore, the only possible cause of relative acceleration is a different ballistic coefficient. Starting from this hypothesis, the acceleration is then defined within three

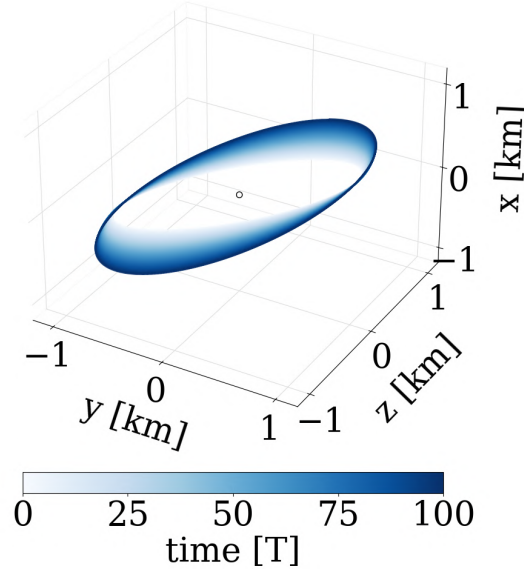


Figure 3.5: Relative orbit evolution under J_2 effect in the RTN frame

experimental constants [111]:

$$\mathbf{a}_{\text{d-drag}} = \begin{bmatrix} 0 \\ A \cos u + B \sin u + C \\ 0 \end{bmatrix} \quad (3.47)$$

where A, B, C are empirically computed constant values. Note that the only non-zero component is along the tangent direction, as only the absolute velocity of the satellites is retained from eq. 3.39. Indeed, for a circular orbit, the contribution due to the Earth's angular velocity would result in a non-null $\hat{\mathbf{N}}$ -oriented acceleration. However, demonstrated in [26], its influence on relative dynamics is negligible for inter-satellite distances that are small enough.

The authors also show that the secular influence on ROE is given by a close relationship between the constants A, B, C and the rates of the semi-major axis and the relative eccentricity defined by:

$$\begin{cases} A = na\delta\dot{e}_x \\ B = na\delta\dot{e}_y \\ C = \frac{n}{2}a\delta\dot{a} \end{cases} \quad (3.48)$$

For completeness, it should be mentioned that density-based models are still the most widely used. An example is the one developed by Koenig et al. [112], based on a closed-

with:

$$L_A = -\frac{21}{2}n\gamma(3\cos^2 i_c - 1), \quad L_I = -\frac{21}{2}n\gamma\sin(2i_c), \quad Y_A = \frac{21}{4}n\gamma\sin^2 i_c, \quad Y_I = 3n\gamma\sin^2 i_c,$$

$$c_\varphi = \cos(\dot{\varphi} dt), \quad s_\varphi = \sin(\dot{\varphi} dt), \quad c_u = \cos(u - u_0), \quad s_u = \sin(u - u_0)$$

4 | Principles

This chapter introduces the theoretical principles that allow the optimisation problem to be formalised. To this purpose, three main aspects are covered:

- the performance metric, determined by the HoA.
- The characteristics of satellite formations that allow acquisitions without risking a collision. Indeed, the overall objective of this thesis is the design of interferometrically optimal formations that are also operationally viable, and thus, safe.
- The mathematical formulation of the optimisation problem.

After having derived the parametric expression for HoA as a function of the ROEs in section 4.1, the requirements to obtain a safe formation -*i.e.*, without the risk of mutual collisions - are presented in section 4.2. The mathematical formulation of the HoA optimisation problem is therefore set in two separate scenarios: in the first, simplified one, a pair of satellites is considered with the objective of maintaining a single homogeneous HoA over an arbitrary range of latitudes. In the second, the formulation is extended to include the availability of an arbitrary number of satellites. Both scenarios are presented in section 4.3. Finally, section 4.4 is dedicated to the modifications and precautions necessary to account for acquisition modes other than bi-static SAR interferometry.

4.1. Parametric Expression of the Height of Ambiguity

As explained in Chapter 2, the HoA is a key parameter in determining interferometric performance. Furthermore, it is also the most significantly influenced by the acquisition geometry and, consequently, by the relative dynamics of the imaging satellites. Unless fixed baselines were employed –which would require a significantly greater control effort (Scala et al. [35]) –the mutual positions vary over time and, with them, the HoA. In fact, the HoA depends on the effective baseline (see eq. 4.2), which in turn is determined by the acquisition geometry. The ROEs are particularly convenient for formulating changes

to the HoA through time because of the intuitive geometric interpretation of the relative motion they provide (see Chapter 3). Indeed, by introducing a parametric expression of the HoA relying on them, the impact that satellite dynamics has on interferometry can be conveniently understood.

The HoA is defined accordingly to Chapter 2, where $m = 1$ is selected for the bistatic case.

$$\text{HoA} = \frac{\lambda r_0 \sin \theta_i}{B_{\perp}}. \quad (4.1)$$

The relative geometry does not influence any of the terms in the numerator (wavelength, slant range and incidence angle), but only the effective baseline $B_{\perp}$. Accordingly with D'Errico [16], in across-track interferometry the effective baseline is computed from the separation in the radial (x) and across-track (z) directions, projected onto the radar line of sight. As such, it nullifies when the satellites are aligned in that direction. Considering the geometry depicted in Figure 4.1, it reads:

$$\begin{aligned} B_{\perp}(\theta_l, u) &= |B(u) \cdot (\sin(\alpha - \theta_l))| = \\ &= |B(u) \cdot (\sin \alpha \cos \theta_l - \cos \alpha \sin \theta_l)| = \\ &= |-x(u) \sin \theta_l + z(u) \cos \theta_l|, \end{aligned} \quad (4.2)$$

where, it is reminded, the mean argument of latitude u is the only time-dependent parameter. To express the effective baseline in terms of the ROEs, the mapping connecting them to the coordinates in the Hill frame for a harmonic solution of the HCW in eq. 3.31 can be employed. Considering relative orbits bounded and centred with respect to the chief (*i.e.*,

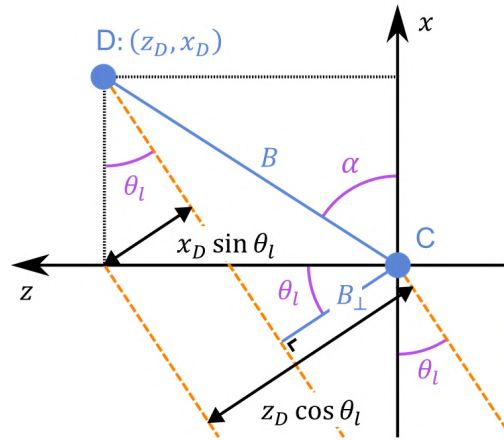


Figure 4.1: Relative geometry of a chief-deputy pair in the $x - z$ plane of the RTN, adapted from D'Errico [16].

$\delta a = 0$, $\delta \lambda = 0$, in agreement with D'Amico [27]) and employing the amplitude/phase form of the eccentricity and inclination vectors, it reads:

$$B_{\perp} = | -a_c \delta e \cos(u - \varphi) \sin \theta_l + a_c \delta i \sin(u - \vartheta) \cos \theta_l |. \quad (4.3)$$

A clear interpretation of how the amplitudes and phases of the relative eccentricity and inclination vectors influence the effective baseline is given by the four plots in Figure 4.2. Here the magnitudes and phases are varied one at a time, while keeping the other parameters fixed at the representative values: $a_c \delta e = 300$ m, $a_c \delta i = 600$ m, $\varphi = \pi/2$, $\vartheta = 3\pi/2$. In particular, the four charts display the effect of a change in the parameters: δe , δi , φ , ϑ respectively on $B_{\perp}$. Note above all that $\varphi = \vartheta = \pi/2 \pm \pi$: As explained in Chapter 3, this choice guarantees stability with respect to the first zonal harmonic of the geopotential. Moreover, it leads to a purely radial separation between satellites at $u = 90^\circ$ (approximately at the poles), and only in the across-track direction at the equator, consistently with TanDEM-X (Krieger et al. [15]).

Finally, substituting the last expression into the definition of HoA in eq. 4.1, the parametric formula sought is:

$$\text{HoA} = \frac{\lambda r_0 \sin \theta_i}{a_c |\delta e \cos(u - \varphi) \sin \theta_l + \delta i \sin(u - \vartheta) \cos \theta_l|}. \quad (4.4)$$

The expression in eq. 4.4 is of capital importance for this work, as it links the HoA directly to the orbital geometry, making it possible to design constellations with predictable DEM accuracy. Another advantage is that it confines the variation of the HoA due to the elapsed time in the mean argument of latitude u only, while the dependence on the ROEs is determined by the parameters δe , δi , φ , ϑ . Moreover, it highlights the π -periodicity

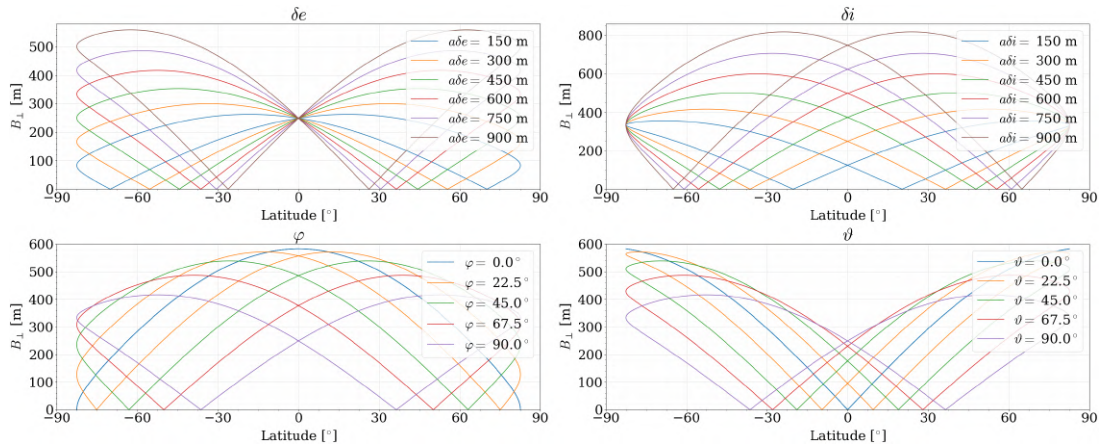


Figure 4.2: Sensitivity of the effective baseline to amplitudes and phases of $\Delta \mathbf{e}$, $\Delta \mathbf{i}$ vectors.

exhibited by the HoA relative to the mean argument of latitude. Finally, it makes clear that the HoA is not defined for every value of u , but instead tends to an asymptote at latitudes corresponding to a null effective baseline. Figure 4.3 displays the behaviour of the effective baseline and of the HoA relative to the mean argument of latitude. In the representative scenario considered, the effective baseline nullifies in correspondence of $u \approx 137^\circ \pm 180^\circ$. The consequence is that, at those mean arguments of latitude, the relative geometry does not allow interferometry to be performed.

Since the time evolution of the HoA is uniquely determined by the magnitudes and phases of the eccentricity and relative inclination vectors, those must be regarded as the decision variables of any HoA optimisation problem. However, not all possible combinations result in operational feasibility, since the collision avoidance is, in principle, not guaranteed. For example, as demonstrated by D’Amico and Montenbruck [29] and explained in Chapter 3, the constraint $\varphi = \vartheta + k\pi$ can be added to achieve passive safety, the defining condition of the Helix formation (see Table 3.1).

For this particular type of relative orbit, the mean argument of latitude corresponding to an asymptote for the HoA reads:

$$\tan(u - \varphi) = (-1)^{k+1} \frac{\delta e}{\delta i} \tan(\theta_l). \quad (4.5)$$

The Helix was conceived starting from the e-i vector separation method introduced by D’Amico and Montenbruck [29] to ensure the passive safety of a formation composed of two satellites: a chief and a deputy. However, when considering formations composed of multiple satellites, it is necessary to examine how this requirement can be extended to

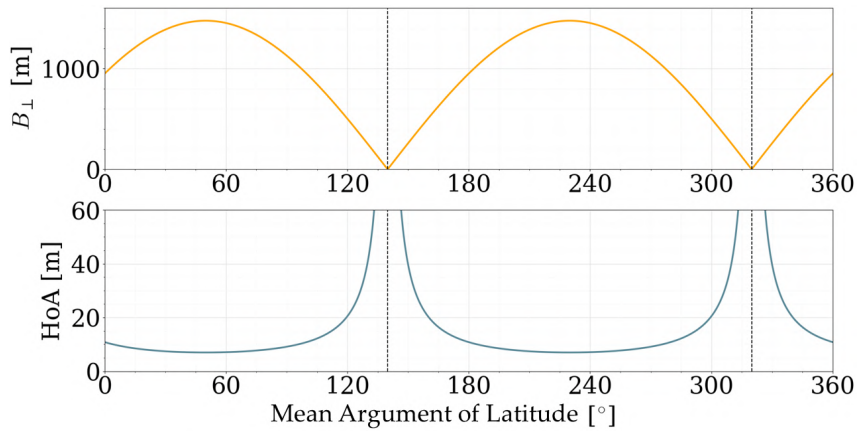


Figure 4.3: Examples of effective baseline and HoA against mean argument of latitude in an SSO at altitude 500 km.

guarantee the passive safety of all possible pairs.

4.2. Safety Requirements

This section introduces the collision avoidance requirements. First, the condition to ensure a minimum mutual distance between each pair is presented; then, it is shown how to scale the passive safety concept to allow for N-satellite constellations.

4.2.1. Minimum distance constraint

A first study on the safety of a swarm of N satellites was conducted by Koenig and D'Amico [32], who derived the necessary and sufficient condition to maintain a minimum separation $a_c \delta r_{min}$ in the x-z plane between each pair. In its adimensional form, it reads:

$$|\Delta \mathbf{e} \cdot \Delta \mathbf{i}| \geq \delta r_{min} \sqrt{\delta e^2 + \delta i^2 - \delta r_{min}^2}, \quad (4.6)$$

and defines a hyperbolic exclusion zone in the state plane of the ROEs.

Adding the hypothesis that $\vartheta_j = \pm\pi/2$ for each deputy j , each satellite pair must satisfy the condition:

$$\left(\sin^2 \varphi_{i,j} - \frac{\delta r_{min}^2}{\delta i_{i,j}^2 - \delta r_{min}^2} \cos^2 \varphi_{i,j} \right) \delta e_{i,j}^2 \geq \delta r_{min}^2. \quad (4.7)$$

As illustrated in Chapter 3, the Helix formation with $\vartheta = \pm\pi/2$ is particularly convenient because it allows for eliminating the secular effect of J_2 on δi_y . Nevertheless, a further development that relaxed this hypothesis was provided by Scala et al. [33], who introduced an extended version of the ROEs: the eROEs. Specifically, they foster the systematic modelling of the relative dynamics for all possible satellite pairs. Denoting with $\Delta\delta\alpha$ the difference between the ROEs of the two deputies relative to the chief, the eROEs of pair i, j read:

$$\delta \boldsymbol{\alpha}_{i,j}^E = \begin{bmatrix} \Delta\delta a \\ \Delta\delta\lambda + \frac{\Delta\delta i_y}{\tan i_c} \left(\frac{\cos(\delta i_{x,i} + i_c)}{\cos i_c} - 1 \right) \\ \Delta\delta e_x \\ \Delta\delta e_y \\ \Delta\delta i_x \\ \Delta\delta i_y \frac{\sin(\delta i_{x,i} + i_c)}{\sin i_c} \end{bmatrix} \quad (4.8)$$

Notably, additional terms are introduced other than the simple differences between the ROEs of the two deputies, as a consequence of the explicit derivation of eROEs from Keplerian elements. This indeed preserves the closure of ROE algebra, *i.e.*, maintains the

physical meaning of deputy-deputy ROEs. Consequently, when dealing with large formations or with large along-track separations, the eROEs allow avoiding the accumulation of linearisation errors. However, for small separations and negligible inclination differences, it is reasonable to introduce the approximation: $\sin(\delta i_{x,i} + i_c) \approx \sin i_c$, thus making the pairwise components equivalent to the simple subtraction of the chief-referenced ones: $\delta \alpha_{i,j} \approx \delta \alpha_j - \delta \alpha_i$. In the rest of the work, this last assumption is retained, allowing for more practical treatment of the reciprocal state of two deputies as a linear combination of their chief-related ROEs.

4.2.2. Generalised Helix formation

The practical utility of the Helix formation is that it allows for maximising the minimum distance in the x-z plane of the RTN, given the magnitudes of the relative eccentricity and inclination vectors. To do so, it introduces a constraint on the phases of the chief-related ROEs: the generalisation of this concept to a constellation of N satellites is thus introduced in the following. More precisely, the aim is to establish a generalised Helix in which analogous conditions are guaranteed for each possible pair of satellites in the formation. For this purpose, the differences between the eccentricity and relative inclination vectors of the i, j deputies are considered, which are defined as follows:

$$\Delta \mathbf{e}_{i,j} = \Delta \mathbf{e}_i - \Delta \mathbf{e}_j = \begin{bmatrix} \delta e_i \cos(\varphi_i) - \delta e_j \cos(\varphi_j) \\ \delta e_i \sin(\varphi_i) - \delta e_j \sin(\varphi_j) \end{bmatrix} \quad (4.9)$$

$$\Delta \mathbf{i}_{i,j} = \Delta \mathbf{i}_i - \Delta \mathbf{i}_j = \begin{bmatrix} \delta i_i \cos(\varphi_i + n\pi) - \delta i_j \cos(\varphi_j + m\pi) \\ \delta i_i \sin(\varphi_i + n\pi) - \delta i_j \sin(\varphi_j + m\pi) \end{bmatrix} \quad (4.10)$$

It is here reminded that the symbols $\delta e, \delta i, \varphi$ represent, respectively, the magnitude of the relative eccentricity vector, the magnitude of the relative inclination vector, and the phase of the relative eccentricity vector. The subscripts denote the deputies to whom they refer. To ensure the mutual passive safety, all vector pairs of this kind must be (anti-)parallel for each couple of satellites. This can be equivalently expressed by equating their cross-product to zero:

$$\begin{aligned} & \Delta \mathbf{e}_{i,j} \times \Delta \mathbf{i}_{i,j} = 0 \\ \Leftrightarrow & (-1)^m \delta e_i \delta i_j (\sin \varphi_i \cos \varphi_j - \cos \varphi_i \sin \varphi_j) + (-1)^n \delta e_j \delta i_i (\sin \varphi_j \cos \varphi_i - \cos \varphi_j \sin \varphi_i) = 0 \\ \Leftrightarrow & \delta e_i \delta i_j (\sin \varphi_i \cos \varphi_j - \cos \varphi_i \sin \varphi_j) - (-1)^{n-m} \delta e_j \delta i_i (-\sin \varphi_j \cos \varphi_i + \cos \varphi_j \sin \varphi_i) = 0 \\ \Leftrightarrow & (\delta e_i \delta i_j \pm \delta e_j \delta i_i) \cdot \sin(\varphi_i - \varphi_j) = 0. \end{aligned} \quad (4.11)$$

The last expression indicates that it is, in fact, possible to obtain a generalised Helix under two possible conditions:

1. All deputies have the same orientation of the eccentricity vector: $\varphi_i = \varphi_j + k\pi \forall i, j$; combining it with the definition of Helix, it entails all the relative eccentricity and inclination vectors for each pair to lie on the same line.
2. All deputies have identical ratios between the magnitude of the relative eccentricity and the relative inclination: $\delta e_i / \delta i_i = K \forall i$. In other words, all relative orbits must be scaled versions of each other.

More precisely, the second category effectively exhibits passive safety only when there is a difference in the first term of the product, which occurs if $n - m$ is even. Converting this condition to the mutual orientation of the $(\Delta \mathbf{e}_{i,j}, \Delta \mathbf{i}_{i,j})$ vectors of all deputy pairs, it means that they can only have all the same direction, or all opposite directions. This rule is in agreement with simple geometric considerations illustrated in the three plots of Figure 4.4. Indeed, in the left and central plots, the vector differences between the relative eccentricity and inclination vectors of two chief-deputy pairs can be (anti-)parallel, while this is not possible in the right plot, where one pair has vectors pointing in the same direction, the other pair in opposite directions.

The requirement on the phases for a classical Helix formation has been derived to maximise the distance in one of the x and z directions of the RTN when the other is minimal for a chief-deputy pair (D'Amico and Montenbruck [29]). This also holds for the generalised Helix concept, for which the differentials of the positions in x and z directions are considered:

$$\begin{cases} \delta x_{1,2} = \delta e_1 \cos(u - \varphi_i + n\pi) - \delta e_2 \cos(u - \varphi_j + m\pi) \\ \delta z_{1,2} = \delta i_1 \sin(u - \varphi_i + n\pi + n\pi) - \delta i_2 \sin(u - \varphi_j + m\pi + m\pi) \end{cases} \quad (4.12)$$

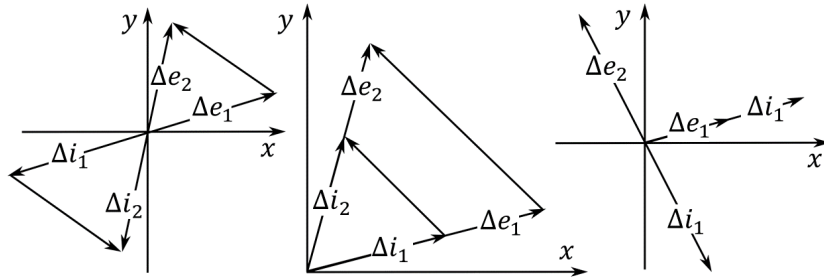


Figure 4.4: Left and Centre: optimal configuration compatible with maximum safety; Right: incompatible configuration.

To maximise one of the two when the other reaches zero, one can impose, for instance, $\delta z = 0, \delta x' = \frac{d\delta x}{du} = 0$, even though the same result holds inverting the roles of x and z :

$$\begin{cases} \delta x'_{1,2} = \delta e_1 \sin(u - \varphi_i + n\pi) - \delta e_2 \sin(u - \varphi_j + m\pi) = 0 \\ \delta z_{1,2} = \delta i_1 \sin(u - \varphi_i + n\pi) - \delta i_2 \sin(u - \varphi_j + m\pi) = 0 \end{cases} \quad (4.13)$$

$$\Leftrightarrow \begin{cases} \delta i_1 \sin(u - \varphi_i + n\pi) = \delta i_2 \sin(u - \varphi_j + m\pi) \\ \delta e_1 \sin(u - \varphi_i + n\pi) = \delta e_2 \sin(u - \varphi_j + m\pi) \end{cases} \quad (4.14)$$

Eq. 4.14 is verified for the two previously found conditions: $\varphi_j = \varphi_i + k\pi$ or $\delta e_i/\delta i_i = \delta e_j/\delta i_j$. This result proves indeed that the conditions given by eq. 4.11 are compliant with the passive safety condition in the x-z plane, thus providing an effective extension of the classical two-satellite Helix concept.

Considering the example of a formation of four satellites, in the Figures 4.5, 4.6 three possible geometries of the vectors $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ are reported: as expected, the two are collinear for each of the six possible pairs.

At last, in Figure 4.7, two practical implementations of the generalised Helix concept for a four satellite formation are shown. Here, the relative orbits between all satellite pairs are displayed on the x-z plane of the RTN for both of the two identified passive safety conditions. In the left plot, the formation is designed so that the $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ vectors lie on the same line under the specific condition $\varphi = \vartheta = \pi/2 \pm \pi$. This is highlighted by the initial positions of the satellites, located at an angular distance of $\pm\pi/2$ from the x-axis. On the other hand, the right plot reports a configuration in which the xz-projected orbits are scaled versions of each other in the particular case with $\delta e_i/\delta i_i = 1$.

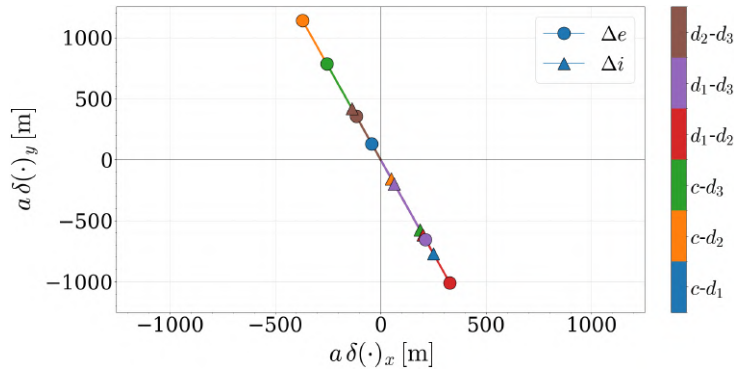


Figure 4.5: Example of a possible geometry of the relative e/i vectors for the maximum safety configuration with equal phase differences for a four satellite formation.

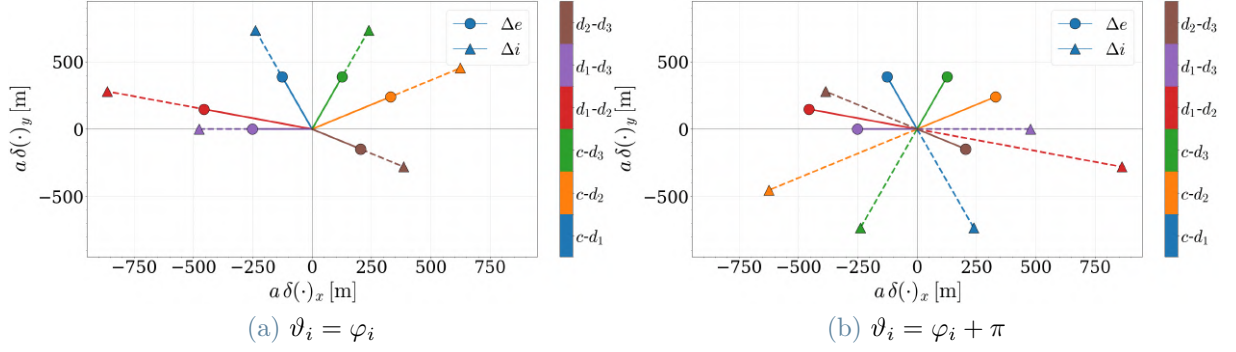


Figure 4.6: Two possible geometries of the relative e/i vectors for the maximum safety configuration with equal shape orbits for a four satellite formation.

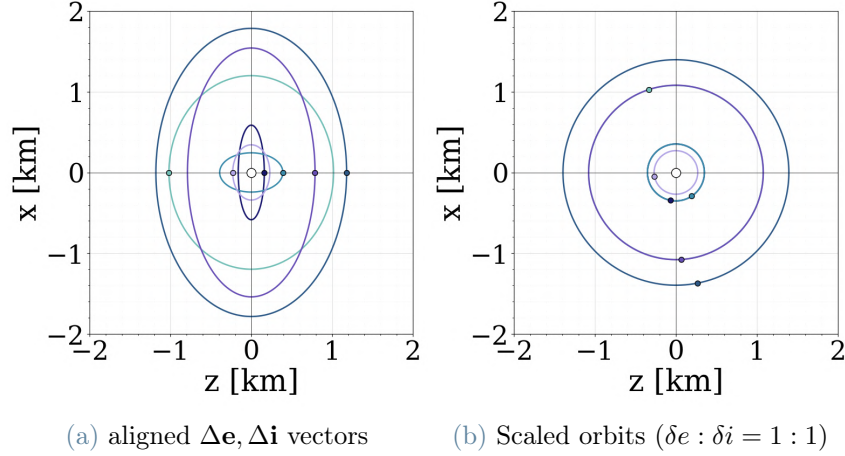


Figure 4.7: x-z projection of relative orbits between each pair of a four satellite generalised Helix formation.

4.3. Mathematical Formulation of the Optimisation Problem

This section introduces the mathematical formulation of the optimisation problem, the solution of which is presented in Chapter 5. The problem with a single baseline is illustrated first, and is subsequently extended to allow for arbitrarily large constellations. To do so, it is initially assumed that each deputy forms a Helix formation together with the chief; later in this section, it is shown that the result holds for generic harmonic solutions of the HCW.

Moreover, it is considered that the formation acquires images with a fixed angle of incidence θ_i at each latitude. This hypothesis is then partially relaxed in Chapter 7, where

the optimal strategy for the definition of this angle is addressed.

4.3.1. Single-baseline problem

The problem is initially formulated for a single pair of satellites, *i.e.*, a single baseline. The aim is to maintain an HoA defined in eq. 4.4 as close as possible to a nominal value HoA^* . Consequently, the optimality problem consists of minimising the J functional as reported in the following expression:

$$\begin{aligned} \arg \min_{\delta \alpha} J &:= \int_{\mathcal{U}} (\text{HoA}(\delta \alpha, u) - \text{HoA}^*)^2 du = \\ &= \int_{\mathcal{U}} \left(\frac{K}{|- \delta e(u - \varphi) \sin \theta_l + \delta i(u - \varphi + k\pi) \cos \theta_l|} - \text{HoA}^* \right)^2 du, \end{aligned} \quad (4.15)$$

where the constant $K := \frac{\lambda r_0 \sin \theta_i}{a_c}$ embedding all the constant parameters is introduced.

The set $\mathcal{U}$ contains the mean arguments of latitude at which the acquisitions have to take place: by setting $\mathcal{U} = [0, 2\pi]$, the entirety of the orbit would be considered. Nevertheless, note that because of the π -periodicity of the effective baseline with respect to u (see eq. 4.4), the same result is obtained with $\mathcal{U} = [-\pi/2, \pi/2]$. This symmetry is particularly convenient in view of an analytical approach to the problem since, as shown in Chapter 5, it can lead to a practical closed-form derivation of the optimal set of ROEs.

However, the aforementioned presence of the asymptote for the HoA (see eq. 4.5) causes the integral to always diverge if the mean arguments of latitude within $\mathcal{U}$ form an interval wider than half a period. Consequently, to find a satisfactory solution to the problem, precautions must be taken to eliminate those mean arguments of latitudes that cause the HoA to be undefined. Despite the said convenience from a computational viewpoint, the interval $\mathcal{U} = [0, \pi]$ is not needed for a remote sensing mission with global coverage as its objective. Because of the π -periodicity of the HoA, it is indeed sufficient to set $\mathcal{U} = [0, \pi/2]$ to acquire once all the latitudes overflown by the formation. In fact, with this choice, the points in the northern hemisphere will be mapped in the ascending part of the orbit, those in the southern hemisphere in the descending part. Equivalently, one could choose $\mathcal{U} = [\pi/2, \pi]$, for which the acquisition of the two hemispheres would simply be swapped between the ascending and descending sections of the orbit. An illustration of this concept is provided in Figure 4.8. In either case, because of Earth's rotation, the entire surface would be mapped after a sufficient number of periods, thus making this second option tailored to global Earth observation missions.

In the case of a non-polar orbit, perfect global coverage is not achievable, for the maximum

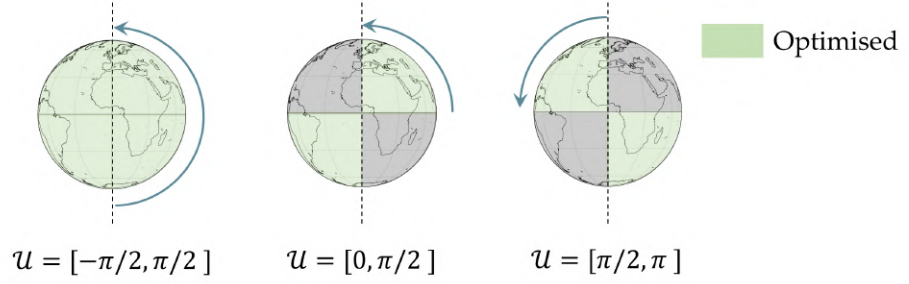


Figure 4.8: Latitudes considered in the optimisation problem for a circular polar orbit.

latitude overflow is equal to the inclination of the orbit i_c . Nevertheless, the condition $\mathcal{U} = [0, \pi/2]$ is still valid to account for all the points on the ground over which the formation passes. In fact, referring to Figure 4.9, one can apply the theorem of sines to the spherical triangle PNM, obtaining the following connection between the true argument of latitude $\tilde{u}$ –defined as the sum of the argument of perigee and true anomaly –and the geocentric latitude (see Vallado [89]):

$$\sin \phi = \sin \tilde{u} \sin i_c. \quad (4.16)$$

The maximum value $\phi = i_c$ is thus obtained when $\tilde{u} = \pi/2$. However, two clarifications are due at this point.

First, eq. 4.16 applies to the true argument of latitude and not to the mean one, which is considered in the integration. The two are equivalent only in the case of a circular orbit of the chief, an assumption retained hereafter in compliance with the HCW hypotheses. Nevertheless, a thorough derivation of the correct integration interval defined by the mean argument of latitude for quasi-circular orbits is reported in Appendix A. Specifically, it is shown that when the true argument of latitude is $\tilde{u} = \pi/2$, the mean one is given by:

$$u = \frac{\pi}{2} - 2e \cos \omega + \frac{3}{4}e^2 \sin(2\omega) + \mathcal{O}(e^3), \quad (4.17)$$

and clearly the two are equal for $e \rightarrow 0$. At last, Figure 4.10 displays the effective baseline and the HoA traced with respect to the geocentric latitude.

The second observation is that the geocentric and geodetic latitudes do not coincide: their mutual relationship can be calculated as in the appendix of the *Astrodynamic Almanack* [114]. The consequence is that the maximum latitude observed does not correspond to the orbit’s inclination: the difference is the order of 0.2° in latitude, *i.e.*, about 20 km at the Earth’s surface for an SSO in LEO [114]. The maximum is reached anyway

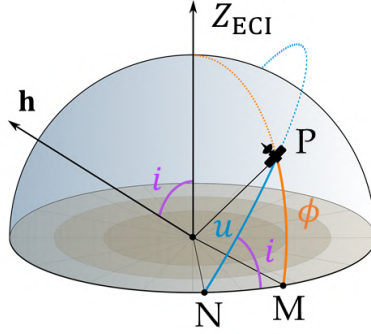


Figure 4.9: Geometry relating the geocentric latitude to the argument of latitude for a non-polar orbit.

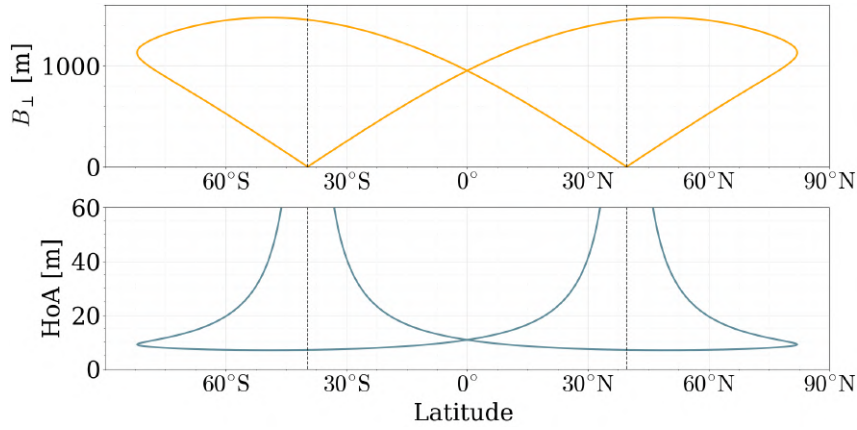


Figure 4.10: Example of effective baseline and HoA against geocentric latitude in a SSO at height 500 km.

at $u = \pi/2$, thus making the interval $\mathcal{U} = [0, \pi/2]$ consistent with global mapping without redundancies.

4.3.2. Multi-baseline problem

The solution to the single-baseline problem allows for determining the set of ideal configurations to maintain a nominal HoA with a two-satellite formation. The aim of this work, however, is also to investigate the improvement that a greater number of satellites — and thus more baselines — can bring. Extending the problem setup of the previous scenario, two target values for the HoA are considered: HoA_1^* and HoA_2^* . It is therefore necessary to solve a multi-objective optimisation problem where the objective functions are given by the L^2 distances from the target HoAs, and the decision variables are the ROEs of the satellite formation. For a more practical formalisation it is convenient to define the set of deputies $\mathcal{D}$ and the set of satellites pairs $\mathcal{P}$ (if a formation is made of N satellites with

one chief and $N - 1$ deputies, $\mathcal{D}$ has cardinality $|\mathcal{D}| = N - 1$, while $|\mathcal{P}| = N(N - 1)/2$. Additionally, $\delta\alpha_d$ is used to indicate the set of relative orbital elements of the d -th deputy with respect to the chief, whereas HoA_p is defined as the height of ambiguity given by the p -th satellite pair. Moreover, it is assumed that for each value of the mean argument of latitude, the images acquired by any pair of satellites can be combined to form an interferogram. The multi-objective optimisation reads:

$$\arg \min_{\delta\alpha_d, d \in \mathcal{D}} (F_1(\delta\alpha_d), F_2(\delta\alpha_d)), \quad (4.18)$$

where the objective functions are defined as follows:

$$F_i = \int_{\mathcal{U}} \min_{p \in \mathcal{P}} (\text{HoA}_p(\delta\alpha_d, u) - \text{HoA}_i^*)^2 du. \quad (4.19)$$

As in most multi-objective optimisation problems, it is not possible to obtain the optimal solution for all the objectives simultaneously. Intuitively, if the configuration had all the available baselines designed to obtain a HoA^* , this would inevitably worsen the error on the second. The aim hence becomes to identify non-dominated or Paretian solutions, *i.e.*, solutions for which no improvement with respect to one objective can be achieved without worsening another (see Veldhuizen et al. [115]).

A common strategy to handle these types of problems is through the introduction of scalarisation methods (see Hwang and Masud [116]), which allow the reduction to a single-objective optimisation dependent on the *parameters of the scalarisation*. Non-dominated solutions can then be found by iteratively solving the reformulated problem for different values of the parameters of the scalarisation. Specifically, three of the most widely used and established scalarisation methods are reported in the following.

Linear scalarisation: The linear scalarisation (see Emmerich and Deutz [117]) consists of formulating the problem as a linear combination of the individual objectives, each multiplied by a coefficient.

$$\begin{cases} J = w_1 F_1(\delta\alpha_d) + w_2 F_2(\delta\alpha_d), & w \geq 0 \\ w_1 + w_2 = 1 \end{cases} \quad (4.20)$$

The choice of the coefficients w_i allows to give more importance to either of the two objectives and, by making them vary in the interval $[0, 1]$ it is possible to obtain (part of) the Pareto front. This scalarisation, intuitive as it is, is regarded as the preferred one throughout this work. Furthermore, although the goal is to find all non-dominated

solutions, a significant part of the results shown in Chapter 5 were obtained through adopting this scalarisation, with the coefficients defined as:

$$w_i = \frac{\text{HoA}_i^*}{\sum_{i=1}^2 \text{HoA}_i^*}. \quad (4.21)$$

This choice is motivated by the need to attribute the same importance to the relative errors committed with respect to each of the HoA^* , leading to an impartial balance between the two objectives.

The main issue faced by this technique is that it allows reaching only the points belonging to the convex hull of the Pareto front [117]. For this reason, alternative strategies have been considered.

Chebyshev scalarisation: The single-objective problem is in this case written as (Miettinen [118]):

$$J = \max_i w_i |F_i(\delta\alpha_d) - z_i^*|, \quad (4.22)$$

where z_i^* is a reference point, and again w_i are the parameters of the scalarisation. The advantage of this method is that it ensures obtaining all the non-dominated solutions provided a careful choice of z_i^* . More precisely, this is guaranteed if the reference is the utopia point, *i.e.* $z_i^* = \min_{d \in \mathcal{D}} F_i(\delta\alpha_d)$, [118]. Its determination, hence, requires solving the optimality problems for each of the objective functions alone. However, since all the objectives are certainly non-negative and assuming that the point $[F_1, F_2] = [0, 0]$ is sufficiently close to the Pareto front, it is reasonable to use the origin as a surrogate for the utopia point.

ε -constraint method: This method consists of optimising one of the two objectives while the other is constrained to stay below the threshold ε (see again Emmerich and Deutz [117]):

$$\begin{cases} \arg \min_{\delta\alpha_d, d \in \mathcal{D}} F_1(\delta\alpha_d) & \text{s.t.} \\ F_2(\delta\alpha_d) < \varepsilon \end{cases} \quad (4.23)$$

It is the most computationally expensive method, although the small size of the problem (only two objectives) partially alleviates the issue. Nevertheless, it theoretically allows for generating all non-dominated solutions, provided that the values of ε are chosen finely enough.

Availability of a closed-form solution

The proposed scalarisation methods can contribute to simplifying the problem, but realistically, they are not sufficient to allow a purely analytical approach. The main obstacle to the existence of a possible closed-form solution is the presence of the min operator within the cost function in eq. 4.19, which entails a piecewise integration. For simplicity, one may consider the case with three available baselines (three satellites); each of the objective functions then becomes:

$$F_i = \int_{\mathcal{U}_1} (\text{HoA}_1 - \text{HoA}_i^*)^2 du + \int_{\mathcal{U}_2} (\text{HoA}_2 - \text{HoA}_i^*)^2 du + \int_{\mathcal{U}_3} (\text{HoA}_3 - \text{HoA}_i^*)^2 du. \quad (4.24)$$

However, after rearranging the terms via trigonometric identities, the consistency of the regions $\mathcal{U}_p$ in which the p -th pair of satellites is selected requires the switching points on the boundaries to satisfy conditions of the type:

$$\frac{1}{A_i \cos(\cdot)_i} \pm \frac{1}{A_j \cos(\cdot)_j} = C \Leftrightarrow A_j \cos(\cdot)_j \pm A_i \cos(\cdot)_i = C A_i A_j \cos(\cdot)_i \cos(\cdot)_j, \quad (4.25)$$

which is a nonlinear transcendental equation with products and linear combinations of trigonometric functions involving the decision variables themselves. As such, the structural transcendental nature makes a fully analytical solution not possible. Furthermore, note how the derivatives of functions of the form $\frac{1}{\cos(\cdot)}$ scale like $\frac{1}{\cos^2(\cdot)}$. This causes small changes in the argument near zero to produce large changes in the target, which degrades the stability of gradient-based methods that would often have to handle ill-conditioned matrices. For this reason, as better detailed in the chapter 5, methods of these two kinds are employed in the resolution: 1) heuristic, involving the exploitation of the solution obtained for a formation of only two satellites; or 2) purely numerical, based on genetic algorithms.

Problem size reduction

As with any numerical optimisation method, the speed and quality of the solution depend largely on the size of the problem being considered. It is therefore useful to reduce as much as possible the number of decision variables on which the solution depends. To this purpose, one can reformulate the expression of the HoA given in eq. 4.4 as a function of specific combinations of the ROEs in order to make the numerical algorithms employed more efficient.

The following derivation makes use of the two trigonometric identities:

$$\begin{cases} A \cos x + B \sin x = \sqrt{A^2 + B^2} \cos(x + \arctan(-B/A)) \\ A \cos(x + \alpha) + B \cos(x + \beta) = \sqrt{A^2 + B^2 - 2AB \cos(\beta - \alpha)} \cos\left(x + \arctan\left(\frac{A \sin \alpha - B \sin \beta}{A \cos \alpha - B \cos \beta}\right)\right) \end{cases} \quad (4.26)$$

which are written in the particular case in which $A, B > 0$. Without prejudice to generality, one can assume that the look-angle is $\theta_l \in [0, \pi/2]$: this is equivalent to considering that the radar is always right looking (see Figure 4.1), but similar considerations apply for the opposite case. In the following, the assumption of a Helix formation is retained, but later in this section, it is shown how the case with relative eccentricity and inclination vectors in a non-parallel alignment can be equivalently treated.

Using the first identity to rewrite the expression of the effective baseline for the pair chief-first deputy, it yields:

$$\begin{aligned} B_{\perp,1} &= | -\delta e \cos(u - \varphi_1) \sin \theta_l + \delta_1 \sin(u - \varphi_1) \cos \theta_l | \\ &= \sqrt{\delta e_1^2 \sin^2 \theta_l + \delta i_1^2 \cos^2 \theta_l} \cos\left(u - \varphi_1 + \arctan\left(-\frac{\delta i_1}{\delta e_1 \tan \theta_l}\right)\right), \end{aligned} \quad (4.27)$$

where the subscript $(\cdot)_i$ indicates the quantities referring to the pair chief- i -th deputy.

It is convenient to introduce the two variables maximum effective baseline: $B_{\perp max,1} := \sqrt{\delta e_1^2 \sin^2 \theta_l + \delta i_1^2 \cos^2 \theta_l}$, and angular shift: $\delta_1 := \varphi_1 + \arctan\left(\frac{\delta i_1}{\delta e_1 \tan \theta_l}\right)$. Additionally, considering the first pair as a reference, the mutual phase shift $\delta_{1,j}$ of every other deputy with respect to the first one is defined as follows:

$$\delta_{1,j} := \delta_1 - \delta_j \quad (4.28)$$

The effective baseline for the chief- j -th deputy pair therefore reads:

$$B_{\perp,j} = B_{\perp max,j} |\cos(u - \delta_1 - \delta_{1,j})|. \quad (4.29)$$

Eq. 4.29 reformulates the link between the dynamics expressed by means of the ROEs and the effective baseline more favourably. More importantly, it highlights how the ROEs contribute to determining the constant amplitude and phase of the function $B_{\perp}(u)$, which in turn is defined by a simple oscillating term in the mean argument of latitude. Consequently, it allows expressing the position of each deputy with respect to the chief with a reduced number of variables (the magnitude and phase of the effective baseline instead of

those of the relative eccentricity and inclination vectors).

However, the pairs composed of two deputies still need to be taken into account. By applying again the first identity in 4.26 the following holds:

$$B_{\perp,i,j} = |-(x_j - x_i) \sin \theta_l + (z_j - z_i) \cos \theta_l| = \quad (4.30)$$

$$= |B_{\perp max,i} \cos(u - \delta_i) - B_{\perp max,j} \cos(u - \delta_j)|. \quad (4.31)$$

Finally, one can introduce the quantities:

$$\begin{cases} B_{\perp max,1,j} = \sqrt{B_{\perp max,1}^2 + B_{\perp max,j}^2 - 2 B_{\perp max,1} B_{\perp max,j} \cos(\delta_j - \delta_1)} \\ \xi_{1,j} = \arctan \left(-\frac{B_{\perp max,j}(\delta_j - \delta_1)}{B_{\perp max,1} - B_{\perp max,j} \cos(\delta_j - \delta_1)} \right) \end{cases} \quad (4.32)$$

so that, leveraging the second equation in 4.26, the effective baseline of the deputy-deputy pairs can also be expressed in amplitude-phase form:

$$B_{\perp,1,j} = B_{\perp max,1,j} |\cos(u - \delta_1 + \xi_{1,j})|. \quad (4.33)$$

Eq. 4.33 represents a significant result for the purpose of this work, as it demonstrates that each effective baseline can be directly mapped to the ROEs of the corresponding chief-deputy pair (or functions thereof) by first expressing it in its amplitude-phase form.

The problem size for a constellation of N_{sat} satellites is hence reduced from $3 \cdot (N_{sat} - 1)$ to $2 \cdot (N_{sat} - 1)$ non-redundant decision variables. The solution to the optimisation is therefore determined in terms of 1) the maximum effective baselines $B_{\perp max,j}$, which provides information about the size of the relative orbits, 2) δ_1 , the angular position of the first deputy along its orbit, and 3) $\delta_{1,j}$, the angular separation between the first baseline and all the others. Remarkably, the choice to use the mutual angular shift $\delta_{1,j}$ between the deputies rather than simply the chief-related δ_j is mainly due to a more straightforward formulation of the coupling with passive safety requirements discussed in the following chapters.

The proposed derivation has been obtained by imposing a concordant orientation of the eccentricity and relative inclination vectors for each chief-deputy pair. However, it can be shown that this does not neglect any potential optimal solutions. In fact, the condition $\varphi = \vartheta + \pi$ has the only effect of changing the sign of the z coordinate, which could be seen as equivalent to adding a minus sign to the magnitude of the relative inclination. In any case, the optimal solution in terms of maximum effective baseline and phase shift is not

affected, whereas the ROE set achieving this solution, naturally, is. To make this evident, one can assume that, in a formation made of a chief and two deputies, the phase of the relative inclination vector of the second deputy relative to the chief is: $\vartheta_2 = \varphi_2 + n\pi$. If the optimal phase shift is δ_{12} , the phase of the relative eccentricity reads:

$$\begin{cases} \varphi_2 = \varphi_1 - \delta_{12} + \arctan\left(-\frac{\delta i_2}{\delta e_2 \tan \theta_l}\right) - \arctan\left(-\frac{\delta i_1}{\delta e_1 \tan \theta_l}\right) & \text{if } n \text{ even} \\ \varphi_2 = \varphi_1 - \delta_{12} - \arctan\left(-\frac{\delta i_2}{\delta e_2 \tan \theta_l}\right) - \arctan\left(-\frac{\delta i_1}{\delta e_1 \tan \theta_l}\right) & \text{if } n \text{ odd} \end{cases} \quad (4.34)$$

where the only difference is in the sign of the term $\arctan\left(-\frac{\delta i_2}{\delta e_2 \tan \theta_l}\right)$. The sign change widens the set of possible configurations, achieving the optimality in terms of HoA, but does not introduce new solutions. In other words, once the optimised set of decision variables $\{B_{\perp max}^*, \delta_1^*, \delta_{1,j}^*\}$ that uniquely solves the optimisation problem has been determined, the mapping given by eqs. 4.29 and 4.33 would allow multiple compliant formations. For some of these, the $\Delta \mathbf{e}, \Delta \mathbf{i}$ vectors would be concordant, whereas discordant for others.

In conclusion, once the optimal solution expressed with the new reduced set of variables has been obtained, multiple sets of compliant ROEs can be defined, either by exploiting the aforementioned mapping or by reversing the directions of the eccentricity and relative inclination vectors. It is worth remarking, however, that all these solutions, although different from a formation-flying perspective, are perfectly identical in terms of meeting interferometric requirements concerning the HoA.

4.3.3. Generalisation to harmonic relative motion

All the previously shown derivations refer to the case of a helix formation. However, a similar approach can be used to show that solving the problem in the new set of variables also allows to account for generic harmonic solutions of the HCWs listed in Table 3.1. Therefore, without introducing constraints on the connection between φ and ϑ , the effective baseline can be written as follows:

$$\begin{aligned} B_{\perp} &= \delta e \sin \theta_l \cos(u - \varphi) + \delta i \cos \theta_l \sin(u - \vartheta) = \\ &= (\delta e \sin \theta_l \cos \varphi - \delta i \cos \theta_l \sin \vartheta) \cos u + (\delta e \sin \theta_l \sin \varphi + \delta i \cos \theta_l \cos \vartheta) \sin u = \\ &= B_{\perp, max} \cos(u - \delta) = B_{\perp, max} (\cos u \cos \delta + \sin u \sin \delta). \end{aligned} \quad (4.35)$$

The aim is to determine the quantities $B_{\perp max}$ and δ such that, if possible, the two formulations are equivalent for all u . This is achieved by simply imposing that the corresponding

coefficients are equal:

$$\begin{cases} B_{\perp max} \cos \delta = \delta e \sin \theta_l \cos \varphi - \delta i \cos \theta_l \sin \vartheta \\ B_{\perp, max} \sin \delta = \delta e \sin \theta_l \sin \varphi + \delta i \cos \theta_l \cos \vartheta \end{cases} \quad (4.36)$$

from which it yields:

$$\begin{cases} B_{\perp max} = \sqrt{\delta e^2 \sin^2 \theta_l + \delta i^2 \cos^2 \theta_l + \delta e \delta i \sin 2\theta_l \sin(\varphi - \vartheta)} \\ \delta = \arctan_2(\delta e \sin \theta_l \sin \varphi + \delta i \cos \theta_l \cos \vartheta, \delta e \sin \theta_l \cos \varphi - \delta i \cos \theta_l \sin \vartheta) \end{cases} \quad (4.37)$$

Therefore, since, for the general harmonic case, one can express the effective baselines in terms of magnitude and phase, all possible harmonic solutions to the relative dynamics are already embedded in the same optimal set of variables. Simply, once the optimal solution has been found, one can use the mapping given by eq. 4.37 to design a formation respectful of the harmonic solutions of the HCWs. Hence, it has been proven that letting φ and ϑ be unconstrained does not introduce new degrees of freedom into the optimisation.

4.4. Alternative Acquisition Strategies

The formulation of the optimisation problem presented in Section 4.3 refers to single-pass, multi-static, across-track interferometry. In this section, the methodology is extended to two notable cases: 1) pursuit monostatic acquisition mode and 2) SAR tomography (see Chapter 2).

However, it is first necessary to consider a possible strategy for forming the interferogram that has not yet been taken into account, which allows obtaining an additional HoA, the so-called *synthetic* or *dual* HoA (Krieger et al. [72]).

4.4.1. Synthetic HoA

If at least two interferograms with different HoAs are available, it is possible to combine them to obtain a third one with a new HoA, which is referred to as a synthetic HoA, HoA_{syn} . Considering that a target at altitude h generates an interferometric phase (see Chapter 2):

$$\varphi_{int} = \frac{2\pi}{\text{HoA}} h, \quad (4.38)$$

one demodulates two interferograms with respect to each other in order to obtain a third whose phase is given by the difference between the two original ones: $\Delta\varphi_{int} = \varphi_{int1} -$

φ_{int2} [72]. The HoA of the third interferogram is therefore given by:

$$\text{HoA}_{syn} = \left(\frac{1}{\text{HoA}_1} - \frac{1}{\text{HoA}_2} \right)^{-1}. \quad (4.39)$$

A synthetic HoA is created, for instance, in the TanDEM-X mission by combining two interferograms generated about a year apart, making the resulting one more robust to phase unwrapping issues [72]. This is an aspect of considerable relevance for the optimisation problem since, if two interferograms with HoA given by: $\left(\frac{1}{\text{HoA}_1^*} - \frac{1}{\text{HoA}_2^*} \right)^{-1}$ and HoA_1^* were available, this would be entirely equivalent to the desired pair: HoA_1^* , HoA_2^* . Nevertheless, it can be shown that if three or more satellites are available, it is useless to combine their interferograms, as the corresponding synthetic HoAs would always be redundant. More precisely, given two pairs of satellites generating interferograms with two distinct HoA, there always exists a third pair whose HoA is precisely the synthetic one obtained from the other two.

Without losing generality, some simplifying hypotheses are introduced: 1) the first and second interferograms are created between the chief and the deputies 1 and 2, giving the two HoA: HoA_1 and HoA_2 , respectively, with $\text{HoA}_1 < \text{HoA}_2$. 2) the geometry is such that $B_{\perp,1} > B_{\perp,2} > B_{\perp,12}$, as shown in Figure 4.11.

The remaining baseline (the one between the two deputies) yields:

$$B_{\perp,12} = |(x_1 - x_2) \sin \theta_l - (z_1 - z_2) \cos \theta_l| = B_{\perp,1} - B_{\perp,2}, \quad (4.40)$$

and the corresponding HoA therefore reads:

$$\text{HoA}_{12} = \frac{K}{B_{\perp,12}} = \frac{K}{B_{\perp,1} - B_{\perp,2}} = \frac{K}{\frac{K}{\text{HoA}_1} - \frac{K}{\text{HoA}_2}} = \left(\frac{1}{\text{HoA}_1} - \frac{1}{\text{HoA}_2} \right)^{-1} = \text{HoA}_{syn}. \quad (4.41)$$

Remarkably, a different combination of signs in eq. 4.41 can be obtained by a suitable exchange of subscripts.

Eq. 4.41 confirms that the formulation of the optimisation problem considered so far remains perfectly valid even taking into account the synthetic HoA. No changes have to be introduced, since for each satellite triplet, the HoA of every pair is equivalent to the synthetic one of the other two. However, note that this is true when having more than two satellites in formation; in the case of TanDEM-X, the possibility of introducing this third baseline is crucial, since it does not correspond to any physical baseline. In this case, the role of the synthetic HoA is equivalent to that of an additional interferogram,

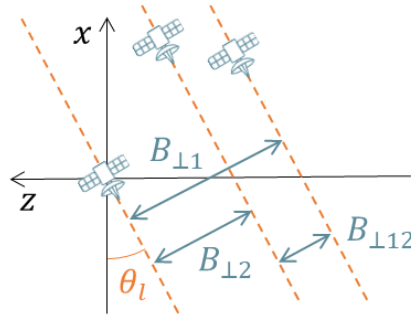


Figure 4.11: Relative geometry and effective baselines in the x-z plane for a three-satellite formation.

generated thanks to a fictitious third satellite.

This result seems to suggest that there are no differences between the formation of two or three satellites. The HoAs generated are identical indeed, but the essential aspect is that for two satellites only it is necessary to generate two DEMs in succession, with a time separation that, for TanDEM-X, is about a year (Krieger et al. [72]). Therefore, there is the possibility that the acquired areas have changed during this timespan, leading to a coherence loss (see Chapter 2). This issue is instead not present with three satellites operating in single-pass mode, for which the acquisitions would be simultaneous.

4.4.2. Pursuit monostatic mode

Even though this work is mainly aimed at systems operating in the bistatic acquisition mode, SAR missions frequently have the capability to switch between different imaging techniques. This is also the case with TanDEM-X, which, while nominally operating as a bistatic system, can change to pursuit monostatic or alternating bistatic modes (see Chapter 2) for calibration or as a backup solution (Krieger et al. [72]). The pursuit monostatic is particularly delicate from a formation design perspective, as the satellites have a significant separation in the along-track direction. In the context of across-track interferometry, small phase shifts in the y-direction of the RTN can be compensated via beam forming or by slightly modifying the satellite's attitude, as done with the yaw-steering technique introduced by Krieger et al. [59]. However, when the distance is significant, these strategies are no longer sufficient, and the satellites' orbital trajectories must be defined consistently with imaging needs. The critical requirement for the orbital elements initialisation is to ensure that the satellites fly over the exact same area during the same orbit. In other words, the orbits have to be designed so that the satellites, even if separately, travel along the same ground-track, thus compensating for the Earth's

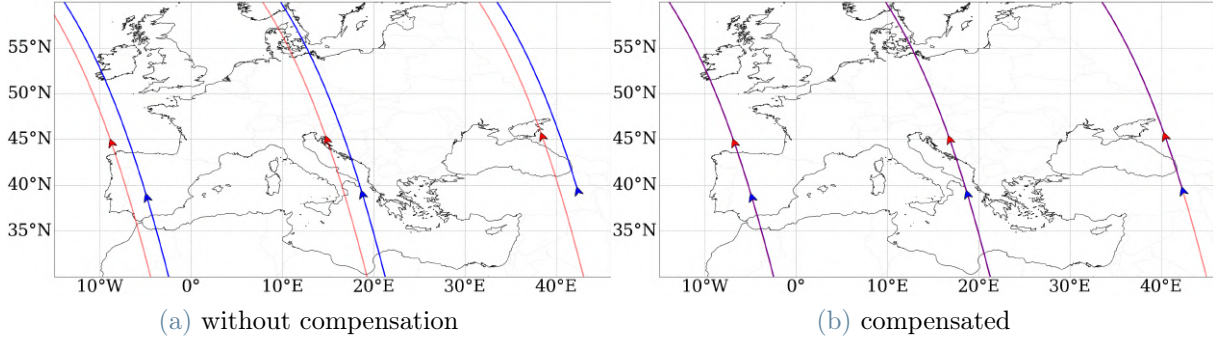


Figure 4.12: Compensation of Earth’s rotation for the pursuit monostatic acquisition mode over three periods for SSO at 500 km altitude.

rotation. If a nominal along-track separation $\delta\lambda_0 = \Delta u + \Delta\Omega \cos i_c$ is present, then one can employ a separation between the right ascensions of the ascending nodes $\Delta\Omega$ of satellites’ orbits to compensate for it. For the considered scenario (LEO, almost circular orbits, absence of perturbations), it is reasonable to state that $\delta\dot{\lambda}$ is equal to the mean motion n . Given Earth’s angular velocity $\omega_\oplus$, during the time span $\Delta t = \frac{\delta\lambda_0}{\delta\dot{\lambda}}$ it rotates by an angle $\Delta\Omega = \omega_\oplus \Delta t$. The necessary requisite for overlapping ground tracks, therefore, reads:

$$\frac{\delta\lambda_0}{\delta\dot{\lambda}} = \frac{\Delta\Omega}{\omega_\oplus} \Leftrightarrow \Delta\Omega = \frac{\omega_\oplus \Delta u}{n - \omega_\oplus \cos i_c}. \quad (4.42)$$

The compliance with this condition is hence the key passage to generalise the results found for bi-static systems to other kinds of acquisition modes.

Figure 4.12 displays two satellite formations: the difference in the RAANs given by eq. 4.42 is imparted in the right plot, leading to the superposition of the ground tracks; no precaution is taken on the left to compensate for Earth’s rotation.

4.4.3. Extension to SAR tomography

In the context of interferometry, the availability of interferograms with different HoAs acquired simultaneously is a crucial benefit to be attributed to the use of distributed concepts (see Chapter 2). The same idea becomes even more valid for SAR tomography, where it is necessary to obtain a multitude of images to reconstruct the three-dimensional structure of the acquired scene (Reigber et al. [82]).

In the following, a method to extend the validity of the formulation tailored to interferometry introduced in section 4.3 to SAR tomography is presented.

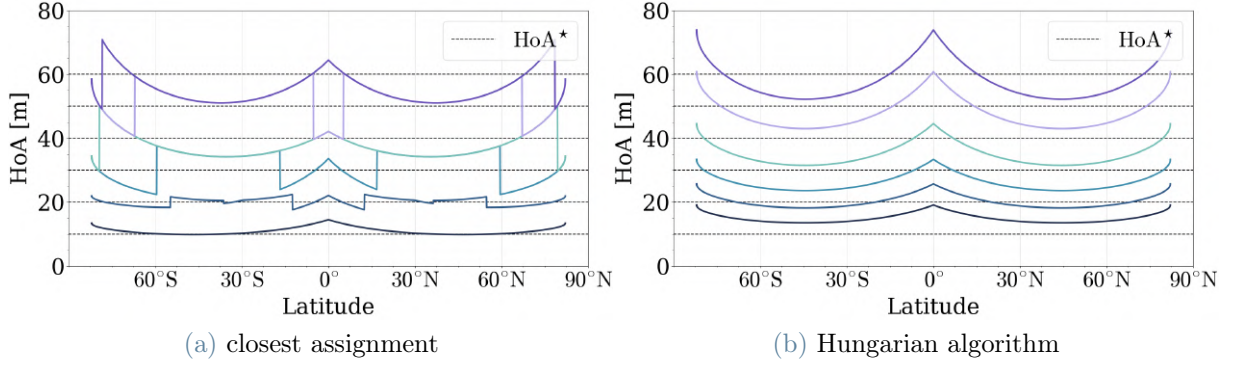


Figure 4.13: Examples of different assignment approaches for SAR tomography.

The key difference with respect to InSAR is that for tomographic SAR, there are more than two objective functions, each corresponding to a different HoA^* value. Simply selecting the best effective baseline available for each of them — as has been done so far — could result in the same pair being required to cover two distinct HoA^* . Indeed, such an assignment would lead to the creation of redundant baselines, which would not bring any contribution to the tomography. The mathematical formulation of the problem provided in section 4.3 is prone to this undesirable effect, as the min operator within the integral simply selects the baseline closest to the desired value. Figure 4.13 shows an example of two possible assignments for a four satellite formation targeting six different HoA^* . Specifically, in Figure 4.13a is reported a result obtained with the original formulation is reported: the overall L^2 norm error is minimised, but there are fewer than six different HoA available at each latitude. In Figure 4.13b an assignment consistent with tomographic imaging needs is displayed instead: here six different HoAs are always available.

To consider this aspect in the optimisation, the target values must be injectively linked to the satellite pairs. In other words, one needs to add the two following constraints:

- each target value is assigned exactly one satellite pair, and
- each satellite pair is used at most once per latitude value.

A convenient way to achieve this is through the definition of a mapping $\phi(u) : \mathcal{T} \rightarrow \mathcal{P}$, *i.e.*, an injective function from the set of target values $\mathcal{T} \ni \text{HoA}^*$ to the set of satellite pairs $\mathcal{P}$. The min operator in eq. 4.19 is thus substituted by the following:

$$\min_{\phi(u)} \sum_{\tau=1}^{N^{\circ} \text{HoA}^*} |\text{HoA}_{\phi(u)}(\delta\alpha, u) - \text{HoA}_{\tau}^*|^2. \quad (4.43)$$

The function $\phi(u)$ can be represented as a binary assignment matrix $A \in \{0, 1\}^{n \times p}$ where

n is the number of target values, p the number of satellite pairs, and the element $a_{ij}(u)$ is 1 if the i -th target value is assigned to the j -th satellite pair, 0 otherwise. This entails the solution of an assignment problem - also known as a linear sum assignment problem (Stanimirovic et al. [119]) - at each u value. As such, a variety of algorithms exist in the literature to provide efficient solutions. One of these is the Hungarian algorithm initially introduced by Kuhn [120], which is reported in its complete form in Appendix B. Below is provided a pseudo-algorithm describing the proposed methodology in its essential steps.

Algorithm 4.1 Optimal Assignment of Satellite Pairs to HoA^* over $\mathcal{U}$

Require: Functions $\{\text{HoA}_1, \dots, \text{HoA}_p\}$, targets $\{\text{HoA}_1^*, \dots, \text{HoA}_n^*\}$, interval $\mathcal{U}$

Ensure: Assignment matrices $A(u)$ with elements $a_{i,j}(u)$ for each $u \in \mathcal{U}$

1: **for** each $u \in \mathcal{U}$ **do**

2: Construct the cost matrix $C(u) \in \mathbb{R}^{n \times p}$ with elements

$$c_{ij}(u) = |\text{HoA}_j(x) - \text{HoA}_i^*|^2$$

3: Solve

$$\min_{A \in \{0,1\}^{n \times p}} \sum_{i=1}^n \sum_{j=1}^p a_{ij} c_{ij}(u)$$

subject to

$$\sum_{j=1}^p a_{ij} = 1 \quad \forall i, \quad \sum_{i=1}^n a_{ij} \leq 1 \quad \forall j$$

4: Store the optimal assignment $A(u)$

5: **end for**

6: **return** $\{A(u) : u \in \mathcal{U}\}$

In conclusion, having introduced the setup of the optimisation problem together with the safety-related constraints and the modifications allowing for extensions to different contexts, in Chapter 5 the adopted resolution strategies are investigated in detail.

5 | Optimisation Problem

This chapter presents the methods adopted and the results obtained for the optimisation problem introduced in Chapter 4. Initially, an analytical solution to the single-baseline problem is derived in section 5.1, using a flexible approach that can be tailored to achieve global coverage or to improve the performance over a more limited geographic area. Two strategies are subsequently proposed to scale the found solution up to multiple-baseline formations. The first one is presented in section 5.2, and is based on a heuristic leveraging the analytical result derived for two satellites. The solution obtained is then refined numerically in section 5.3. Finally, the same numerical methods are applied to investigate both the feasibility of SAR tomography via natural harmonic solutions of the HCW and the effect of the transformation to the Zero-Doppler frame.

5.1. Single-baseline Optimisation

In this section, the solution to the problem of maintaining a nominal HoA for a two-satellite formation is presented. Before dealing with the more complete Helix formation, the particular case of the xz-PCO, passively safe thanks to the application of the eccentricity/inclination vector separation method, is considered. In fact, as explained in Chapter 3 the xz-PCOS can be thought as a specific case of Helix formation for which the magnitudes of the relative eccentricity and inclination vectors are equal. The crucial aspect in the illustrated derivations is the choice of the set $\mathcal{U}$ of mean arguments of latitude for which one is interested in maintaining the nominal HoA. The case $\mathcal{U} = [0, \pi]$ is considered first: given the π -periodicity of the HoA with respect to u , it implies taking into account the entire orbit of the formation. Subsequently, it is shown how, starting from this scenario, one can obtain the optimal solution consistently with global coverage needs.

5.1.1. Complete orbit optimisation for an xz-PCO formation

In the following, the optimal solution for the particular case of an xz-PCO ($\delta e = \delta i, \varphi = \vartheta + k\pi$) and $\mathcal{U} = [0, \pi]$ is derived. Since the function to be integrated in eq. 4.15 is

π -periodic, the choice of the interval of mean arguments of latitude allows introducing an arbitrary shift in the integration variable without modifying the result of the integral, and thus the solution to the optimum problem. In fact, this removes any dependencies on the phase of the relative eccentricity vector φ in the integration, thereby reducing the problem to a single dimension. It is therefore necessary to derive only the optimal value of the magnitude of the relative eccentricity vector, which in turn corresponds to the magnitude of the inclination one $\delta e = \delta i$.

By introducing the variable $\xi = u - \varphi + (-1)^k \theta_l$, where the sign of θ_l depends on whether the angles φ and ϑ have an offset of π or not, the cost function defined in eq. 4.15 can be simplified as follows:

$$J = \int_0^\pi \left(\frac{K}{|\delta e (\cos(u - \varphi) \sin(\theta_l) + \sin(u - \varphi + k\pi) \cos(\theta_l))|} - \text{HoA}^\star \right)^2 du \quad (5.1)$$

$$\Leftrightarrow J = \int_0^\pi \left(\frac{K}{\delta e |\sin(u - \varphi + \theta_l)|} - \text{HoA}^\star \right)^2 du \quad (5.2)$$

$$\Leftrightarrow J = 2 \int_0^{\pi/2} \left(\frac{K}{\delta e \sin(\xi)} - \text{HoA}^\star \right)^2 d\xi \quad (5.3)$$

where the variable $\xi := u - \varphi + \theta_l$ is introduced for readability. The presence of an asymptote in eq. 4.4 in correspondence with $\xi = k\pi$, however, causes the integral to diverge. To address this issue, some modifications are required in the mathematical description. Recognising that performing interferometry is not feasible at those latitudes for which the HoA is indefinite, their contribution is removed by multiplying the integrand with suitable window functions.

Rectangular window: Given the argument of latitude corresponding to an asymptote, the rectangular window is defined as equal to 0 when $u \in (\alpha - \varepsilon, \alpha + \varepsilon)$ and 1 everywhere else. Hence, this removes from the integration all the mean arguments of latitude in a circular neighbourhood of the asymptote with radius ε while preserving the rest. By implementing this type of window, the objective function becomes:

$$J = \int_\varepsilon^{\pi/2} \left(\frac{K}{\delta e \sin \xi} - \text{HoA}^\star \right)^2 d\xi, \quad (5.4)$$

and the solution to the optimisation problem reads:

$$\delta e(\varepsilon) = \delta i(\varepsilon) = \frac{K}{\text{HoA}^\star \tan(\varepsilon) (\log(\cos \xi) - \log(\sin \xi))}. \quad (5.5)$$

Sinusoidal window: The rectangle function introduces an abrupt separation between considered and neglected latitudes, as up to $u = \varphi$ they are accounted for and then completely cancelled out. A possible alternative is to use a sinusoidal window $w_s = (\sin \xi)^{2p}$, which more smoothly decreases to 0 approaching the asymptotes, giving less importance to the latitudes the closer they are to $\xi = k\pi$. Note that $2p$ must be greater than or equal to 2, otherwise the integral would remain divergent. Also, the higher p , the less importance is given to the latitudes close to the asymptotes. Taking $p = 1$ it yields:

$$J = \int_0^{\pi/2} \sin^2 \xi \left(\frac{K}{\delta e \sin \xi} - \text{HoA}^* \right)^2 d\xi, \quad (5.6)$$

and the optimal magnitudes $\delta e, \delta i$ read:

$$\delta e = \delta i = \frac{K \pi}{2 \text{HoA}^*}. \quad (5.7)$$

Gaussian window: As a last alternative, the Gaussian window: $w_g(\xi) := 1 - \exp\left(-\frac{\sin^2(\xi)}{\sigma^2}\right)$ is considered. It can be regarded as the synthesis between the two others, as it is not as abrupt as the rectangle, but allows preserving more latitude values than the sinusoidal. By selecting σ , it is also possible to regulate how aggressive this window is in terms of neglecting latitudes close to the asymptotes. The solution to the optimisation becomes slightly more convoluted, but can still be expressed in closed form as follows:

$$\delta e(\sigma) = \delta i(\sigma) = \frac{\int_0^{\pi/2} \left(1 - \exp\left(-\frac{\sin^2(\xi)}{\sigma^2}\right) \right) \frac{d\xi}{\sin^2(\xi)} \frac{K}{\text{HoA}^*}}{\int_0^{\pi/2} \left(1 - \exp\left(-\frac{\sin^2(\xi)}{\sigma^2}\right) \right) \frac{d\xi}{\sin(\xi)}}. \quad (5.8)$$

Figure 5.1 shows how the different window functions shape the behaviour of the optimised HoA relative to the mean argument of latitude. Note that, with the considered window's parameters, the Gaussian window is the least selective with the latitudes considered, as it preserves the widest interval of arguments of latitude. However, the consequence is that the closeness to the target value $\text{HoA}^* = 10$ m of the HoA values attained along the orbit is worse compared to the other, more selective, alternatives.

5.1.2. Complete orbit optimisation for a Helix formation

Building upon the results found for the particular case of the xz-PCO, the Helix formation with $\delta e \neq \delta i$ is treated in the following. As expected, the limitation given by the presence

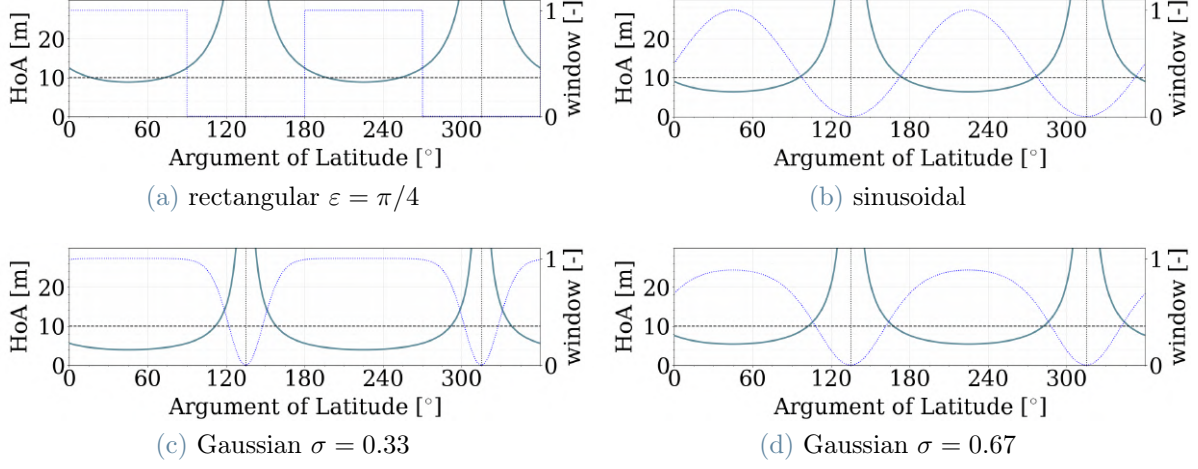


Figure 5.1: Optimised HoA under different window functions for xz-PCO, $\text{HoA}^* = 10$ m.

of the asymptotes for the HoA is also present in this case. Two alternative approaches are considered for minimising the difference between the nominal and the achieved height of ambiguity (HoA):

- The optimisation is carried out within a rectangular window of the orbit: a convenient choice as such windows allow to select regions of interest for Earth observation, like $u \in [0, \pi/2]$ for global coverage.
- Instead of minimising the RMSE between the achieved HoA and the nominal one, the optimisation minimises the deviation of their ratio from unity. This means that the effective baseline is employed as a surrogate variable, avoiding issues related to the presence of asymptotes in the HoA. The downside is that this method tends to favour shorter baselines due to the reciprocal relationship between HoA and $B_{\perp}$.

In the following, the analytical solution is presented for both cases, starting from the latter. Please note that, especially for the rectangular window case, only the most relevant steps of the derivation are reported here to preserve readability. The missing algebraic passages are collected in Appendix A.

Optimisation of the effective baseline

The variable $\beta = u - \varphi$ is introduced and $B_{\perp}^*$ is used to denote the effective baseline defined in eq. 4.2 corresponding to the nominal HoA^* . Then the modified objective function reads:

$$J_B = \int_0^{\pi} (|\delta e \cos \beta \sin \theta_l + \delta i \sin \beta \cos \theta_l| - B_{\perp}^*)^2 d\beta. \quad (5.9)$$

Computing the square and solving the indefinite integral, it yields:

$$J_B = 4\sqrt{\delta e^2 \sin^2 \theta_l + \delta i^2 \cos^2 \theta_l}(-2B^*) + 2\pi B_\perp^{*2} + \pi(\delta e^2 \sin^2 \theta_l + \delta i^2 \cos^2 \theta_l). \quad (5.10)$$

In order to minimise J_B , it is necessary to evaluate it in correspondence to its stationary points, which satisfy the condition:

$$\nabla J_B = \begin{bmatrix} \partial_{\delta e} J_B \\ \partial_{\delta i} J_B \end{bmatrix} = \begin{bmatrix} 2\pi\delta e \sin^2 \theta_l - 8B_\perp^* \frac{\delta e \sin^2 \theta_l}{\sqrt{e^2 \sin^2 \theta_l + \delta i^2 \cos^2 \theta_l}} \\ 2\pi\delta i \cos^2 \theta_l - 8B_\perp^* \frac{\delta e \cos^2 \theta_l}{\sqrt{e^2 \sin^2 \theta_l + \delta i^2 \cos^2 \theta_l}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5.11)$$

where $\partial_{(\cdot)}$ stands for $\frac{\partial}{\partial(\cdot)}$.

By definition, the Helix formation does not admit solutions with null δe or δi , thus the magnitudes can be cancelled, leading to two identical equations. As a consequence, there is no unique solution to the problem, but rather a set of optimal points drawing an ellipse in the $(\delta e, \delta i)$ plane defined by the following equation:

$$\frac{\delta e^2}{\left(\frac{4B_\perp^*}{\pi \sin \theta_l}\right)^2} + \frac{\delta i^2}{\left(\frac{4B_\perp^*}{\pi \cos \theta_l}\right)^2} = 1. \quad (5.12)$$

Optimisation with the rectangular window

The optimal solution employing the rectangular window is found analogously. The entire procedure is reported in Appendix A, and provides the locus of points $(\delta e, \delta i)$ given by:

$$\frac{\delta e^2}{\left(\frac{K}{\text{HoA}^* \sin \theta_l} \frac{\log(\tan^2(\varepsilon/2))}{2 \tan \varepsilon}\right)^2} + \frac{\delta i^2}{\left(\frac{K}{\text{HoA}^* \cos \theta_l} \frac{\log(\tan^2(\varepsilon/2))}{2 \tan \varepsilon}\right)^2} = 1, \quad (5.13)$$

which again consists of an ellipse in the δe - δi plane. The variables φ and ϑ , instead, do not influence the optimal solution. This is a consequence of the optimisation being carried out over an entire period of the HoA function, which makes the shift introduced by those two angles irrelevant. As an additional proof of the validity of the last result, it is observed that, by imposing $\delta e = \delta i$ and leveraging logarithmic properties, the solution perfectly matches that of the xz-PCO given in eq. 5.5.

Eq. 5.13 provides an efficient tool for the design of InSAR-optimal satellite formations. Indeed, not only does it give a simple relation to define optimal formations, but it also

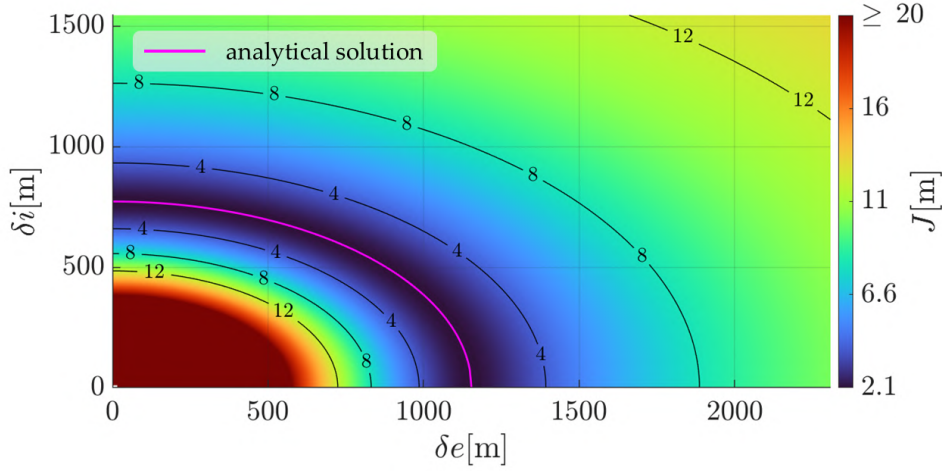


Figure 5.2: Normalised L^2 -norm error for $\text{HoA}^* = 20$ m with color scale saturated above 95th percentile, analytical curve traced in magenta. Results for wavelength $\lambda = 3$ cm, look angle $\theta_l = 33.8^\circ$, height 512 km.

provides a clear metric to measure how far a particular configuration is from optimality.

Figure 5.2 shows the square root of the cost function normalised by the integration interval: $\sqrt{\frac{J}{\pi - 2\varepsilon}}$, is reported in the $(\delta e, \delta i) \geq 0$ plane given $\varepsilon = \pi/4$. The said operations performed on J do not change the optimal result, but allow for a direct evaluation of the result from the chart, as the cost coincides with the continuous version of the *Root Mean Square Error* (RMSE). As expected, the analytically found ellipse in magenta perfectly matches the numerical minimum. Moreover, note that the iso- J curves are scaled versions of the optimal ellipse. This enables a convenient classification of different configurations, as their closeness to optimality can be evaluated from the scale ratio between the ellipse to which the corresponding point belongs and the optimal one.

At last, the choice of retaining only the rectangular window in the generalisation from the xz-PCO to the Helix formation is justified. Indeed, by tuning the parameter ε , it is possible to obtain the solution for cases more relevant to InSAR, *i.e.*, achieving global coverage or targeting specific latitudes rather than optimising over the whole orbit. This formulation, therefore, offers greater flexibility, which can be exploited by remote-sensing missions interested in the accurate mapping of specific geographic areas. An example is illustrated in Figure 5.3 where a value $\varepsilon \approx 75^\circ$ was chosen, achieving the best performance at the latitudes corresponding to Central Europe. This is because the optimisation interval is given by: $u \in [-45^\circ + \varepsilon, 135^\circ - \varepsilon]$. Note that the absolute error on the HoA is well below 25 cm over the area of interest.

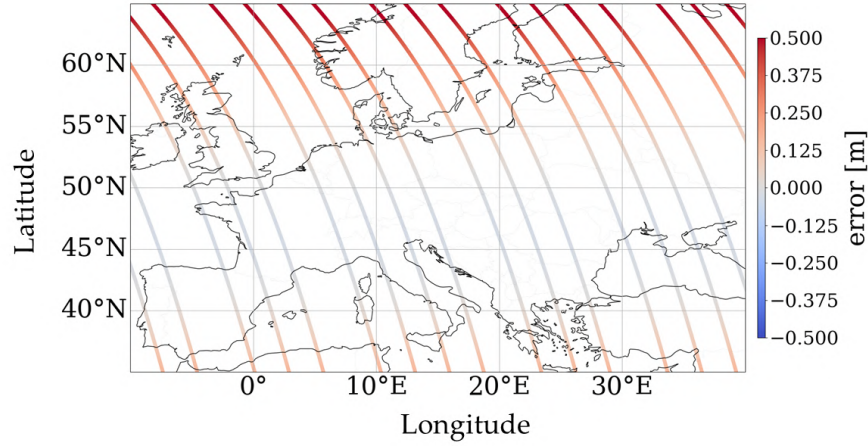


Figure 5.3: Absolute error for the HoA obtained with the rectangular window over the Central Europe region.

5.1.3. Global coverage with a Helix formation

As presented in Chapter 4, the choice $\mathcal{U} = [0, \pi]$ leads to redundancies in the observed regions. More precisely, using this integration interval, each latitude value is considered twice: once when the formation flies over it in the ascending part of the orbit and once in the descending part. This certainly has the advantage of acquiring each point on the Earth's surface from two different imaging geometries, which could be useful in the case of particularly rough surfaces, since it would allow for attenuating the effects of layover and shadowing (see Chapter 2). However, these represent isolated cases, for which it would be more convenient to locally adapt the geometry. For a mission aiming at global coverage, it is therefore much more appropriate to consider each latitude only once in the integral calculation. This can be easily achieved by choosing $\mathcal{U} = [0, \pi/2]$. Indeed, evaluating the cost function in eq. 5.1.2 over the entire orbit (*i.e.*, $\mathcal{U} = [0, 2\pi]$), the presence of asymptotes in the HoA curve required modifications to the formulation, as done with the introduction of suitable windows in section 5.1.1. Instead, by selecting an interval of width $\pi/2$ (corresponding to $\varepsilon = \pi/4$) the integral in eq. 4.15 is ensured to remain well defined. Under this condition, the original cost function can be applied directly, and the global coverage solution follows from eq. 5.13. The optimal result obtained in this case is precisely that displayed in Figure 5.2. The restriction of the integration interval, however, has a downside: over the full orbit, the periodicity of $\text{HoA}(u)$ made the solution independent of the orbital phase angles φ and ϑ . Once the integration is limited to a shorter interval, the exact alignment of the $\text{HoA}(u)$ curve with respect to that interval

becomes relevant. Such an alignment depends on the phases of the $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ vectors, which are no longer irrelevant for the optimal solution. This new dependence on the phases φ and ϑ can be readily taken into account by observing that the curve $\text{HoA}(u)$ must be symmetric within the interval $u \in [0, \pi/2]$ at optimality, which entails that its minimum is found for $u = \pi/4$.

The corresponding value of φ can then be obtained leveraging the following trigonometric identity:

$$E \sin(u - \varphi) + I \cos(u - \varphi) = \sqrt{E^2 + I^2} \cos(u - \delta), \quad \delta = \varphi + \arctan \frac{E}{I}, \quad (5.14)$$

whereas $\vartheta = \varphi \pm \pi$ for a Helix formation. Since the minimum of the $\text{HoA}(u)$ function is obtained at $u = \delta + k\pi$, the phases must be compliant with the requirement $\delta = \pi/4$. In conclusion, the ROEs must satisfy the following condition to minimise the error between the actual and nominal HoA values globally:

$$\begin{cases} \sqrt{\delta e^2 \sin^2 \theta_l + \delta i^2 \cos^2 \theta_l} = \frac{2}{\log(3 + 2\sqrt{2})} \frac{K}{\text{HoA}^\star} \approx 1.1346 \frac{K}{\text{HoA}^\star} \\ \varphi + \arctan \left(\frac{\delta e}{\delta i \tan \theta_l} \right) = \frac{\pi}{4} \end{cases} \quad (5.15)$$

As a result of the optimisation, two of the three decision variables $\delta e, \delta i, \varphi$ (ϑ being constrained by the Helix definition) are fixed, whereas the third can vary. More precisely, it is always possible to obtain the solution by fixing one of the two magnitudes in a completely arbitrary way and solving the problem to find the other magnitude and φ . This is because the arctangent operator induces the presence of constraints on φ , which has to respect the condition:

$$-\frac{\pi}{4} < \varphi < \frac{3\pi}{4}. \quad (5.16)$$

At last, a comparison is conducted between the solution found with the two methods: optimisation with HoA and optimisation with the effective baseline. Also in this second case it is necessary to restrict the integration interval to a width of $\pi/2$ and appropriately fix the phase φ . By doing so, the optimality condition obtained by cancelling the gradient reads:

$$\sqrt{\delta e^2 \sin^2 \theta_l + \delta i^2 \cos^2 \theta_l} B_\perp^\star = \frac{4\sqrt{2}}{2 + \pi} B_\perp^\star \approx 1.1002 \frac{K}{\text{HoA}^\star}, \quad (5.17)$$

which, as expected, is slightly lower than the exact one because of the reciprocal relationship between the HoA and the effective baseline.

5.2. Multi-baseline Optimisation: Heuristic Solution

As explained in Chapter 4, the presence of the min operator and the absolute values in the objective function does not allow for analytically determining the gradient and, in general, makes it difficult to solve the problem using gradient-based methods. In this section, an approximate solution based on the exact result obtained for a two-satellite formation is provided. The fundamental idea is to employ the following heuristic:

1. In a formation consisting of a chief c and two deputies d_1, d_2 , the two deputies are positioned so that the pairs $c - d_1, c - d_2$ satisfy the condition in eq. 5.15 for each of the two objective values HoA^* .
2. The relative RMSE for each of the two HoA^* is computed.
3. A new deputy d_i is added to the formation so that the pair $c - d_i$ is optimised with respect to the HoA^* with the worst relative RMSE; if the two are equal, the decision is based on the absolute RMSE.
4. If multiple pairs $c - d_j$ share the same target HoA^* , their phases φ_j are recomputed so that the functions HoA_j are equally spaced in the interval $u \in [0, \pi/2]$. This is done by imposing that they respectively reach their minimum at the argument of latitude δ_j , given by:

$$\delta_j = \frac{\pi}{4N}(2j - 1)$$

5. The procedure is iterated until a formation with the desired number of deputies is achieved.

In its essence, this algorithm ensures that all chief-deputy baselines are distributed to provide the most homogeneous coverage of the objectives. However, by its definition, the proposed heuristic does not consider pairs consisting of two deputies.

The usefulness of this approach also lies in the provision of an explicit upper bound on the error that can be committed by any optimal solution. In fact, this can be calculated by selecting the value of ε so that each of the N^* pairs covering a HoA^* is active for a range of arguments of latitude with length $\pi - 2\varepsilon = \frac{\pi}{2N^*}$, from which $\varepsilon = \frac{\pi}{2}(1 - \frac{1}{2N^*})$. By substituting this value into eq. 5.13, the upper bound of the RMSE yields:

$$\text{RMSE}(N^*) = \frac{\pi}{2N^*} - \frac{2}{b^*} \tanh^{-1} \left(\frac{2 \sin \frac{\pi}{4N^*}}{1 + \sin^2 \frac{\pi}{4N^*}} \right) + \frac{2}{b^{*2}} \tan \frac{\pi}{4N^*}, \quad (5.18)$$

where

$$b^* = -\frac{2}{\tan(\frac{\pi}{2}(1 - \frac{1}{2N^*}))} \frac{1}{\log(\tan^2(\frac{\pi}{4}(1 - \frac{1}{2N^*})))}$$

and $\tanh^{-1}(\cdot) := \frac{1}{2} \log\left(\frac{1+(\cdot)}{1-(\cdot)}\right)$ is the inverse hyperbolic arctangent function.

In Figure 5.4, it is shown how the RMSE relative to a single HoA^{*} behaves as the number of satellite pairs increases. In practice, given a set of N chief-deputy pairs, if M of these are dedicated to one objective and $N-M$ to the other, the graph provides the upper bound for the percentage RMSE independently of the particular value HoA^{*}. From the chart, it is already evident that a formation of three satellites certainly guarantees an error of less than 11% on both targets, since it is possible to freely assign a different chief-deputy pair to each of the two objectives.

The results attained with the heuristic proposed are listed in the Tables 5.1, 5.2 for different ratios HoA₁^{*} : HoA₂^{*}. As expected, the RMSE is always equal to or better than the predicted upper bound from Figure 5.4. More precisely, the equality holds in the cases where no deputy-deputy pair is used, like the example shown in Figure 5.5. The chart shows the HoA as a function of the geocentric latitude ϕ : since it was obtained for an SSO with inclination $\sim 98^\circ$, it is defined in the interval $\phi \in \sim [8^\circ, 82^\circ]$. As explained in Chapter 4, the ROEs of the optimal solution are fixed apart from one degree of freedom whose choice does not influence the result in terms of HoA, but changes the geometry of the formation. Figure 5.7 shows a possible way to obtain the result in Figure 5.6, in which the ROEs were chosen to obtain xz-PCOs. Two satellites follow the same trajectory relative to the chief: the radius of the orbit depends indeed only on the magnitudes of the eccentricity and relative inclination vectors, which, for two deputies with the same target HoA^{*}, are identical. Furthermore, the peculiar rhomboid geometry reflects the need to introduce differences in the phases of these vectors, so as to homogeneously distribute the

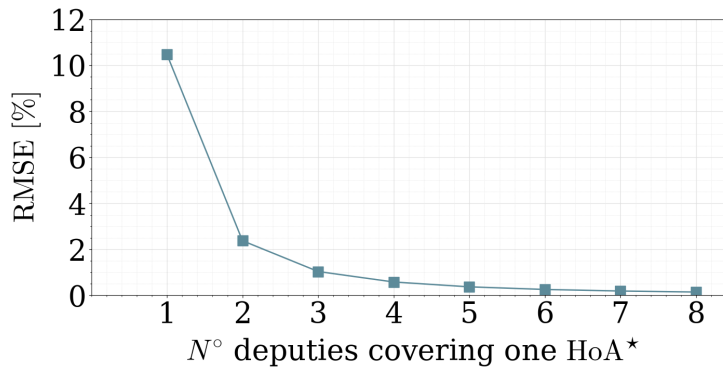


Figure 5.4: Upper bound of the relative RMSE given the number of dedicated satellite pairs.

different values of effective baselines along the latitude range. For this reason, the two innermost deputies are in symmetrical positions with respect to the other: each of them must, in fact, handle half of the mean arguments of latitude from the equator to the poles, whereas the outermost one must manage the entire hemisphere alone ¹.

$\text{HoA}_1^* : \text{HoA}_2^*$	$B_{\perp, \max}(c - d_1)$ [m]	$B_{\perp, \max}(c - d_2)$ [m]	$B_{\perp, \max}(d_1 - d_2)$ [m]	RMSE(HoA_1^*)[%]	RMSE(HoA_2^*)[%]
1 : 2	1166.65	583.32	583.32	10.45	10.45
1 : 3	1166.65	388.88	777.77	10.45	10.45
1 : 4	1166.65	291.66	874.99	10.45	10.45
1 : 5	1166.65	233.33	933.32	10.25	10.45

Table 5.1: Results from the heuristic optimisation for a three satellite formation; $c - d_i$ indicates the chief- i -th deputy pair, d_i, d_j the pair i -th, j - deputy.

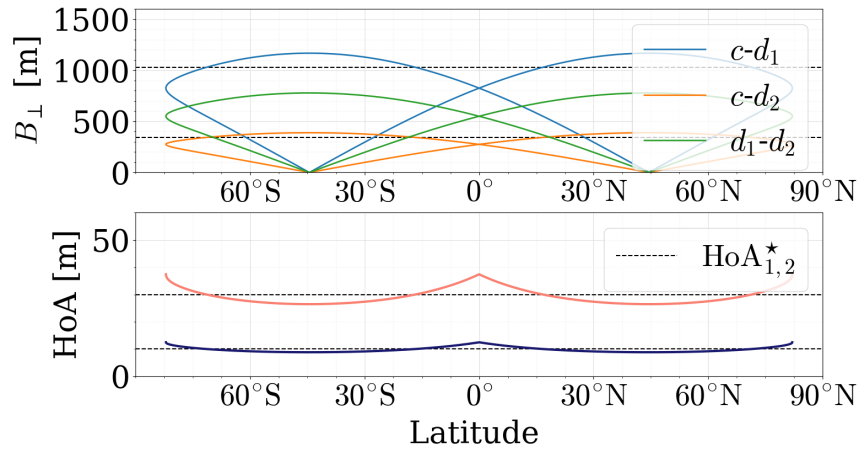


Figure 5.5: Optimised HoA from the heuristic solution for a three satellite formation.

$\text{HoA}_1^* : \text{HoA}_2^*$	$B_{\perp, \max}^{c-d1}$ [m]	$B_{\perp, \max}^{c-d2}$ [m]	$B_{\perp, \max}^{c-d3}$ [m]	$B_{\perp, \max}^{d1-d2}$ [m]	$B_{\perp, \max}^{d1-d3}$ [m]	$B_{\perp, \max}^{d2-d3}$ [m]	RMSE(HoA_1^*)[%]	RMSE(HoA_2^*)[%]
1 : 2	528.17	1166.65	528.17	708.14	285.84	708.14	10.45	2.31
1 : 3	352.11	1166.65	352.11	852.06	190.56	852.06	10.45	2.36
1 : 4	211.27	1166.65	211.27	974.82	114.33	974.82	10.33	2.36
1 : 5	528.17	1166.65	528.17	708.14	285.84	708.14	9.00	2.36

Table 5.2: Results from the heuristic optimisation for a four satellite formation; $c - d_i$ indicates the chief- i -th deputy pair, d_i, d_j the pair i -th, j - deputy.

¹The relevant orbital and imaging parameters used as benchmarks for the representative cases illustrated are also collected in Chapter 6, Table 6.3, where the maintenance of the nominal relative configurations is assessed.

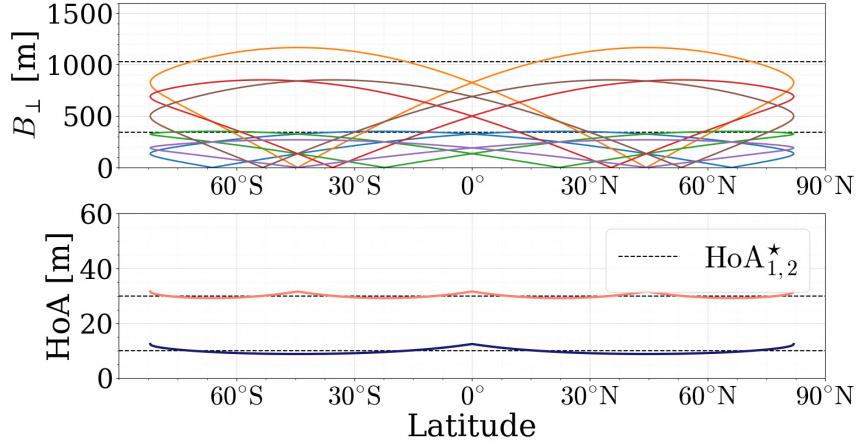


Figure 5.6: Optimised HoA from the heuristic solution for a four satellite formation

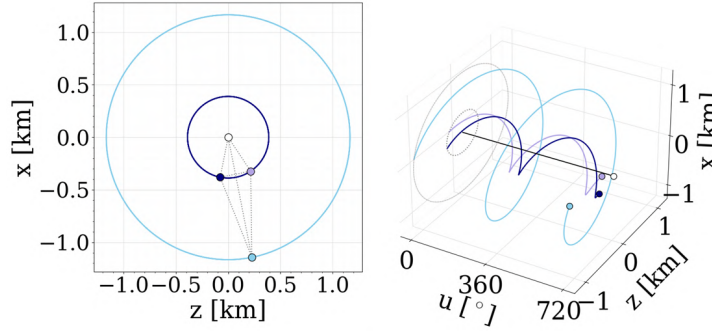


Figure 5.7: Circular projected relative trajectory solution to the heuristic for a four satellite formation and objective ratio 1 : 3. Left: time evolution; Right: projection onto the xz plane.

5.3. Multi-baseline Optimisation: Numerical Solution

This section illustrates the numerical methods employed in the resolution of the optimisation problem, aiming to provide an improvement over the heuristic approach. Specifically, methods based on genetic algorithms are considered, given their simplicity of implementation and their superiority over gradient-based methods in handling problems with multiple local minima. Indeed, the one considered falls into this category: given a set of effective baselines, a local minimum is reached whenever one or more of them reach the optimal value for a target HoA^* . At this point, only completely changing the assignments of each baseline to the target values would allow for further improvement. The consequence is the introduction of multiple disconnected local minima.

First, single-objective optimisation algorithms are considered: they rely on one of the

scalarisations illustrated in Chapter 4. Additionally, other methods capable of directly generating (an approximation of) the Pareto front (Veldhuizer et al. [115]) are taken into account. In the remainder of this section, the results following an extension of the problem formulation to SAR tomography are shown. Finally, the changes induced by the transformation into the zero-Doppler frame are illustrated.

5.3.1. Reduction to single-objective optimisation

Following scalarisation, the multi-objective problem can be conveniently reduced to a single-objective one. To then obtain the Pareto front, it would be required to separately compute each non-dominated solution for different values of the parameters of the scalarisation [117]. To date, a wide variety of algorithms for solving single-objective optimisation problems are available, some more suitable than others depending on the specific characteristics of the problem itself, such as non-convexity, dimensionality, and presence of local minima. In general, the aim is to obtain a good combination of efficiency in the solution and a lower probability of getting stuck in local minima. For this reason, algorithms such as the well-known *Nelder-Mead* or *Simplex* method [121], which, although not based on the calculation of the gradient, still suffer from this issue. Furthermore, the problem has a low number of objectives (two, one for each of the HoA^{*}) and a size (*i.e.*, number of decision variables) that depends on the number of satellites considered, medium to low in the most relevant cases. Therefore, some of the most commonly used algorithms tailored to this type of problem are employed in the resolution, and a comparison of the results obtained with each of them is provided². The considered algorithms are listed below.

- *Covariance Matrix Adaptation Evolution Strategy* (CMAES): This is one of the most well-established and first evolution strategy algorithms to be developed (see Hansen and Ostermeier, 2001 [122]), considered as the benchmark against which to compare the others.
- *Self-Adaptive Differential Evolution* (SADE): This algorithm dynamically adapts its control parameters (mutation and crossover strategies) during the run, thus removing the need for manual fine-tuning (see Qin et al. [123] and Adnan et al. [124] for the foundational algorithm and application overview).
- *Artificial Bee Colony* (ABC): Its main advantage is the lower sensitivity to local minima. However, it generally provides slower convergence than the others (see

²All the experiments were conducted on a machine with a Intel® Core™ Ultra 7 165H (16 cores, 22 Threads, 3.8 Hz), 32 GB RAM, running Windows 11 (64-bit). The code was implemented in Python 3.12.3 with Pygmo 2.19.7.

Algorithm	Three satellites			Four satellites		
	RMSE HoA ₁ [*] [%]	RMSE HoA ₂ [*] [%]	elapsed time	RMSE HoA ₁ [*] [%]	RMSE HoA ₂ [*] [%]	elapsed time
CMAES	7.0223	10.45	4.39 min	6.35	4.41	6.91 min
SADE	7.0228	10.45	3.51 min	3.60	2.37	6.78 min
ABC	7.0234	10.45	9.31 min	3.60	2.37	17.83 min
SADE + Compass	7.02	10.45	4.90 min	3.59	2.36	8.50 min

Table 5.3: Convergence speed and effectiveness of single-objective optimisation algorithms, computed after 1000 generations of an initial population of 200 individuals.

Karaboga [125]): a result that is confirmed in the comparison.

After having performed the global search employing either one of those three, a further refinement of the solution is achieved by employing the local *Compass Search* algorithm (see Kolda et al. [126]).

The results attained through a linear scalarisation³ of the initial multi-objective problem are presented in the following. Specifically, they have been found considering parameters of the scalarisation proportional to the target HoAs^{*} : $w_i = \frac{\text{HoA}_i^*}{\text{HoA}_1^* + \text{HoA}_2^*}$, so as to give equal relative weight to each of the two objectives. Table 5.3 summarises the performance analysis of the single-objective optimisation algorithms described above. Note that ABC is slightly slower than the others, while SADE is the most efficient, especially for a bigger problem size. Furthermore, there is a significant reduction of the Root Mean Square Error (RMSE) over the solution found with the previously proposed heuristic, which is due to the better exploitation of the deputy-deputy baselines.

Figure 5.8 displays the effective baselines and selected HoA found with the combination of SADE and Compass algorithms. The same for Figure 5.9, which refers to a four-satellite formation instead. A convenient representation of the performance related to the specific geographical areas acquired is proposed in Figures 5.10 and 5.11, which show the absolute error with respect to the nominal HoA^{*} value with respect to the ground-track points over a time span of five orbital periods, for formations of three and four satellites, respectively. Note that the problem setup used in this case involves acquiring images of the Northern Hemisphere in the ascending part of the orbit and vice versa for the Southern Hemisphere, achieving global coverage. In these two quarters of the orbit, the error remains in the order of a meter or less, while the points where the satellites are located in an unfavourable geometry (*i.e.*, which lead to vanishing effective baselines) are

³It is reminded that the single-objective function obtained from a linear scalarisation of a multi-objective problem with objective functions $F_1, F_2, \dots, F_N$ is given by $\sum_{i=1}^N w_i F_i$, with the scalar quantities w_i being the parameters of the scalarisation

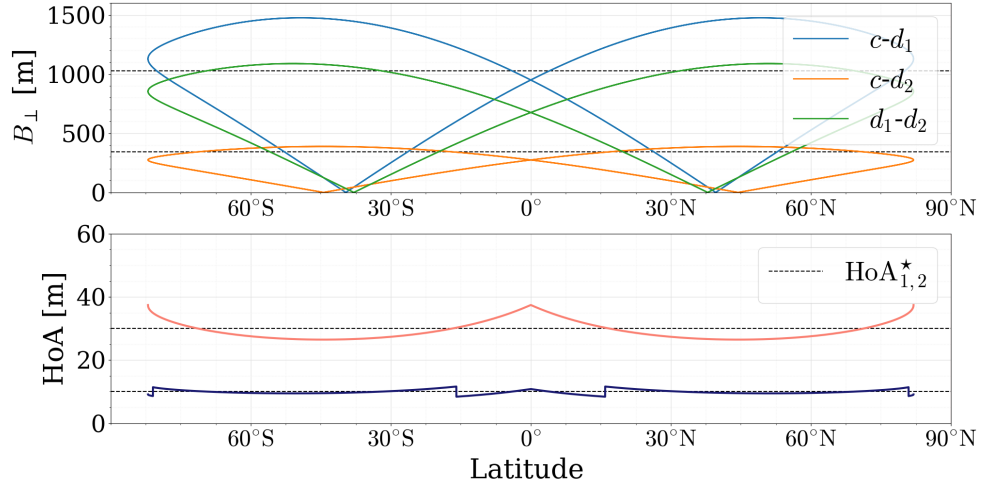


Figure 5.8: Optimal solution for a three satellite formation in SSO: SADE + Compass algorithms with $\text{HoA}_{1,2}^* = 10\text{m}, 30\text{m}$. Top: effective baselines; Bottom: chosen HoA.

not used for imaging. Naturally, the reverse setup would have been equally possible.

The previously presented results are extremely promising, as they demonstrate that even with a reduced number of satellites, it is possible to achieve a uniform HoA throughout the mapping of the entire Earth’s surface. This, in turn, demonstrates that multi-baseline formations can effectively overcome the limitation of the robustness/accuracy trade-off affecting single-baseline concepts.

5.3.2. Multi-objective optimisation

Section 5.3.1 demonstrated how the linear scalarisation with proportional weights can produce more than satisfactory results; however, it does not allow for exhaustively examining all the trade-off solutions between the two objectives. Hence, there is a need for algorithms capable of efficiently generating homogeneously distributed Paretian solutions. The following ones are thus taken into account.

- *Non-Dominated Sorting Genetic Algorithm 2* (NSGA2): This is a seminal algorithm for multi-objective optimisation, well-known for maintaining diversity and elitism (see Deb et al. [127]). As such, it is regarded as the standard throughout this work.
- *Multi-Objective Evolutionary Algorithm based on Decomposition* (MOEAD): As explained previously, one of the simplest ways to generate the Pareto front is to solve single-objective optimisation problems arising from scalarisation with different parameter values, a procedure that, in itself, would be extremely computationally expensive. The MOEAD algorithm (see Zhang and Lee [128]) is based precisely

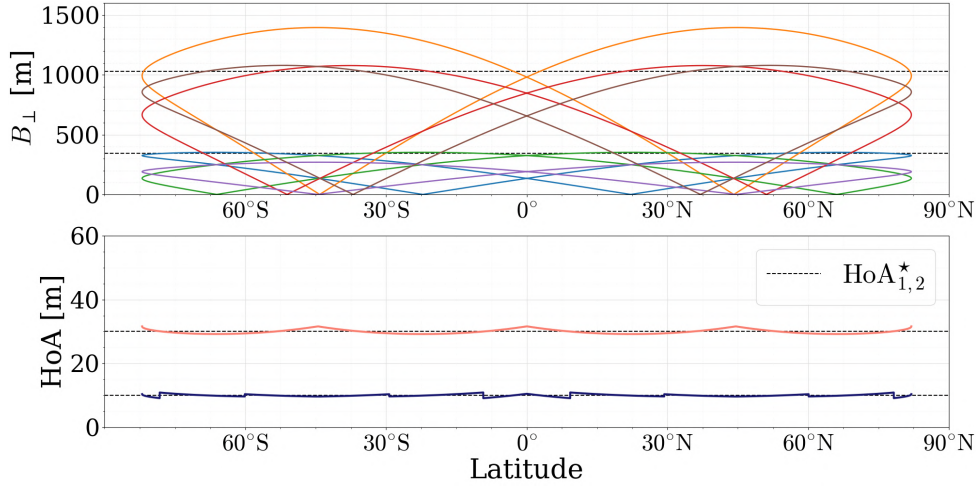


Figure 5.9: Optimal solution for a four satellite formation in SSO: SADE + Compass algorithms with $HoA_{1,2}^* = 10\text{m}, 30\text{m}$. Top: effective baselines; Bottom: chosen HoA.

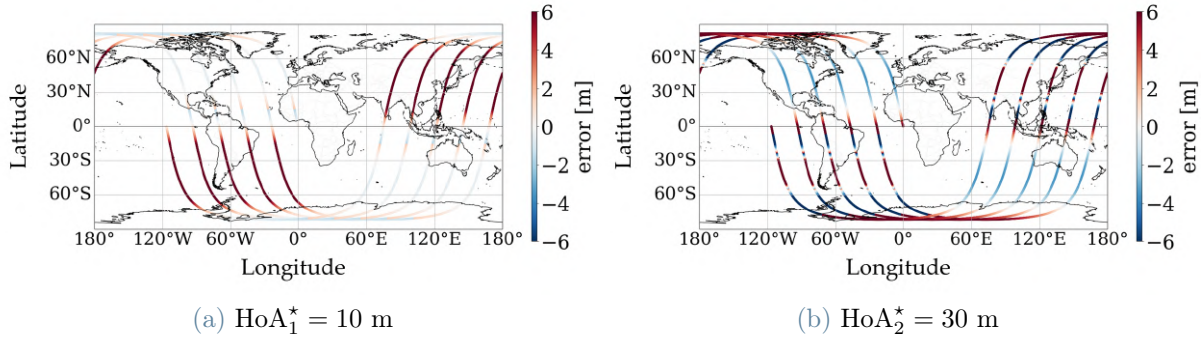


Figure 5.10: Absolute error versus ground track for a three-satellite formation over five periods; colour scale saturated above 6 m.

on increasing the efficiency of this approach by collaboratively solving scalar problems obtained via the described linear and Chebyshev scalarisations. As such, it is expected to confirm the solutions obtained via the single-objective formulation described in the previous section. It is also foreseen to maintain elitism in the results, but since diversity is heavily controlled by weight selection, a less homogeneous distribution of non-dominated solutions is awaited.

- *Non Dominated Sorting Particle Swarm Optimisation* (NSPSO): It is the multi-objective extension of the well-known particle swarm optimisation (see Liu [129]), fast, but difficult to set up so as to maintain diversity and elitism in the Pareto solutions.

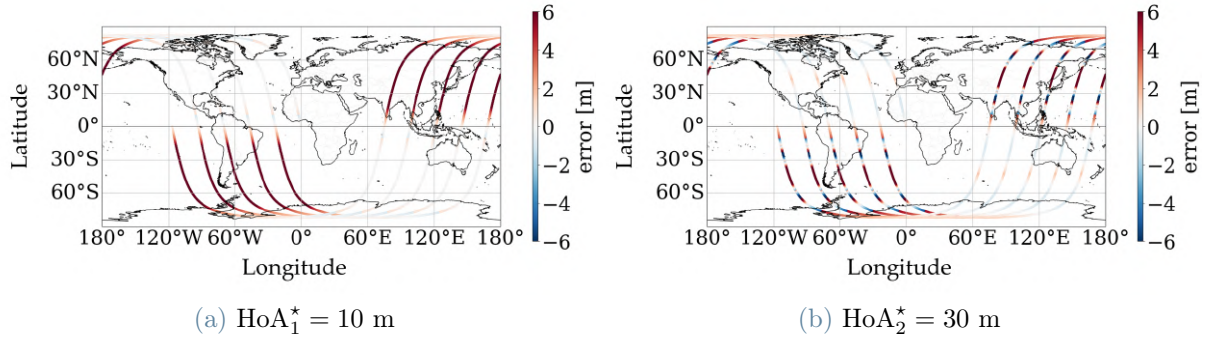


Figure 5.11: Absolute error versus ground track for a four-satellite formation over five periods; colour scale saturated above 6 m.

Algorithm	elapsed time (three satellites)	elapsed time (four satellites)
NSGA2	3.67 min	10.41 min
MOEAD	2.75 min	8.30 min
NSPSO	2.86 min	7.53 min

Table 5.4: Convergence speed of multi-objective optimisation algorithms, computed after 1000 generations of an initial population of 200 individuals.

Table 5.4 summarises their performance in terms of speed, for which no significant difference is observed. On the other hand, the results are highly dissimilar regarding the closeness to the true Pareto front, displayed in Figure 5.12 for three satellites and in Figure 5.13 for four satellite formations. In particular, the Paretian solutions of the three proposed algorithms and of a fourth obtained by their hybridisation are shown. The comparison reveals that because of its greater setup complexity, the NSPSO is outperformed by the other two in terms of closeness to the true Pareto front. NSGA2 generates the most homogeneous distribution of Paretian solutions and has been proven to be effective even after a smaller number of generations. Finally, MOEAD tends to find points more concentrated in clusters, especially after a short time, but it also identifies non-dominated solutions not recovered by the other two algorithms.

The derived Pareto fronts are indeed of great usefulness in the context of InSAR mission design. For some systems, maintaining a lower HoA for greater accuracy could be a much more stringent requirement than the robustness to phase unwrapping afforded by a higher HoA. As an example, this could hold for formations that use short baselines or long wavelengths, which lead to already large HoAs. Conversely, for other types of missions,

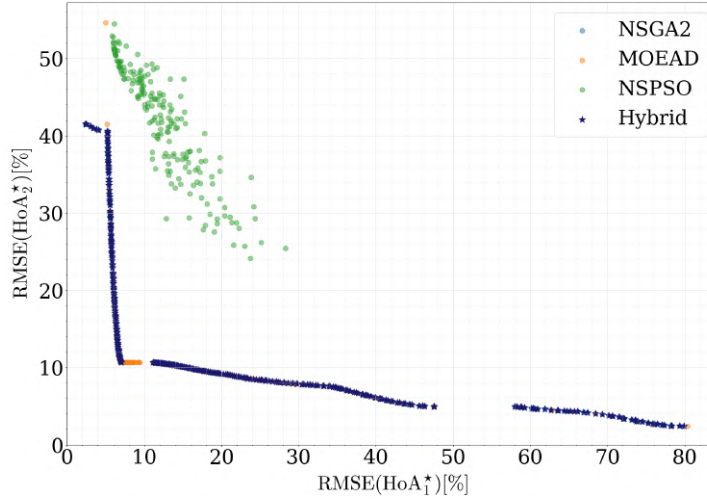


Figure 5.12: Comparison between algorithms for Paretoian solutions; three satellite formation with $\text{HoA}_{1,2}^* = 10\text{m}, 30\text{m}$.

the opposite could be equally true. Hence the convenience of the proposed charts, which enable selecting the priority to give to each objective tailored to mission-specific needs.

5.3.3. SAR tomography optimisation

In extending the proposed approach to tomography, a larger number of HoA^* is required. The satisfactory results obtained with two objectives, especially for formations with more than three satellites, suggest that it is indeed possible to increase the number of HoA^* and still achieve acceptable performance in terms of coverage of the nominal HoA values. While for interferometry the role of the two HoA s was fundamentally different (one for accuracy and one for phase unwrapping), in the case of tomography, they are all equivalent (Knael and Cardillo [79]). Since there is no longer interest in trade-offs between objectives with different purposes, it is appropriate to no longer consider the problem as a multi-objective optimisation. In this case, in fact, the greatest possible uniformity between the errors related to each objective is sought. For this reason, only the linear scalarisation with coefficients inversely proportional to each HoA^* is considered. Furthermore, since the combination of SADE and Compass Search has proven to be the best choice in terms of balancing reach of optimality and time efficiency, it is the only one employed here. Theoretically, it would be possible to arbitrarily choose the reciprocal ratios between all HoA^* , but two specific options are considered in the following, allowing for homogeneity in the distribution of the HoA [82]. The first one involves target values ordered in an arithmetic progression, whereas the second in geometric progression. A mere three satellite formation does not represent a meaningful scenario, since a maximum

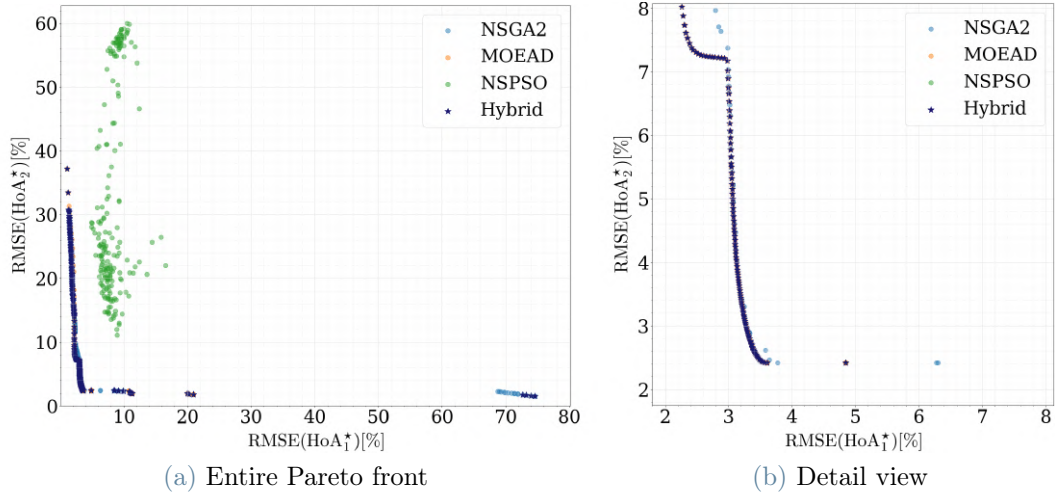


Figure 5.13: Comparison between algorithms for Paretian solutions; four satellite formation with $\text{HoA}_{1,2}^* = 10\text{m}, 30\text{m}$.

of three baselines could be achieved, only two of which are independent. Hence, it is considered a formation of four satellites to allocate the six available baselines in the best possible way. Starting with the arithmetic progression, one can consider a target HoA_1^* and subsequently add new values until the complete set: $\text{HoA}_1^* \cdot \{1, 2, 3, 4, 5, 6\}$ is reached. For each of the partial sets, the optimality problem is then solved with the Hungarian algorithm, illustrated in Chapter 4, replacing the min operator in the objective function. An analogous approach is adopted for the geometric progression, starting with a value HoA_1^* and progressively extending the problem to include: $\text{HoA}_1^* \cdot \{1, 2, 4, 8, 16, 32\}$.

The results are reported in Tables 5.5, 5.6 and in the Figures 5.14, 5.15, respectively for arithmetic and geometric progressions. They show a clear worsening in performance in terms of the RMSE as the number of targets increases, as expected. However, this deterioration is much more pronounced in the case of geometric progression, while the arithmetic progression maintains acceptable values for up to six targets. The reason for this discrepancy lies in the fact that adopting effective baselines in geometric progression entails some satellites being very close and others at a significantly larger distance, making it impractical to achieve effective baselines in between. The arithmetic progression, on the other hand, requires satellites more evenly distributed in the x-z plane, allowing for better performance.

RMSE [%]				
	3 objectives	4 objectives	5 objectives	6 objectives
HoA_1^*	10.53	10.81	30.40	53.69
2 HoA_1^*	4.38	6.41	6.03	10.92
3 HoA_1^*	2.50	7.84	9.03	15.02
4 HoA_1^*	–	6.29	8.50	14.92
5 HoA_1^*	–	–	7.19	10.75
6 HoA_1^*	–	–	–	10.55

Table 5.5: Results of the optimisation for SAR tomography; objectives in arithmetic progression.

RMSE [%]				
	3 objectives	4 objectives	5 objectives	6 objectives
HoA_1^*	8.22	19.65	30.94	72.95
2 HoA_1^*	5.81	9.90	16.07	10.61
4 HoA_1^*	4.47	9.58	24.80	48.45
8 HoA_1^*	–	8.45	14.62	24.17
16 HoA_1^*	–	–	10.68	15.44
32 HoA_1^*	–	–	–	10.72

Table 5.6: Results of the optimisation for SAR tomography; objectives in geometric progression.

5.3.4. Results in Zero-Doppler frame

As anticipated in Chapter 3, from the interferometric standpoint, it is more appropriate to represent the relative dynamics in the ZD frame. In this case, the considerations made on the possible reduction of the problem's size are no longer valid. In fact, the equations of motion become coupled, making it impossible to simply consider the dynamics in the x-z plane. While eq. 4.29 allows to reformulate the effective baseline as a function of the latitude argument in terms of modulus and phase (both independent of u) in the RTN, this no longer holds after the change of reference frame. Consequently, the optimisation becomes more computationally demanding, since each ROE represents a separate decision variable. Additionally, the HoA ceases to be π -periodic, as its behaviour is different in

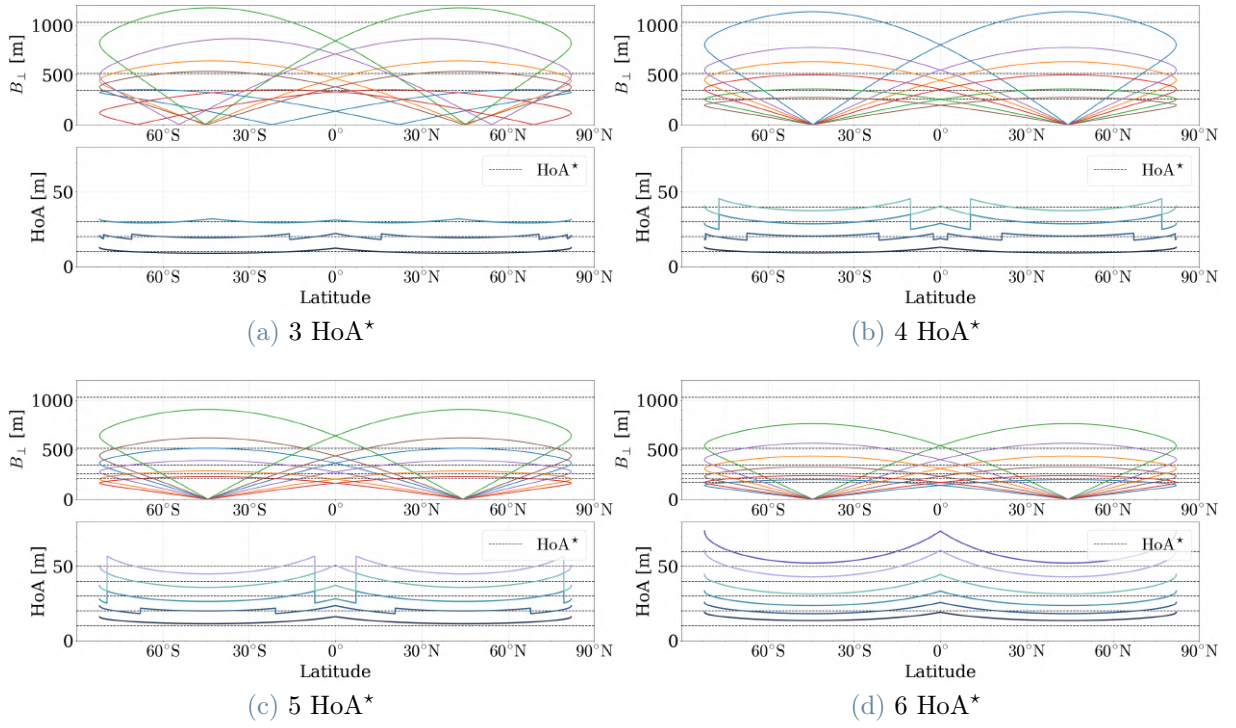


Figure 5.14: Optimised HoA for targets in arithmetic progression.

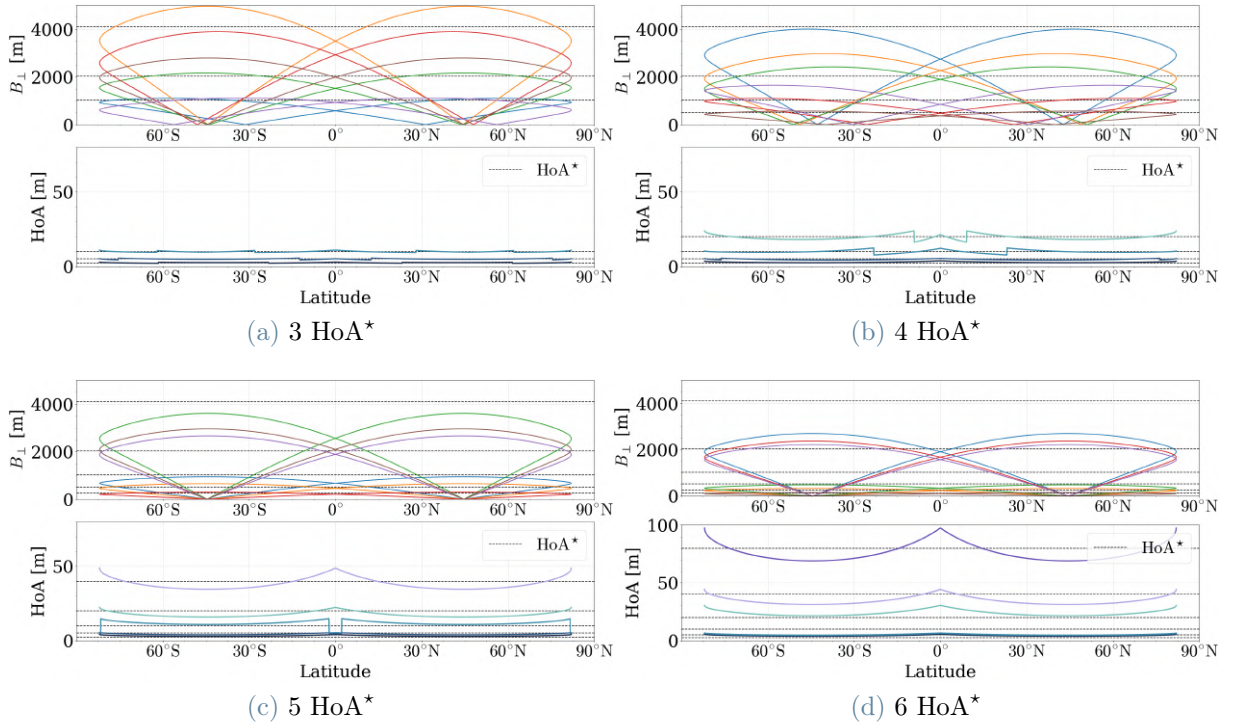


Figure 5.15: Optimised HoA for objectives in geometric progression.

Orbit type	minimum distance [m]	RMSE(HoA ₁ [*])[%]	RMSE(HoA ₂ [*])[%]	Computational time [min]
RTN-optimised	269,43	5.54	8.02	4.90
ZD-optimised	68.14	4.97	7.44	35.40

Table 5.7: Optimisation in the Zero-Doppler frame. The computational time is evaluated for the SADE algorithm with 1000 generations and an initial population of 200 individuals.

the two hemispheres because of the influence of Earth's rotation speed. The integration is thus slowed down because the two intervals: $u \in [0, \pi/2]$ and $[\pi, 3\pi/2]$ must be considered separately. At last, as better detailed in Chapter 6, the generalised Helix formation is compatible with optimality, while the same cannot be ensured for the formulation in the ZD frame.

In Table 5.7, the optimal results obtained for one of the cases also considered in the classical scenario (see Table 5.3) are shown, with a HoA^{*} ratio of 1 : 3 for a formation of three satellites. The best solution found by the genetic algorithm is superior to that obtained previously in terms of RMSE due to the greater number of degrees of freedom available: the Earth's rotation is indeed exploited to improve performance. The problem, however, is that the minimum distance between any pair of satellites decreases significantly compared to the cases examined in the classical representation. Conversely, the second row of the table reports the results obtained from the optimised safe formation in the classic RTN, with the performance reevaluated in the ZD. It is observed that the gap between the two solutions is not significant ($\sim 0.5\%$), suggesting that the initial formulation is perfectly valid, especially considering its compliance with more strict safety requirements. The relative orbit and the HoA trend in the optimal case are shown in Figure 5.7. It is evident here that the asymmetry introduced by the Earth's rotation on the HoA exists between the two hemispheres.

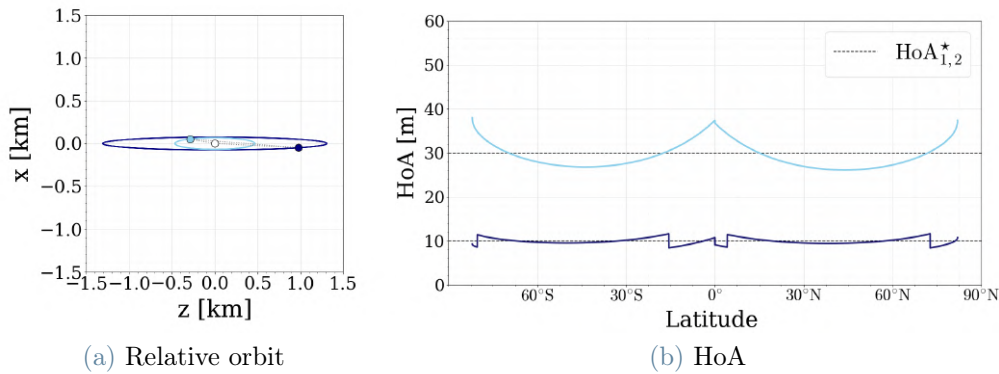


Figure 5.16: HoA for a three-satellite formation optimised in the ZD frame.

In conclusion, this chapter has shown how to determine the set of ROEs for both global and local coverage following HoA optimality. The only assumption considered is that each pair of the chief and one of the deputies forms a helix formation. Chapter 6 provides a more in-depth discussion of how optimality conditions combine with safety requirements, also considering pairs of two deputies.

6 | Formation Maintenance

In the previous chapters, the requirements that the ROEs must respect to maintain a maximum safety configuration—the generalised Helix (Chapter 3)—and to guarantee compliance with the nominal HoA (Chapter 4) have been derived. In this chapter, the possibility of designing formations that respect both constraints is discussed in section 6.1. Having obtained the set of compliant configurations, section 6.2 examines how they are affected by the introduction of orbital perturbations. Finally, section 6.3 presents the control strategies used to mitigate the effect of disturbances, together with an estimate of the required formation maintenance costs.

6.1. Optimal Generalised Helix Formations

This section aims to understand whether it is possible to design a generalised Helix formation that is also interferometrically optimal in terms of coverage of the nominal HoA. As illustrated in section 4.3 of Chapter 4, solving the optimisation problem uniquely determines the maximum effective baseline $B_{\perp max,j}$ and the angular shifts $\delta_1, \delta_{1,j}$. Therefore, recalling the mapping defined in eq. 4.3.2, the ROEs of each deputy j must comply with the following relationships, assuming that $\vartheta_j = \varphi_j + n\pi$:

$$\begin{cases} \delta i_j = \sqrt{\frac{B_{\perp max,j}^2}{\cos^2 \theta_l} - \delta e_j^2 \tan^2 \theta_l} \\ \varphi_1 = \delta_1 \pm \arctan\left(\frac{-\delta i_1}{\delta e_1 \tan \theta_l}\right) \\ \varphi_j = \delta_{1,j} - \delta_1 + (-1)^n \arctan\left(\frac{\delta i_j}{\delta e_j \tan \theta_l}\right) \end{cases} \quad (6.1)$$

Chapter 4 has shown that the passive safety ensured by the generalised Helix can be achieved under two possible conditions: 1) All deputies' relative orbits projected onto the x-z plane are scaled versions of each other, or 2) all pairs of chief-deputy have the same phases of the eccentricity and inclination vectors modulo π . In the following, the compatibility of both those solutions with the equations in 6.1 is discussed.

6.1.1. Safe configuration with scaled orbits

Without losing generality, the first deputy is taken as the reference, and denoted with the $(\cdot)_1$ subscript. The condition each deputy has to respect in compliance with this first passively safe configuration is: $\delta e_1 \delta i_j = \delta e_j \delta i_1$, where j identifies one of the other deputies in the formation with $j \neq 1$. Thus, substituting it in the first equation in 6.1, it yields:

$$\left(\frac{B_{\perp max,j}^2}{\cos^2 \theta_l} - \delta e_j^2 \tan^2 \theta_l \right) \delta e_1 = \left(\frac{B_{\perp max,1}^2}{\cos^2 \theta_l} - \delta e_1^2 \tan^2 \theta_l \right) \delta e_j = 0 \quad (6.2)$$

$$\Leftrightarrow \delta e_j = \frac{B_{\perp max,j}}{B_{\perp max,1}} \delta e_1. \quad (6.3)$$

Consequently, all the magnitudes of the eccentricity vector must be in fixed ratios given by the maximum effective baselines with respect to that of the first deputy. Recalling that the optimisation leaves a partial degree of freedom between the magnitudes δe_j , δi_j and the phase φ_j , the passive safety condition in eq. 6.3 requires fixing one of the two magnitudes. If they were instead both fixed by the optimisation, it would no longer be possible to guarantee compliance with eq. 6.3 through an appropriate choice of φ_j , as this variable does not influence the solution. Once the passive safety requirement has been imposed for all deputies, only a single degree of freedom is left for the entire formation, given by δe_1 .

Regarding the phase of the relative eccentricity vectors, by enforcing the respect of the last equation of 6.1, the following requirement is obtained:

$$\varphi_j = \delta_1 - \delta_{1,j} + (-1)^n \arctan \sqrt{\frac{B_{\perp max,1}}{\delta e_1^2 \sin^2 \theta_l} - 1}. \quad (6.4)$$

Specifically, eq. 6.4 provides the phase of the chief-related relative eccentricity vector for the j -th deputy at optimality.

6.1.2. Safe configuration with aligned $\Delta \mathbf{e}$, $\Delta \mathbf{i}$ vectors

This second possible passively safe configuration entails $\varphi_i = \varphi_j + k\pi \forall i, j$. For simplicity, the quantities $E_j := \delta e_j \sin \theta_l$ and $I_j := \delta i_j \cos \theta_l$ are employed in the following. By substituting the definition of δ_1 into the last equation of 6.1, it yields:

$$\delta_{1,j} = k\pi + \arctan \left(\pm \frac{I_j}{E_j} \right) - \arctan \left(\pm \frac{I_1}{E_1} \right), \quad k \in \mathbb{Z}, \quad (6.5)$$

where the plus-minus signs in the arctangent functions depend on the relative orientation (parallel or anti-parallel) between the relative eccentricity and the relative inclination vectors. At this point, the condition given by the first equation of 6.1 is imposed, obtaining:

$$\pm \arctan \left(\frac{\sqrt{B_{\perp max,j}^2 - E_j^2}}{E_j} \right) = \delta_{1,j} + \arctan \left(\pm \frac{I_1}{E_1} \right) + k\pi \quad (6.6)$$

Analogously to the case of orbits in scale ratios, all the deputies must be compliant with the condition in eq. 6.6, which leaves only one degree of freedom in the choice of ROEs. To proceed rigorously, it is necessary to examine the eight cases given by the different possible combinations of signs and parities of k coming from eq. 6.6 to derive the necessary condition for an optimal configuration. In fact, the presence of the arctangent operator causes the solution to change based on the sign of all the terms and the value of $\delta_{1,j}$ obtained following the optimisation. Specifically, it is necessary to verify that the term on the right side in eq. 6.6 belongs to the interval $[-\pi/2, \pi/2]$. If this holds one can then invert eq. 6.6 to retrieve E_j as:

$$E_j = \frac{B_{\perp max,j}}{\sqrt{1 + \tan^2 \left(\delta_{1,j} - k\pi + \arctan \left(\pm \frac{I_1}{E_1} \right) \right)}} \quad (6.7)$$

Table 6.1 lists all the conditions that must be met by the chief related ROEs for each possible combination of the signs and parity of k , given:

$$\delta e_T = \frac{B_{\perp max,1}}{\sin \theta_l \sqrt{1 + \tan^2 \delta_{1,j}}}, \quad \delta e_C = \frac{B_{\perp max,1}}{\sin \theta_l \sqrt{1 + \cot^2 \delta_{1,j}}}$$

. The solutions also depend on the quadrant to which $\delta_{1,j}$ belongs, so, once its value has been fixed by optimisation, only a portion of all possible combinations of signs and parities can actually be used to obtain a passively safe configuration.

At last, the ROEs of the j -th deputy relative to the chief must respect the following set of conditions:

$$\begin{cases} \delta e_j = \frac{B_{\perp max,j} / \sin \theta_l}{\sqrt{1 + \tan^2 \left(\delta_{1,j} - \arctan^2 \left(\frac{B_{\perp max,1} / \cos^2 \theta_l - \delta e_1^2 \tan^2 \theta_l}{\delta e_1 \tan \theta_l} \right) \right)}} \\ \varphi_j = \delta_1 \pm \arctan \sqrt{\frac{B_{\perp max,1}^2}{\delta e_1^2 \sin^2 \theta_l} - 1} + k\pi \\ \delta e_1 \text{ s.t. conditions in Table 6.1} \end{cases} \quad (6.8)$$

$(\vartheta - \varphi)_1$	$(\vartheta - \varphi)_j$	$\varphi_j - \varphi_1$	$\delta_{1,j} \in [0, \pi/2]$	$\delta_{1,j} \in [\pi/2, \pi]$	$\delta_{1,j} \in [\pi, 3\pi/2]$	$\delta_{1,j} \in [3\pi/2, 2\pi]$
0	0	0	$\delta e_1 \leq \delta e_T$	–	–	$\delta e_1 \geq \delta e_C$
0	0	π	–	$\delta e_1 \geq \delta e_C$	$\delta e_1 \leq \delta e_T$	–
0	π	0	$\delta e_1 \geq \delta e_T$	$\delta e_1 \leq \delta e_C$	–	–
0	π	π	–	–	$\delta e_1 \geq \delta e_T$	$\delta e_1 \leq \delta e_C$
π	0	0	–	–	$\delta e_1 \leq \delta e_C$	$\delta e_1 \geq \delta e_T$
π	0	π	$\delta e_1 \leq \delta e_C$	$\delta e_1 \geq \delta e_T$	–	–
π	π	0	$\delta e_1 \geq \delta e_C$	–	–	$\delta e_1 \leq \delta e_T$
π	π	π	–	$\delta e_1 \leq \delta e_T$	$\delta e_1 \geq \delta e_C$	–

Table 6.1: Conditions for safety and optimality.

Two examples of passively safe relative orbits that generate optimal solutions among the ones listed in Table 5.2 are depicted in Figure 6.1. In the left and central plots, a configuration is shown where the relative orbits are such that $\delta e_i / \delta i_i = 1$ with three and four satellites, respectively. The right plot shows instead a formation where all pairs have the relative eccentricity and inclination vectors lying in the same direction for the particular case $\varphi_i = \vartheta_i = \pi/2 + k\pi$: as better detailed in the next section, this solution has the advantage of being robust to J_2 perturbation. The particular configuration shown in the central plot is similar to the one obtained with the heuristic (see Figure 5.6), but with different radius values and small phase shifts that allow for a more effective exploitation of the available baselines.

In conclusion, the proposed methodology demonstrates that for both conditions that guarantee maximum passive safety, it is possible to determine a set of optimal configurations.

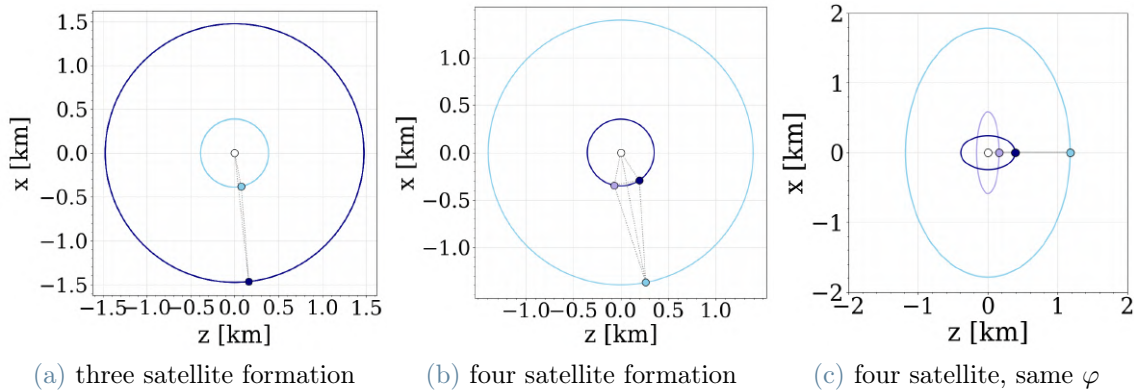


Figure 6.1: Examples of generalised Helix orbits projected onto the x-z plane generating the optimal solution for $[\text{HoA}_1^*, \text{HoA}_2^*] = [10 : 30]$.

Specifically, they are defined up to one degree of freedom, which can be determined so as to respect a minimum mutual distance constraint by enforcing compliance with eq. 4.6.

As an example, consider Figure 6.2, which displays the minimum distance in the x-z plane for each pair of satellites in an optimised formation of three. In particular, the left plot refers to the safe condition with parallel $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ vectors, while that on the right refers to scaled relative orbits. Note that, since the optimisation was run for parallel $\Delta \mathbf{e}, \Delta \mathbf{i}$ vectors, in accordance with Table 6.1, the phase shift must be: $\delta_{1,2} \in (0, \pi/2)$. Therefore, in the case displayed in the left plot, the minimum distance is defined only for $\delta e_1 \leq \delta e_T$. The value of δe_1 needed to maintain a certain minimum separation can be set so that none of the curves is, for that point, below the desired value. While the trend of the curves in the first case is determined by the tabulated conditions, the second has an intuitive interpretation: the minimum distance is precisely given by δe_1 for the first deputy and proportional to it for other deputies, as long as $\delta e_1 < \delta i_1$. In this region, the minimum distance is in fact obtained in a purely radial direction when the latitude argument $u = \varphi_1$ is reached. Beyond this point, the minimum distance is obtained in the z-direction when $u = \varphi_1 + \pi/2$ and depends on δi_1 , which in turn is related to δe_1 through the first equation in 6.1. Finally, the δr_{min} goes back to zero when $\delta i_1 = 0$ or $\delta e_1 = B_{\perp max,1} / \sin \theta_l$. The maximisation of the minimum distance corresponds therefore to the condition $\delta e = \delta i$, which entails relative circular orbits when projected onto the x-z plane (i.e. xz-PCOs, see Chapter 3).

In Table 6.2 are listed the minimum distances between any pair of satellites for different optimal solutions of the HoA optimisation problem. Note that the one corresponding to relative orbits that are scaled versions of each other is always better.

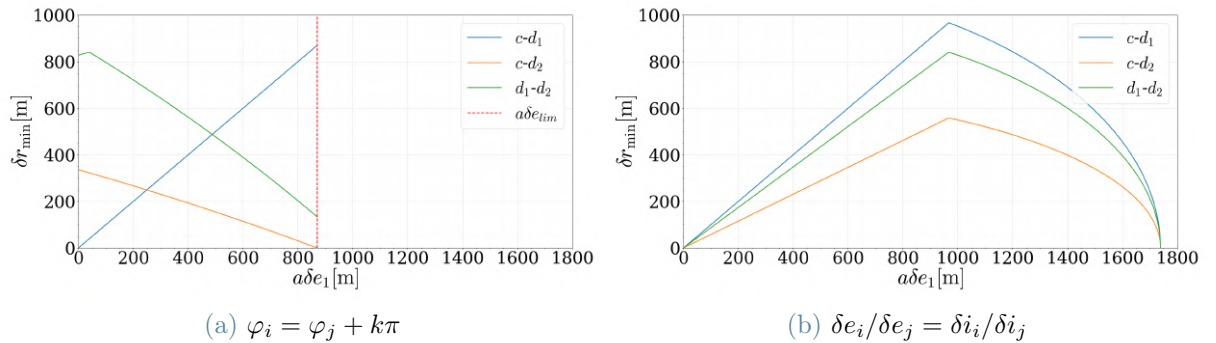


Figure 6.2: Minimum distances between all satellite pairs as a function of the magnitude of one relative eccentricity vector for a three satellite formation with optimum maximum effective baselines: $B_{max1} = 967$ m, $B_{max2} = 558$ m, $B_{max3} = 1367$ m.

HoA _{1,2} [*]	Scaled orbits		Aligned $\Delta\mathbf{e}, \Delta\mathbf{i}$ vectors	
	xyz distance [m]	xz projected [m]	xyz distance [m]	xz projected [m]
10 m, 20 m	524.30	524.30	678.29	150.46
10 m, 30 m	269.43	269.43	242.95	161.59
10 m, 40 m	202.16	202.16	182.28	121.25
10 m, 50 m	211.13	211.13	145.77	96.48

Table 6.2: Minimum distance between any pair of satellites in a formation of four.

It is worth remarking that the results derived in this section suggest a first notable formation type, consisting of nested xz-PCOs. The advantage provided by this geometry is that it allows maximising the minimum distance between any pair of satellites, while preserving interferometric optimality.

6.2. Impact of Perturbed Dynamics

This section describes the conditions for formation maintenance in the presence of orbital perturbation effects. Initially, the contribution of the first zonal harmonic of the geopotential is taken into consideration, showing how it is possible to adapt the stability condition $\vartheta = \pm\pi/2$ derived by D’Amico [29] to the generalised Helix and to the constraints for the optimality of the HoA. Subsequently, the assumptions regarding atmospheric drag and navigation and control errors that are included in the evaluation of formation maintenance costs are illustrated.

6.2.1. Generalised Helix under J_2 Perturbation

As explained in Chapter 3, the effect of J_2 on the ROEs is modelled in this work through the eqs. 3.45 derived by Gaias et al. [26]. It mainly generates a change in the direction of the relative eccentricity vector, which sweeps a circle in the plane $\delta e_x\text{-}\delta e_y$, and a secular evolution of the component δi_y of the relative inclination vectors proportional to the difference in inclination of the orbits. For bounded relative orbits, the condition $\delta a = 0$ holds, and is here retained. The atmospheric drag - which is responsible for a $\delta a \neq 0$ - is included in the dynamical model when calculating the costs necessary for formation keeping. Eqs. 3.45 coincide therefore with those previously derived by D’Amico [27] reported in eq. 3.44.

Considering the reciprocal displacement of the vectors $\Delta\mathbf{e}, \Delta\mathbf{i}$, the passive safety inherent

in the Helix formation is progressively affected since, after a certain time, the vectors will go from being parallel to orthogonal [29]. More precisely, if corrective manoeuvres are not performed, the safety condition given by eq. 4.6 would no longer be valid. Figure 6.3 shows the evolution of the relative eccentricity and inclination vectors in the two cases $\vartheta \neq \pi/2$ and $\vartheta = \pi/2$. Note that the case $\vartheta = \pi/2$ eliminates the secular effect on δi_y as explained in Chapter 3: the effect is a relative orbit always bounded in the out-of-plane direction.

As shown in Chapter 4, the optimality requirements fix the maximum effective baseline $B_{\perp max}$ which in turn sets a maximum limit for the magnitudes $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ in compliance with eq. 4.3.2. Thus, after having enforced the optimality, the minimum distance constraint given in eq. 4.6 can be expressed in terms of a maximum allowable deviation of the two vectors from the parallelism condition.

For chief-deputy pairs, the evaluation of this requirement can be expressed through the absolute value of the cosine of the angle between the relative eccentricity and inclination vectors, which must not fall below a given threshold. This angle is given, for the j -th deputy, by $\xi_j = \varphi_j - \vartheta_j$ and its variation over time is easily calculated from eqs. 3.44. Even for pairs consisting of two deputies, the initial condition of parallelism given by the generalised Helix is easily assessed by the angle between the differences of the relative vectors. However, the relative eccentricity and inclination vectors always refer to the chief, thus it is less immediate to evaluate the angle between their mutual differences.

Considering that the introduced angle ξ can be used as an indicator of how the configuration differs from the nominal Helix because of J_2 , the following considerations aim to understand how and how quickly the formation deviates from its nominal geometry. To

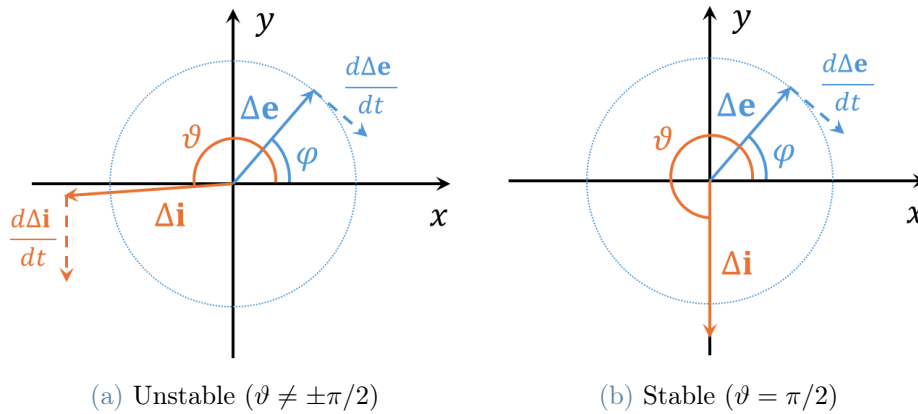


Figure 6.3: Effect of J_2 on relative eccentricity and inclination vectors, adapted from D’Amico and Montenbruck [29].

this purpose, an analytical expression for the angle ξ as it varies with time (or with u) is needed. In Figure 6.4, a sketch of the reference geometry is provided to support the following derivations.

Starting from eqs. 3.44, for a chief-deputy pair the angles φ and ϑ depend on u in the following way:

$$\begin{cases} \varphi = \varphi_0 + \varphi'(u - u_0) \\ \vartheta = \arctan_2(\Delta i_{y0} + 3\gamma \sin^2 i_c (u - u_0) \delta i_{x0}, \Delta i_{x0}) \end{cases} \quad (6.9)$$

The cosine of the angle between the differences of the respective eccentricity and inclination vectors therefore reads:

$$\cos \xi(u) = \cos \left(\varphi_0 + \varphi'(u - u_0) - \arctan_2 \left(\delta i_{y0} + 3\gamma \sin^2 i_0 (u - u_0) \delta i_{x0}, \delta i_{x0} \right) \right). \quad (6.10)$$

Expressed in this form, the angle ξ depends on both the components δi_{y0} and δi_{x0} , which hide the dependency on ϑ . The operator $\arctan_2(\cdot, \cdot)$ differs from the standard $\arctan(\cdot/\cdot)$ because it correctly accounts for the quadrant to which the angle belongs, returning values in $(-\pi, \pi]$. This operator can be expressed as a standard tangent function, provided the addition of a correction of $\pm\pi$. Additionally, since this additive term would be inside a cosine function, it would only affect the overall sign. Thus, eq. 6.10 can be expressed as:

$$\cos \xi(u) = \pm \cos \left(\varphi_0 + \varphi'(u - u_0) - \arctan \left(\tan(\vartheta_0) + 3\gamma \sin^2 i_c (u - u_0) \right) \right), \quad (6.11)$$

providing an expression that depends only on the initial phases of the vectors. By imposing the (anti-)parallel condition that defines the Helix and taking the absolute value, one

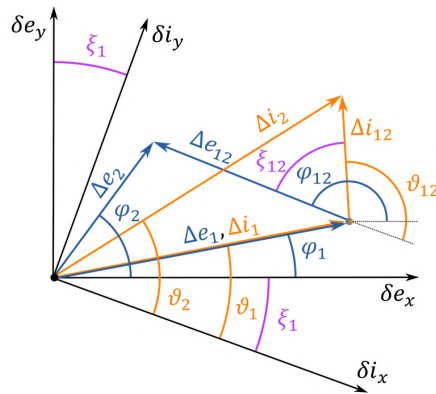


Figure 6.4: Scheme showing the reciprocal orientation of the relative eccentricity and inclination vectors. The subscripts indicate: $(\cdot)_i$ the pair chief - i -th deputy, $(\cdot)_{ij}$ the pair i -th - j -th deputy.

obtains the following equation:

$$|\cos \xi(u)| = |\cos (\varphi_0 + \varphi'(u - u_0) - \arctan (\tan(\varphi_0 + n\pi) + 3\gamma \sin^2 i_c(u - u_0)))|, \quad (6.12)$$

which depends only on φ_0 . Specifically, there is no dependence on the initial magnitudes of the relative eccentricity and inclination vectors: this was expected for δe , which remains constant. Regarding δi , its magnitude changes over time due to the secular effect on δi_y , but, since it is proportional to the inclination difference δi_x , its orientation as time passes depends only on the initial phase ϑ_0 , equal to φ_0 modulo π in a Helix. Notably, eq. 6.12 allows to evaluate the time-behaviour of the difference in the orientations of the relative eccentricity and inclination vectors relative to the chief, and thus to evaluate the best φ_0 to delay as much as possible a decrease in interferometric/safety requirements. As discussed in section 6.1, the two possible conditions for the passive safety of the formation are that all the eccentricity and relative inclination vectors lie on the same line, or that all the orbits relative to the chief are scaled versions of each other. In the first case, all the deputies share the same chief-related phases: $\varphi_{0j} \pm \pi = \varphi_{01} \pm \pi$, so the ξ in eq. 6.12 already expresses the angle between the two vectors as a function of the latitude argument, which is the same for all pairs. In this case, the behaviour of the entire formation due to J_2 can be represented by the single variable $\varphi_{0,1}$.

It can be proved that the same is true in the second case, where $\delta e_j/\delta e_1 = \delta i_j/\delta i_1$. Here, by virtue of eq. 4.28, the initial phase of the chief related eccentricity vector yields for the j -th deputy:

$$\varphi_{j,0} = \varphi_{1,j} + (-1)^n \arctan \left(-\frac{\delta i_{j,0}}{\delta e_{j,0} \tan \theta_l} \right) - (-1)^m \arctan \left(-\frac{\delta i_{i,0}}{\delta e_{i,0} \tan \theta_l} \right) - \delta_{1,j} \quad (6.13)$$

As previously discussed (see eq. 4.11 and Figure 4.6), this configuration is allowed only in the case where $n = m$; since the ratios $\delta e_0/\delta i_0$ are equal, it yields:

$$\varphi_{j,0} = \varphi_{1,0} - \delta_{1,j}. \quad (6.14)$$

Eq. 6.14 shows that the initial phases of the chief-related $\Delta \mathbf{e}$ vectors are indeed fixed for all deputies by the requirements on the optimisation of the HoA, once the phase for the pair chief-first deputy has been chosen.

A further step is necessary in order to consider the deputy-deputy pairs, for which one can demonstrate that the relative phases also do not depend on the magnitudes of the relative vectors, but only on the initial phases. The angle between the relative eccentricity

vectors for the $i - j$ pair is obtained as follows:

$$\begin{aligned} \Delta \mathbf{e}_{i,j} &= \delta e_i \begin{bmatrix} \cos \varphi_i \\ \sin \varphi_i \end{bmatrix} - \delta e_j \begin{bmatrix} \cos \varphi_j \\ \sin \varphi_j \end{bmatrix} = \delta e_{i,j} \begin{bmatrix} \cos \varphi_{i,j} \\ \sin \varphi_{i,j} \end{bmatrix} \\ \Leftrightarrow \tan \varphi_{i,j} &= \frac{\delta e_i \sin \varphi_i - \delta e_j \sin \varphi_j}{\delta e_i \cos \varphi_i - \delta e_j \cos \varphi_j}. \end{aligned} \quad (6.15)$$

Analogously, for the relative inclination it yields:

$$\tan \vartheta_{i,j} = \frac{\delta i_i \sin \vartheta_i - \delta i_j \sin \vartheta_j}{\delta i_i \cos \vartheta_i - \delta i_j \cos \vartheta_j}. \quad (6.16)$$

By virtue of eq. 6.12, the mutual phases for the pairs chief-deputy depend only on their initial values and on the time elapsed. Moreover, the knowledge of the initial phase φ_{10} for the pair chief-first deputy is enough to retrieve that of any other pair by simply introducing the shift $\delta_{1,j}$. Also in this operation no dependence on any parameter other than the initial phases is introduced. Thus, defining the parameter $k_0 := \delta e_{i0}/\delta e_{j0} = \delta i_{i0}/\delta i_{j0}$, the last expression becomes:

$$\begin{cases} \tan \varphi_{1j0} = \frac{\sin(\varphi_{1,0} - \delta_{1,j}) - k_0 \sin(\varphi_{10} - \delta_{1,j})}{\cos(\varphi_{1,0} - \delta_{1,j}) - k_0 \cos(\varphi_{10} - \delta_{1,j})} \\ \tan \vartheta_{1j0} = \frac{\sin(\vartheta_{1,0} - \delta_{1,j}) - k_0 \sin(\vartheta_{10} - \delta_{1,j})}{\cos(\vartheta_{1,0} - \delta_{1,j}) - k_0 \cos(\vartheta_{10} - \delta_{1,j})} \end{cases} \quad (6.17)$$

The factor k_0 does not embed a dependency on the ratio between the magnitudes of relative eccentricity/inclination. Indeed, by combining the requirement that the relative orbits have the same shape with the optimal solution found (see eq. 6.2), it yields: $k_0 = \delta e_i/\delta e_j = B_{\perp max,i}/B_{\perp max,j}$, which is uniquely determined after the optimisation process.

In conclusion, the proposed derivation shows that the influence of J_2 on a generalised Helix can be tuned by one parameter: the initial phase of the relative eccentricity vector of the first deputy relative to the chief $\varphi_{1,0}$. Note that this result holds only because of the specific characteristics of passively safe configurations. In general, the time behaviour of the relative phases also depends on the magnitudes of the $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ vectors, introducing a phase-magnitude coupling that would have led to trade-off choices between HoA optimality and stability to J_2 . This result is particularly convenient, since the optimisation leaves room for one degree of freedom for the entire formation. It has been shown in section 6.1 that a convenient choice could be obtained by maximising the minimum distance between all pairs of satellites. Another notable configuration arises when considering the robustness to the perturbation induced by the geopotential. In fact, it is possible to set

for each pair of deputies: $\varphi_i = \varphi_j = \vartheta_i = \vartheta_j = \pi/2 \pmod{\pi}$. Not only does this allow for maintaining the passive safety provided by the generalised Helix, but, accordingly to eqs. 3.44, it removes the secular effect of J_2 on δi_y .

The expressions for the relative phases in eq. 6.17 can then be inserted into eqs. 6.15 and 6.16, to get the time behaviour of the phase for any specific pair, leading to a slightly more convoluted version of eq. 6.12. Nevertheless, in the specific case of $k_0 = 1$ for which the two deputies travel the same trajectory relative to the chief, one can compute the angle between the two vectors in a much simpler way. In this case, the two expressions in eqs. 6.15 and 6.16 are in the form:

$$\varphi_{ij0} = \arctan_2(\sin \varphi_{i0} - \sin \varphi_{j0}, \cos \varphi_{i0} - \cos \varphi_{j0}) \quad (6.18)$$

By leveraging the trigonometric identities:

$$\begin{cases} \sin \varphi_{i0} - \sin \varphi_{j0} = 2 \cos \left(\frac{\varphi_{i0} + \varphi_{j0}}{2} \right) \sin \left(\frac{\varphi_{i0} - \varphi_{j0}}{2} \right) \\ \cos \varphi_{i0} - \cos \varphi_{j0} = -2 \sin \left(\frac{\varphi_{i0} + \varphi_{j0}}{2} \right) \sin \left(\frac{\varphi_{i0} - \varphi_{j0}}{2} \right) \end{cases} \quad (6.19)$$

eq. 6.18 can be rewritten as follows:

$$\varphi_{ij0} = \arctan_2 \left(\cos \left(\frac{\varphi_{i0} + \varphi_{j0}}{2} \right), \sin \left(\frac{\varphi_{i0} + \varphi_{j0}}{2} \right) \right) \pm \begin{cases} 0 & \text{if } \sin \left(\frac{\varphi_{i0} - \varphi_{j0}}{2} \right) \geq 0 \\ \pi & \text{otherwise} \end{cases} \quad (6.20)$$

By definition of the $\arctan_2$ operator it yields:

$$\begin{cases} \sin \varphi_{ij0} = \cos \left(\frac{\varphi_{i0} + \varphi_{j0}}{2} \right) \\ \cos \varphi_{ij0} = -\sin \left(\frac{\varphi_{i0} + \varphi_{j0}}{2} \right) \end{cases} \quad (6.21)$$

from which $\varphi_{ij0} = \frac{\varphi_{i0} + \varphi_{j0}}{2} + \frac{\pi}{2} + 2n\pi$. The procedure is analogous for ϑ_{ij0} . At last, it is possible to express the angle ξ between the eccentricity and inclination vectors as follows:

$$|\cos \xi_{ij}(u)| = \left| \sin \left(\frac{\varphi_{i0} + \varphi_{j0}}{2} + \varphi'(u - u_0) + \arctan(\vartheta_{0ij} + 3\gamma \sin^2 i_c(u - u_0)) \right) \right| \quad (6.22)$$

6.2.2. Interferometric performance under J_2 perturbation

Having obtained an indicator of how safety degrades due to J_2 , in the following, the same is done regarding interferometric performance. Eqs. 4.3.2 and 4.28 provide a map connecting the parameters responsible for the interferometry - *i.e.*, the maximum effective baseline $B_{\perp max,j}$ and the phase shift δ_j - to the ROEs. The time evolution of the effective baseline and phase shifts is displayed in Figure 6.5. Note that, for a circular orbit a height 500km, $T \approx 1.5h$. The effective baseline follows a growing trend which, however, could be easily contained by choosing an optimal configuration with $\varphi = \pi/2 \pm \pi$, preventing the secular growth of the δi_y component. More complicated to counteract is the effect on the shift δ_j , which tends to cancel out within a certain time. The resulting effect on the interferometric performance is evident in the four plots of Figure 6.6, where the HoA is traced after different time intervals have elapsed, beginning from an initial nominal formation. Note in particular how the asymptote, which was located in correspondence with points at which no acquisition is taken, tends to shift. The effect is a neat decrease in the closeness of the attained HoA to the nominal value, indicated by the black dashed line, for the parts of the orbit of scientific interest. Moreover, the cancellation of the phase differences causes the asymptotes to shift until they concentrate in correspondence with the polar latitudes. On the other hand, the growth of the effective baseline causes the HoA to almost completely nullify after $\sim 10000T \sim 278$ days.

As done for formation safety, it is appropriate to identify a parameter embedding the degradation in scientific performance caused by J_2 . The natural choice is the RMSE, since the L^2 metric according to which the optimisation was carried out is precisely based on it. Figure 6.7 shows the variation of the RMSE in percentage over the periods from 1

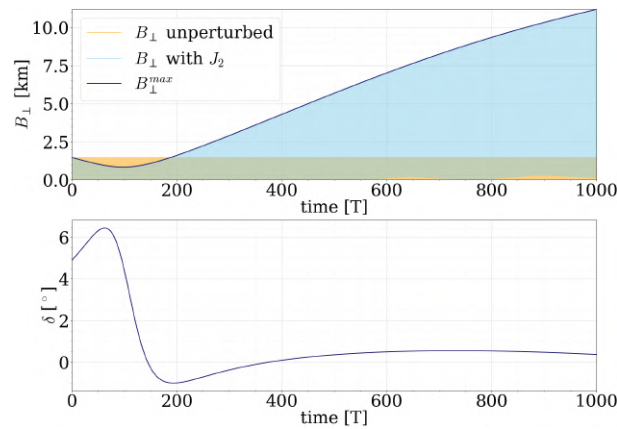


Figure 6.5: Time evolution of the effective baseline and angular shifts under J_2 perturbation.

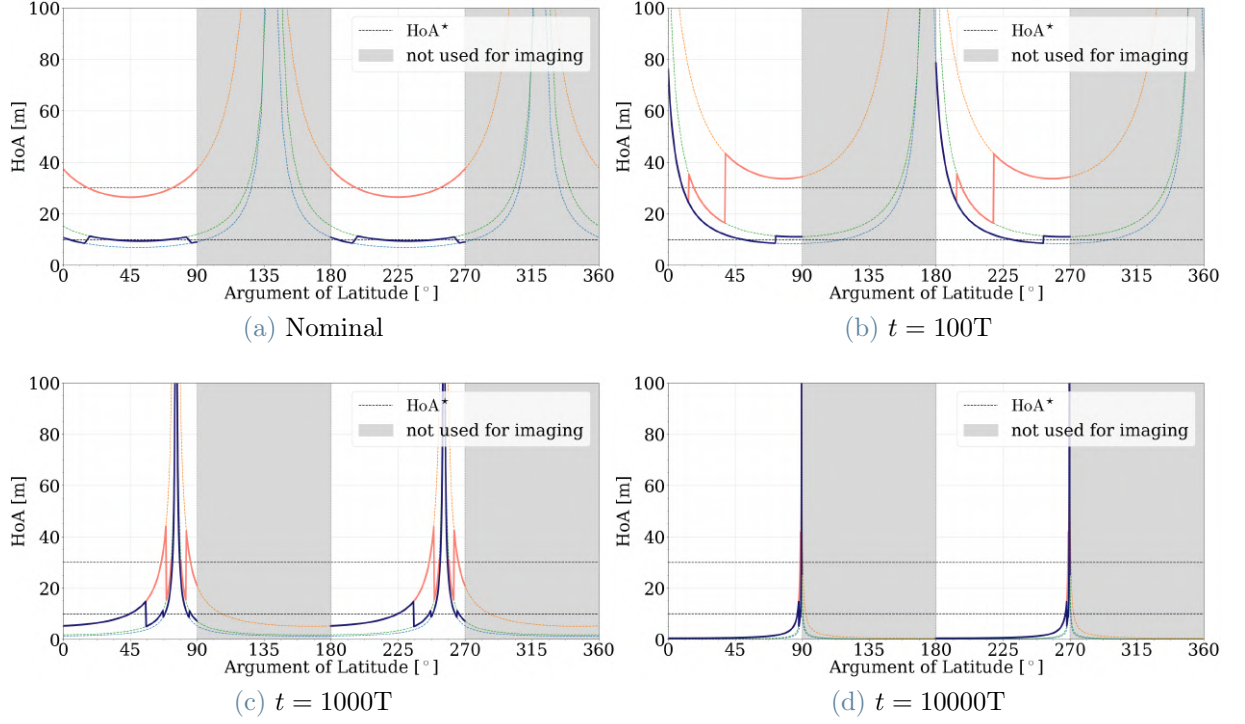


Figure 6.6: Time-lapse of the HoA under J_2 perturbation, starting from an optimal condition.

to 100. Remarkably, the drift of the optimised Helix caused by J_2 perturbation degrades the HoA beyond a 100% RMSE in some tens of days if left uncorrected, motivating the necessity of the control logic that is illustrated in section 6.3.

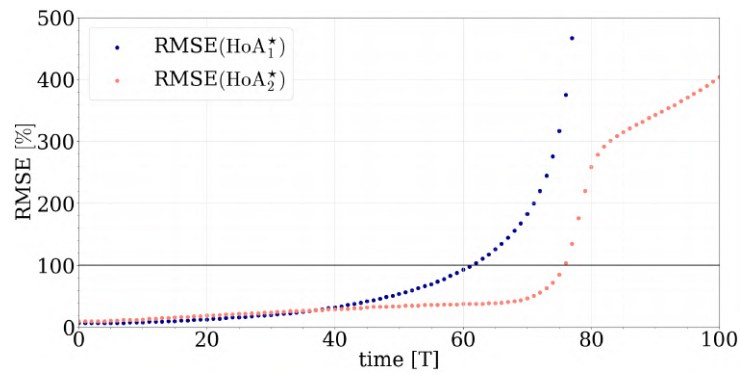


Figure 6.7: RMSE under J_2 perturbation over the first 100 periods (~ 6 days).

6.2.3. Other disturbances

After discussing the effect of the geopotential, this section is dedicated to the other disturbances impacting formation maintenance cost.

Atmospheric drag

Considering the differential drag, it is assumed that it modifies the relative dynamics according to the semi-empirical model used by Gaïas et al. [26] for the AVANTI experiment. Its contribution consists of an acceleration in the tangential direction only, as reported in eqs. 3.47 and 3.48. Regarding the absolute effect — *i.e.*, acting equally on all satellites in the formation — the method considered for its compensation is presented in the following.

As explained in Chapter 3, for quasi-circular orbits it is sufficient to consider only the effect of eqs. 3.40 on the semi-major axis, since these equations show that the eccentricity vector oscillates in direction, whereas its magnitude remains constant. Consequently, the only relevant component of the drag acceleration is the tangential one, which can be formulated in ECI and then rotated to the RTN. Starting from eq. 3.38 (see Vallado [89]), the tangential component of the acceleration in ECI reads:

$$a_T = -\frac{1}{2} C_B \rho |\mathbf{v}_{rel}| (\mathbf{v}_{rel} \cdot \hat{\mathbf{T}}). \quad (6.23)$$

Substituting to the relative velocity its definition given in eq. 3.39 and assuming that the atmosphere rotates in solidarity with the Earth, it is necessary to evaluate the magnitude and the projection onto the tangential direction of: $(\boldsymbol{\omega}_{\oplus} \times \mathbf{r}_c)$, where $\mathbf{r}_c$ is the position of the chief with respect to the centre of the Earth and r_c its norm. To simplify it, one can exploit the product identity: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ obtaining:

$$(\boldsymbol{\omega}_{\oplus} \times \mathbf{r}_c) \cdot \hat{\mathbf{T}} = \boldsymbol{\omega}_{\oplus} \cdot (\mathbf{r}_c \times \hat{\mathbf{T}}). \quad (6.24)$$

The one-unit vector $\hat{\mathbf{T}}$ is orthogonal to the radial direction and to the angular momentum. Thus it can be substituted with: $\hat{\mathbf{T}} = \frac{\hat{\mathbf{N}} \times \mathbf{r}_c}{r_c}$. Leveraging the cross product chain property, it yields:

$$\mathbf{r}_c \times \hat{\mathbf{T}} = \mathbf{r}_c \times \frac{\hat{\mathbf{N}} \times \mathbf{r}_c}{r_c} = \frac{1}{r_c} ((\mathbf{r}_c \cdot \mathbf{r}_c) \hat{\mathbf{N}} - (\mathbf{r}_c \cdot \hat{\mathbf{N}}) \mathbf{r}_c). \quad (6.25)$$

Since $\mathbf{r}_c \cdot \hat{\mathbf{N}} = 0$ because the two vectors are orthogonal, the second term of the acceleration reads:

$$(\boldsymbol{\omega}_{\oplus} \times \mathbf{r}_c) \cdot \hat{\mathbf{T}} = \boldsymbol{\omega}_{\oplus} \cdot (r_c \hat{\mathbf{N}}) = r_c (\boldsymbol{\omega}_{\oplus} \cdot \hat{\mathbf{N}}). \quad (6.26)$$

The tangential component of the atmospheric velocity is thus given by a projection of the Earth's angular velocity onto the direction of the angular momentum. Consequently, the contribution of this term is null for a polar orbit, while it is maximum for an equatorial orbit in which the two vectors are parallel. To compute the magnitude of the relative velocity vector, its radial and normal components are required. The former is simply

given by v_R , because the atmosphere has no velocity component in the radial direction. Regarding the out-of-plane component, the following holds:

$$v_{rel,N} = v_N - (\boldsymbol{\omega}_\oplus \times \mathbf{r}_c) \cdot \hat{\mathbf{N}} = v_N - r_c(\boldsymbol{\omega}_\oplus \cdot \hat{\mathbf{T}}), \quad (6.27)$$

for which the same vectorial properties have been exploited. The magnitude of the relative velocity reads:

$$|\mathbf{v}_{rel}| = \sqrt{v_R^2 + (v_T - r_c(\boldsymbol{\omega}_\oplus \cdot \hat{\mathbf{N}}))^2 + (v_N + r_c(\boldsymbol{\omega}_\oplus \cdot \hat{\mathbf{T}}))^2}. \quad (6.28)$$

Finally, to obtain the final expression for the tangential drag acceleration, one can leverage the circular orbit assumption, for which $v_N = v_R = 0$. Moreover, these two aspects are considered: 1) remote sensing missions aiming at global coverage entail orbits that are reasonably close to being polar (typically SSO); and 2) the contribution due to the Earth's rotation is already small compared to the velocity of the satellites relatively to the ECI. Thus, one can safely neglect the terms with projections of the Earth's angular velocity onto the normal direction. The resulting expression is therefore extremely compact:

$$a_T = \frac{1}{2} C_B \rho v_T \sqrt{v_T^2 + r_c^2(\boldsymbol{\omega}_\oplus \cdot \hat{\mathbf{T}})^2} \quad (6.29)$$

Note that according to eq. 6.29, the acceleration consists of a superposition of a constant value given by the satellite's velocity, and an oscillation with small amplitude and period equal to half the orbital period given by the Earth's rotation. To get the secular decay, it is observed that, in a circular polar orbit, since $\hat{\mathbf{T}}$ is, by definition, tangent to the circular orbit path, it has a uniform variation. Thus, averaging the last term of eq. 6.29 over an orbital period, it yields:

$$\langle (\boldsymbol{\omega}_\oplus \cdot \hat{\mathbf{T}})^2 \rangle = \frac{1}{2} \omega_\oplus^2. \quad (6.30)$$

From the non-singular GVE reported in eq. 3.40, the semi-major axis under atmospheric drag evolves as:

$$\frac{da}{dt} = \frac{C_B \rho v_T}{n} \sqrt{v_T^2 + \frac{(r_c \omega_\oplus)^2}{2}}. \quad (6.31)$$

Typically, a space mission is required to remain within a specific orbital tube: for example, this is defined by a radius of 250 m for TanDEM-X (see Kahle et al. [49]). The control considered is therefore based on an event-triggered logic: a manoeuvre is performed simultaneously on all satellites in the formation as soon as a threshold value is exceeded. The type of manoeuvre considered is illustrated here, so as to derive the cost necessary to maintain the formation within the pre-established orbital tube. Assuming that both the

starting and ending orbits are circular, it is reasonable to approximate the manoeuvre as a Hohmann transfer linking the two (see Hohmann [130], Vallado [89]). For a small entity transfer ($\Delta a/r_1 \ll 1$), the total Δv (Δv_{tot}) is (see Appendix A):

$$\Delta v_{tot} \approx \sqrt{\frac{\mu}{r_1^3}} \cdot \frac{\Delta a}{2}, \quad (6.32)$$

where r_1 is the radius of the starting orbit. Regarding the numerical values employed in the computation of the required control effort, an impulse is performed when the chief satellite falls outside its orbital tube of radius 250 m, *i.e.*, the boundary used for the TanDEM-X mission [49]. As for the ballistic coefficient, the following parameters are considered: a mass at launch of 1350 kg [47], a cylindrical shape of length 5.0 m and diameter 2.4 m (see Wermuth et al. [48]), and a drag coefficient of 3.5 (conservative with respect to the typical envelope used in the literature). The cross-sectional area would be $A_{cross} = (1.2\pi)^2 \text{ m}^2$; however, it can increase due to attitude changes consequent to a different antenna pointing. Thus, it is reasonable to retain a conservative value of 8.0 m^2 . Consequently, the ballistic coefficient is: $C_B \approx 0.021 \text{ m}^2/\text{kg}$. Concerning the density ρ , it is considered that, at the LEO altitude of interest ($\sim 500 \text{ km}$), it can be approximated by its mean value during the maintenance interval. Using an exponential model (Vallado [89]):

$$\rho(h + \Delta h) = \rho(h)e^{-h/H}$$

where h is the height, $\Delta h = \pm 125 \text{ m}$ the height change, and H the local scale height, the density change across $\pm 125 \text{ m}$ is on the order of $0.1 - 0.7\%$. The induced error in the resulting semi-major-axis decay rate is therefore $\ll 1\%$. This is small compared with model and solar-activity uncertainties (several % to tens of %), and so a constant-density approximation (*i.e.*, $\rho(h) \approx \bar{\rho}$) is retained as reasonable for a first-order estimate of formation-keeping cost. (Picone et al. [106]). An example of the strategy used to control the decrease of the variable is reported in Figure 6.8. The left plot illustrates the decay of the semi-major axis caused by atmospheric drag and the corrective action of the control manoeuvre. The strategy keeps the spacecraft within the said orbital tube of 250 m diameter, which corresponds to a tolerance of $\pm 125 \text{ m}$ around the nominal value, indicated by the dashed line. When the chief's semi-major axis decreases beyond the lower bound, an impulse places the spacecraft on a transfer orbit with an intermediate semi-major axis. After half an orbital period, a second impulse circularises the orbit at the upper bound, *i.e.*, 125 m above nominal. In the right plot, the relative distance between the chief and deputy in all three RTN coordinates is displayed. As can be seen, there is no significant discontinuity in correspondence with the manoeuvre, since all satellites

impart impulses of equal magnitude and direction.

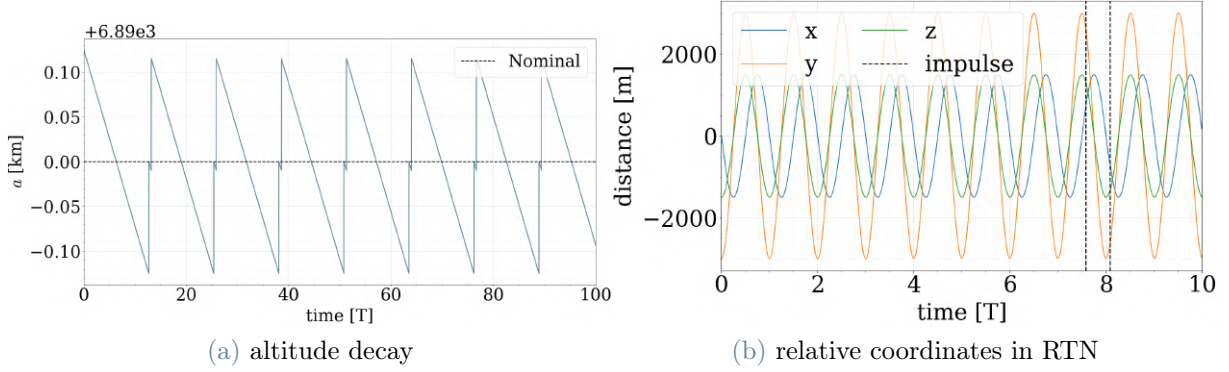


Figure 6.8: Effect of the proposed manoeuvre for the recovery of the semi-major axis consequent to atmospheric drag.

Navigation and control inaccuracies

A detailed modelling of navigation and control systems is beyond the scope of this thesis. However, to obtain more accurate estimates of the control effort required for formation keeping, a simplified approximation of the errors introduced by these systems is introduced. The assumptions listed below are retained hereafter.

- The navigation error due to uncertainties in the knowledge of the relative state of the satellites is modelled as white noise whose standard deviation is obtained by taking the data from the PRISMA mission (see D’Amico and Larsson [40]) as a reference. In this mission, the measurements are derived from the GNSS, considering a standard deviation in the knowledge of the relative state $\sigma_{nav} \approx 10$ cm (Ardaens et al. [131]). After processing these data with an appropriate filter, one can assume that the accuracy increases, reaching a centimetre-level relative positioning knowledge (Persson et al. [39]).
- As for the thrust intensity, a Gaussian noise with standard deviation $\sigma_{int} \approx 1\%$ is assumed. This value is not linked to a specific propulsion system but is representative of the performance typically reported for conventional cold gas and monopropellant thrusters (Harmann and Dartsch [132]).
- In addition to magnitude, a possible misalignment between the commanded thrust vector and the actual one is taken into account. This type of error is typically due to the satellite’s imperfect attitude, as well as possible mounting or firing misalignment. It is simulated by applying a small random orientation error to the

commanded thrust vector given by an isotropic misalignment with standard deviation $\sigma_{ang} \approx 0.05^\circ$: a value compliant with star-tracker-based attitude knowledge, provided accurate mounting and alignment (Kahle et al. [133]).

- Another fundamental characteristic of the propulsion system is given by its minimum impulse bit (MIB), *i.e.*, the minimum thrust intensity that can be produced. To account for it, a value analogous to that of the TanDEM-X mission is considered: ~ 40 mN (Ardaens et al. [134]).
- Finally, thrusters capable of generating a maximum of ~ 1 N thrust are assumed, in compliance with TanDEM-X propulsion system (Ardaens et al. [134]).

6.3. Control Logic

In this section, the strategies and costs in terms of Δv to preserve compliance with interferometric and collision avoidance requirements are illustrated. All the perturbations and inaccuracies described in the previous section are here taken into account. Nevertheless, it is remarked that the results presented are heavily dependent on the particular type and time duration of the mission, on the altitude and, in general, on specific spacecraft characteristics. The Δv obtained in this work is therefore meant to only provide a preliminary guideline. A practical implementation would instead require more detailed and mission-specific analyses. In the following, the cost is assessed by computing:

- An upper bound to necessary control effort, obtained by exactly compensating the perturbing effect.
- A more realistic estimate coming from two different impulsive strategies, which are shown to provide analogous results.

First, an upper bound to the Δv is obtained by ensuring the maintenance of the nominal relative orbit. The perturbed HCW read (Ardaens et al. [134]):

$$\begin{cases} \ddot{x} = 2n\dot{y} + 3n^2x & +u_x + a_x \\ \ddot{y} = -2n\dot{x} & +u_y + a_y \\ \ddot{z} = -n^2z & +u_z + a_z \end{cases} \quad (6.33)$$

where u_i and a_i are the accelerations respectively due to the control and the perturbations in the i -th direction. To obtain the necessary control action, the system $u_i = -a_i \forall i$ is solved. Specifically, for the differential drag, the expression derived by Gaias et al. [26] in eq. 3.47 is employed, whereas for the acceleration imparted by the first zonal harmonic

of the geopotential, it is considered the formulation derived by Willis and D'Amico [135]. The latter gives the relative acceleration components directly in the RTN frame, which read:

$$\begin{bmatrix} a_R^{J_2} \\ a_T^{J_2} \\ a_N^{J_2} \end{bmatrix} = \Gamma \begin{bmatrix} 12 \sin^2 i \sin^2 u - 4 & -4 \sin 2u \sin^2 i & -4 \sin u \sin 2i \\ -4 \sin 2u \sin^2 i & (7 \cos^2 u - 5) \sin^2 i + 1 & \cos u \sin 2i \\ -4 \sin u \sin 2i & \cos u \sin 2i & (5 \cos^2 u - 7) \sin^2 i + 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (6.34)$$

where $\Gamma = -\frac{3\mu_\oplus J_2 R_\oplus^2}{2r_c^5}$ and r_c is the distance between the chief and the centre of the Earth.

Besides the Δv , a slightly more sophisticated approach is considered, based on a condition-triggered logic. It consists of setting thresholds on the minimum distance allowed and on the worsening of scientific performance - which can be reformulated using previously defined indicators $|\cos \xi|$ and RMSE (see section 6.1) - to then perform a manoeuvre as soon as one of those is exceeded. This type of strategy is illustrated in Figure 6.9: in this example, the thresholds 10%, 20%, 0.5, respectively, on the two RMSE and on the cosine value are considered.

The tested strategies rely on impulsive manoeuvres. The variation in the ROE due to a differential manoeuvre of entity: $[\delta v_R, \delta v_T, \delta v_N]^T_{\text{RTN}}$ is obtained from the Lyapunov transformation in eq. 3.29 in agreement with Gaïas and D'Amico [136]:

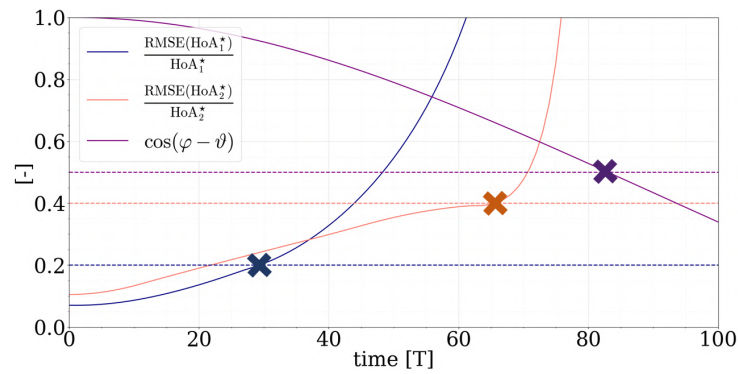


Figure 6.9: Example of manoeuvre trigger logic: the formation is reconfigured as soon as one of the indicators exceeds the corresponding threshold (here after ~ 30 periods).

$$\begin{cases} na_c \Delta \delta a = +2 \delta v_T \\ na_c \Delta \delta \lambda = -2 \delta v_R \\ na_c \Delta \delta e_x = +\delta v_R \sin u_{man} + 2 \delta v_T \cos u_{man} \\ na_c \Delta \delta e_y = -\delta v_R \cos u_{man} + 2 \delta v_T \sin u_{man} \\ na_c \Delta \delta i_x = +\delta v_N \cos u_{man} \\ na_c \Delta \delta i_y = +\delta v_N \sin u_{man} \end{cases} \quad (6.35)$$

where δv_i denotes the adimensional intensity of the pulse in the i -th direction.

In the following, the formation maintenance based on impulsive manoeuvres is addressed, treating first the control in the x-z (radial - across-track) plane, and subsequently that in the y (along-track) direction.

6.3.1. Radial/Across-track control

Both the safety and performance requirements depend only on the x-z dynamics. Indeed, from the scientific perspective, this is true in the context of across-track interferometry, where the along-track separation does not influence the performance. To be precise, this is valid within the capability of the satellite of changing the antenna pointing either through beam steering or mechanical steering (see Fiedler et al. [59]). From the safety standpoint, as the separation along the y-direction is subject to higher uncertainties, ensuring that the mutual distances do not reach zero in the x-z plane is a reasonably conservative hypothesis. For the formation maintenance on the x-z plane, manoeuvres consisting of a pair of tangential impulses are considered. The δv cost can then be formulated in terms of the ROEs starting from the GVE [136]. D'Amico [27] has also shown that this approach leads to the lowest possible δv . The two impulses are computed as:

$$\begin{cases} \delta v_{T1} = \frac{na_c}{4}(\delta a_{man} - \delta a) + \|\Delta \mathbf{e}_{man} - \Delta \mathbf{e}\| \\ \delta v_{T2} = \frac{na_c}{4}(\delta a_{man} - \delta a) - \|\Delta \mathbf{e}_{man} - \Delta \mathbf{e}\| \\ u_{man} = \arctan \left(\frac{\delta e_{y,man} - \delta e_y}{\delta e_{x,man} - \delta e_x} \right) \end{cases} \quad (6.36)$$

where u_{man} is the mean argument of latitude in correspondence with the manoeuvre. Regarding the ROEs, $\Delta \mathbf{e}$ is the relative eccentricity vector, and the subscript $(\cdot)_{man}$ indicates the desired state after the two impulses, whereas its absence indicates the state right before the start of the manoeuvre.

Concerning the out-of-plane control, the δv is given by the desired variation in the relative

inclination vector $\Delta \mathbf{i}$ and can be achieved by a single impulse in the normal direction.

$$\begin{cases} \delta v_{N1} = n a_c \|\Delta \mathbf{i}_{man} - \Delta \mathbf{i}\| \\ u_{N,man} = \arctan \left(\frac{\delta i_{y,man} - \delta i_y}{\delta i_{x,man} - \delta i_x} \right) \end{cases} \quad (6.37)$$

The J_2 perturbation does not influence the difference in the inclinations, *i.e.*, $\Delta \delta i_x = 0$, and the same holds for the atmospheric drag, which is assumed to act only in the tangential direction. Consequently, the out-of-plane manoeuvres are only needed to compensate for the δi_y variation caused by J_2 . Moreover, according to the eq. 6.37, the corresponding impulses would always be given at $u_{N,man} = \pm \pi/2$, so at the highest possible latitude (in either of the two hemispheres).

Note that the relative eccentricity control can be performed using only tangential manoeuvres for half the cost of the relative inclination control, provided that the impulses take place at the best mean argument of latitudes. This makes the out-of-plane control intrinsically more expensive and provides a further motivation to adopt satellite formations with identical inclinations (*i.e.*, $\vartheta = \pm \pi/2$). This would, in fact, cancel the secular evolution of the $\Delta \mathbf{i}$ vector and thus eliminate the need for out-of-plane impulses.

6.3.2. Along-track control

The along-track separation does not significantly influence the interferometric performance; as for the safety, it is already ensured by the preservation of mutual distances in the x-z plane. Nevertheless, the pairs of tangential impulses employed for the radial/across-track formation keeping, unavoidably increase the along-track separation (D'Amico [27]). An additional contribution is then given by the effects of navigation and thrust-direction noise. If a y-separation larger than a certain amount is achieved, the two satellites won't be able to adequately steer in order to image the same area, removing the advantages of a bi-static acquisition (Fieder et al. [59]). It is therefore necessary to introduce a form of along-track control: for this purpose, the two following approaches have been tested.

- Similarly to the solution implemented in the PRISMA mission (D'Amico and Larson [40]), one can exploit the coupling between the relative semi-major axis and the mean argument of longitude: the idea is to use a δa which is slightly different from zero, so that, knowing the temporisation of the in-plane impulses, $\delta \lambda$ oscillates around its nominal value without diverging. The effect is the introduction of a constant offset between satellites in the radial direction. This approach does not require dedicated manoeuvres for the along-track formation maintenance. It is referred to

as $\mathcal{S}_1$.

- An approach more similar to that implemented for the in-plane control, based on performing a manoeuvre whenever a given threshold on $\delta\lambda$ is exceeded. To do so, one could exploit a series of at least three tangential impulses separated by half an orbital period (Gaias and D’Amico [136]). In this case, the manoeuvring schedule is designed so that the δa achieved in between the impulses couples with $\delta\lambda$, bringing the latter to its nominal value after the conclusion of the manoeuvre. Alternatively, if a more precise control of the along-track separation was needed, one could employ a single pair of radial manoeuvres. These would indeed allow for acting directly on $\delta\lambda$, avoiding reliance on the coupling, as it is clear from eq. 6.35. The symbol $\mathcal{S}_2$ is used to denote this second scheme.

The outcome of these two different strategies is reported in Figure 6.10, where the frequent low-intensity tangential pulses required by $\mathcal{S}_2$ to control $\delta\lambda$ are absent for $\mathcal{S}_1$. Note that the absence of radial impulses comes from the fact that only tangential manoeuvres are employed for the control of the ROEs: $[\delta a, \delta\lambda, \Delta e_x, \Delta e_y]$. The small tangential thrusts employed by the strategy $\mathcal{S}_2$ (plot (b)) are absent in plot (a), since the $\mathcal{S}_2$ strategy automatically compensates the along-track displacements. Despite this difference, the two strategies require a similar Δv budget, as is shown by the numerical results in section 6.3.3. The case in the plot (c) is particularly remarkable: it is obtained in the particular case with $\vartheta_j = \pm\pi$. Since this solution eliminates the secular variation of δi_y due to J_2 , the need to perform the more expensive out-of-plane manoeuvres is in fact eliminated.

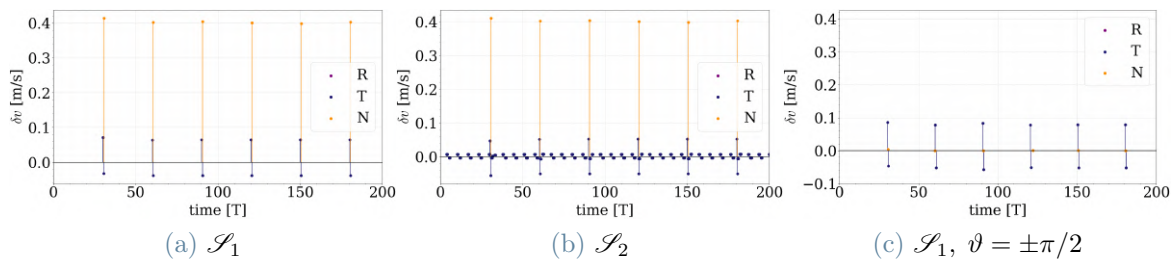


Figure 6.10: Representative examples of manoeuvre temporisation schemes.

6.3.3. Numerical results

To provide an estimate of the formation maintenance cost, the representative scenario described in Table 6.3 is considered. Since the air density is highly affected by solar and geomagnetic activity, three different cases are considered. Specifically, the average density $\bar{\rho}$ is computed from the exponential model of the atmosphere considering a scale height

Orbital parameter	Value
Orbit type	SSO with inclination $\sim 97.5^\circ$
Formation categories	xz-PCO, Helix with identical inclinations
N° satellites	3, 4
Altitude range	[500 km, 580 km]
Average atmosphere density range	$[1, 5] \cdot 10^{-12} \text{ kg/m}^3$
C_B	0.021 [m ² /kg]
ΔC_B	$[0\%, 50\%] \cdot C_B$
Chief's orbital tube diameter	250 m
Baseline range	$\sim [200 \text{ m}, 2000 \text{ m}]$ (from HoA optimisation)
SAR parameter	Value
Incidence angle	37.0°
Look angle	33.8°
Wavelength	0.031 m (X-band)
Nominal HoAs	{10 m, 30 m}

Table 6.3: Orbital and interferometric reference parameters.

of $H = 50 \text{ km}$ as a mid-range value at $\sim 500 \text{ km}$ altitude under moderate solar activity (Vallado [89]). More precisely, the Δv are computed for the interferometrically optimal configurations reported in the Figures 5.8 and 5.9 for three and four satellites respectively, considering the two types of notable generalised Helix configurations presented in Section 6.1:

1. Relative orbits that are scaled versions of each other (specifically, all xz-PCOs).
2. Configuration with aligned $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ vectors, *i.e.*, the J_2 -robust configuration obtained with orbits of identical inclination for all deputies (see Figure 6.1).

The numerical results of the simulations performed are listed in Tables 6.4 and 6.5, where the formation maintenance cost per period (T) for the entire formation is reported. The Δv corresponding to the computed upper bound and to the $\mathcal{S}_1$ and $\mathcal{S}_2$ applied to the optimal xz-PCO are listed in the fifth, sixth, and seventh columns, respectively. Moreover, the last column provides the formation maintenance cost for the J_2 -robust configuration with the $\mathcal{S}_1$ strategy (in this last case, the $\mathcal{S}_2$ is not reported as the numerical results are practically identical). In particular, the results are relative to formations of three and four satellites, which have been shown to provide more than satisfactory results for In-SAR performance. The use of condition-triggered manoeuvres results in a cost reduction of approximately one order of magnitude compared to the upper bound to ensure the maintenance of the nominal relative orbit. Furthermore, the last column reports the minimum cost obtained for the generalised Helix with $\varphi_j = \vartheta_j = \pi/2 \pmod{\pi}$. In fact, this is the only possible configuration that combines the advantage of the maximum safety pro-

$\bar{\rho}$ [kg/m ³]	ΔC_B [m ² /kg]	$a\delta a_0$ [m/s]	Cost for xyz fixed orbits [m/s/T]	Upper bound [m/s/T]	Cost $\mathcal{S}_1$ [m/s/T]	Cost $\mathcal{S}_2$ [m/s/T]	Cost $\vartheta = \pi/2$ [m/s/T]
–	0	0	53.260	0.183	0.017	0.017	0.0004
1×10^{-12}	5×10^{-3}	-2.61×10^{-4}	53.262	0.183	0.019	0.019	0.0018
1×10^{-12}	1×10^{-2}	-5.23×10^{-4}	53.262	0.183	0.021	0.021	0.0033
3×10^{-12}	5×10^{-3}	-7.85×10^{-4}	53.262	0.183	0.022	0.022	0.0051
3×10^{-12}	1×10^{-2}	-1.57×10^{-3}	53.262	0.183	0.028	0.027	0.0099
5×10^{-12}	5×10^{-3}	-1.30×10^{-3}	53.263	0.183	0.026	0.026	0.0083
5×10^{-12}	1×10^{-2}	-2.61×10^{-3}	53.263	0.184	0.034	0.044	0.0168

Table 6.4: Three-satellite formation maintenance cost, optimal configuration in a dawn-dusk SSO, $\mathcal{S}_2$ with along-track manoeuvre after 100 m.

vided by the Helix with the elimination of the secular effect of J_2 on δi_y , thereby removing the need for manoeuvres in non-tangential directions. This robustness to perturbations makes it the most cost-effective alternative, as the required control effort becomes almost negligible, especially when compared to that required to counteract the semi-major axis decay caused by absolute drag. The latter is reported separately in Table 6.6 in order to better appreciate the effects of relative perturbations. Additionally, as it heavily depends on the altitude and the differential ballistic coefficient, the costs are computed for different values of the average density $\bar{\rho}$ and ballistic coefficients C_B .

In addition to the mentioned strategies, the Δv that would be spent to maintain time-fixed baselines is also reported for comparison purposes. Specifically, it is listed in the fourth column of Tables 6.4 and 6.5. This solution would guarantee the same target HoA^{*} everywhere, but it would entail a significantly higher cost (Scala et al. [35]). To maintain time-fixed baselines necessary to impart the additional acceleration:

$$\mathbf{a}_{fix} = [-3n^2x, 0, n^2z]_{\text{RTN}}^T. \quad (6.38)$$

As expected, the resulting Δv is much higher compared to other strategies following the natural relative motion of the satellites. Furthermore, it is also higher than the one obtained by Scala et al. [35]. However, this discrepancy is readily explained by the fact that in this representative example, the inter-satellite distances are up to some kilometres (following the HoA optimisations), while in [35], values up to 100 m are considered.

Figure 6.11 shows the time evolution of the relative orbit of one of the deputies for one of the optimal solutions found. In particular, both the ROEs and the trajectory followed in the x-z plane are illustrated, considering both approaches proposed for along-track formation keeping.

Finally, in Figure 6.12, the mutual distances in the x-z plane are reported for formations

$\bar{\rho}$ [kg/m ³]	ΔC_B [m ² /kg]	$a\delta a_0$ [m/s]	Cost for xyz fixed orbits [m/s/T]	Upper bound [m/s/T]	Cost $\mathcal{S}_1$ [m/s/T]	Cost $\mathcal{S}_2$ [m/s/T]	Cost $\vartheta = \pi/2$ [m/s/T]
–	0	0	83.449	0.278	0.024	0.024	0.0009
1×10^{-12}	5×10^{-3}	-2.61×10^{-4}	83.449	0.278	0.027	0.027	0.0018
1×10^{-12}	1×10^{-2}	-5.23×10^{-4}	83.449	0.2788	0.029	0.029	0.0037
3×10^{-12}	5×10^{-3}	-7.85×10^{-4}	83.449	0.293	0.032	0.032	0.0060
3×10^{-12}	1×10^{-2}	-1.57×10^{-3}	83.449	0.294	0.039	0.039	0.0149
5×10^{-12}	5×10^{-3}	-1.30×10^{-3}	83.449	0.293	0.037	0.037	0.0208
5×10^{-12}	1×10^{-2}	-2.61×10^{-3}	83.450	0.297	0.049	0.049	0.0248

Table 6.5: Four-satellite formation maintenance cost, optimal configuration in a dawn-dusk SSO, $\mathcal{S}_2$ with along-track manoeuvre after 100 m.

Altitude [km]	$\bar{\rho}$ [kg/m ³]	Cost per satellite [m/s/T]
500	1×10^{-12}	0.011
526	3×10^{-12}	0.018
580	5×10^{-12}	0.027

Table 6.6: Absolute drag cost.

optimised for $[\text{HoA}_1^*, \text{HoA}_2^*] = [10 \text{ m}, 30 \text{ m}]$. Specifically, it demonstrates the validity of the applied collision avoidance strategy in optimal three- and four-satellite formations. Having set $a_c \delta r_{\min} = 100 \text{ m}$ as the threshold for the minimum distance allowed in the x-z plane in eq. 4.6, this requirement is respected even in the presence of disturbances and control impulses. Notably, the two plots on the right refer to the passive safe configuration with scaled xz-PCOs. Conversely, the two on the left correspond to a common orientation of the $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ vectors for all satellite pairs. Specifically, the same angle $\vartheta = \pi/2$ has been chosen, leading to the absence of any secular trends, which are instead clearly visible for the xz-PCOs.

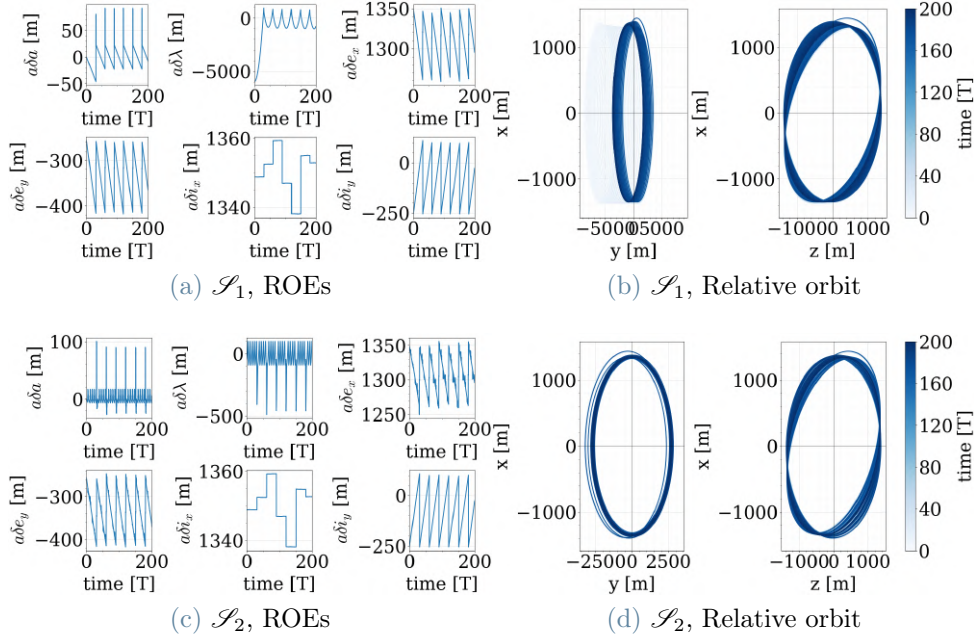


Figure 6.11: Time evolution of ROEs and optimised relative orbit under perturbations and control projected onto the x-z plane of the RTN.

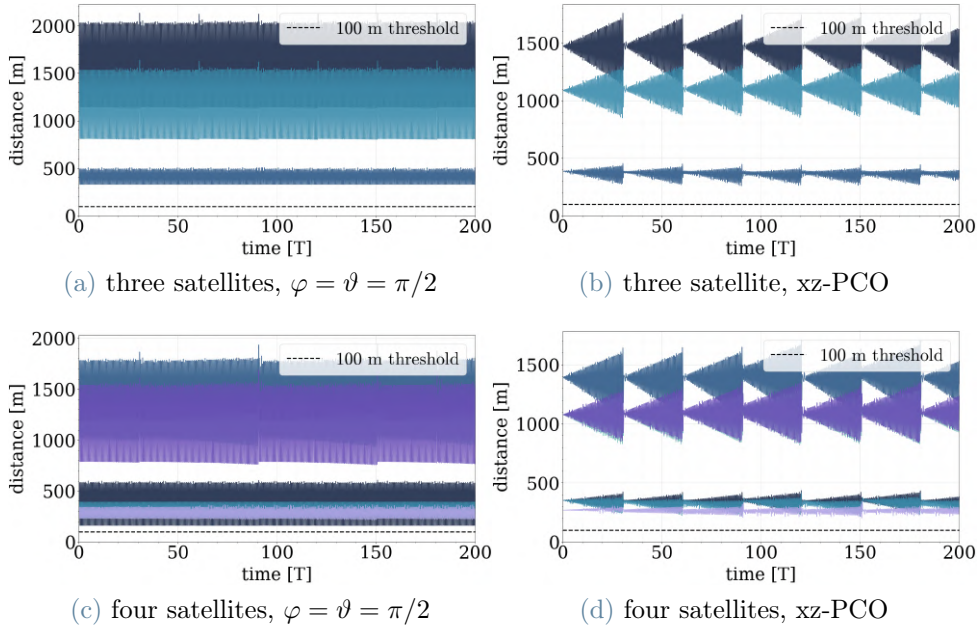


Figure 6.12: Relative distance between each satellite pair in the xz plane subject to all the disturbances illustrated.

7 | Incidence Angle Optimisation

The optimisation presented in the previous chapters is based on the assumption that the incidence angle is fixed. Some systems actually adopt this type of solution due to its simplicity of implementation. However, in general, this angle represents an additional degree of freedom that can be exploited to improve the quality of the DEM. An important complement to the study conducted so far would therefore consist of including this variable in the optimisation process. This Chapter addresses this purpose by proposing an analytical method to minimise the HoA variability at a global level by optimising the range of incidence angles used at each latitude. The approach is primarily aimed at systems, such as TanDEM-X, for which the area to be imaged becomes significantly smaller than the ground access range as the latitude increases. A two-step approach is presented: an initial approximate solution is derived in closed form and subsequently refined. The method is validated against numerical results and compared with simpler strategies involving fixed incidence angles, showing reduced variation of the HoA for different formation parameters.

The Chapter is structured as follows: Section 7.1 presents the background of the problem, also introducing the parametric expression of the HoA derived in terms of the incidence angle. In Section 7.2 is illustrated the methodology adopted to solve the optimisation problem. The validation of the followed approach is conducted in Section 7.3, using TanDEM-X parameters as an example. Finally, Section 7.4 outlines a possible way to extend the method's validity to different kinds of formations, other than those peculiar to TanDEM-X.

7.1. Background and Problem Formulation

An element to consider for the generation of high-quality DEMs is that SAR acquisitions requirements change with latitude. Due to Earth's curvature, in fact, the width of the required ground coverage decreases moving towards the poles, as the ground access ranges of consecutive orbits increasingly overlap, reducing the area that has to be imaged, as sketched in Figure 7.1. Some SAR systems are particularly well-suited to exploit this effect, as they map in subsequent acquisitions multiple narrower swaths to cover a ground

access range of considerable width. In TanDEM-X, for instance, a ground access range of 240 km is mapped during different orbits with swaths of about 30 km for the stripmap acquisition mode by selectively activating proper elevation beams, which point to different parts of the ground access range (Krieger et al. [6]). For this kind of system, the problem of ensuring a uniform HoA can be partially addressed by a thoughtful choice of the non-overlapping portion of the ground access range at each latitude. This, in turn, can be equivalently formulated as the optimal selection of the corresponding incidence angle range. The study presented in this Chapter has the objective of deriving an analytical expression for the range of incidence angles that minimises the variability of the HoA as a function of latitude in the case of Helix formations.

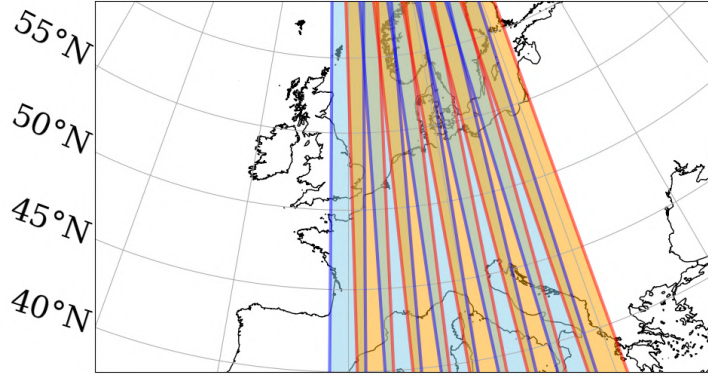


Figure 7.1: Progressive overlapping of a 240 km-wide access range over ten consecutive orbits, displayed in alternating colours for latitudes ranging from 40° N to 60° N, starting from Greenwich meridian.

7.1.1. Model of the relative dynamics

The relative dynamics is formulated by means of the relative eccentricity and inclination vectors $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$, already defined in Chapter 3 in agreement with D’Amico and Montenbruck [29], here reported again for readability.

$$\Delta \mathbf{e} = \delta e \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}; \quad \Delta \mathbf{i} = \delta i \begin{bmatrix} \cos \vartheta \\ \sin \vartheta \end{bmatrix}, \quad (7.1)$$

where δe and δi and φ and ϑ are the magnitudes and phases of $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$, respectively, and the subscripts c and d indicate the chief and the deputy satellites, respectively.

If the assumptions on which the Hill-Clohessey-Wiltshire equations (HCW) are based are valid, bounded, periodic orbits centred on the chief can be expressed in the radial-tangential-normal (RTN) frame using the amplitude-phase formulation of the relative

eccentricity and inclination vectors (D’Amico [24], see Chapter 3):

$$\begin{cases} x(u) = -a_c \delta e \cos(u - \varphi) \\ y(u) = 2 a_c \delta e \cos(u - \varphi) \\ z(u) = a_c \delta i \sin(u - \vartheta) \end{cases} \quad (7.2)$$

where $x(u)$, $y(u)$, and $z(u)$ are the radial, along-track, and across-track components of the baseline, respectively, a_c denotes the semi-major axis of the chief’s orbit, and u stands for the mean argument of latitude defined as the sum of the argument of the perigee and the mean anomaly.

In the following, it is assumed that the chief satellite flies in a circular and polar orbit. The circularity, already required by the HCW equations, implies that the mean and true arguments of latitude are identical; the polar hypothesis ensures that all latitudes are covered. These two assumptions allow considering u as equivalent to the geocentric latitude ϕ over which the satellites pass. In typical SAR missions, sun-synchronous orbits are often employed by imposing an inclination between 96° to 99° depending on the altitude, hence slightly deviating from a perfectly polar orbit. Indeed, the polar assumption, which we retain to display the results without neglecting any latitude value, could be easily relaxed by considering the relationship between u and ϕ reported in Chapter 3, also reported below:

$$\sin \phi = \sin u \sin i . \quad (7.3)$$

For convenience in describing the relative motion, the variable u is initially adopted instead of ϕ . Additionally, a spherical shape of the Earth is assumed, without distinguishing between geocentric and geodetic latitude (see Seidelmann and Urban [114]).

Only Helix orbits are considered in the following: bounded orbits obtained after having imposed the condition $\vartheta = \varphi + k\pi$ (see Chapter 3, Table 3.1). This allows for maintaining a safe distance in the x - z plane, which is crucial considering the higher uncertainties the along-track component is subject to. In fact, this configuration ensures that the vertical coordinate is maximised when the across-track one reaches zero and vice-versa [29].

Moreover, the phases of the relative eccentricity and inclination vectors (φ and ϑ) are often set equal to $\pi/2$ (or $3\pi/2$), like for TanDEM-X. As illustrated in Chapter 3 and further confirmed by the results in Chapter 6, this choice has the merit of providing a more robust stability to the perturbing effect of J_2 because it eliminates the secular evolution of the $\Delta \mathbf{i}$ vector, leading to a less frequent request for a control action to maintain the safety of the formation, as well as to a slower deviation from the nominal configuration [29]. This

assumption is therefore adopted in all the case studies presented in Section 7.3.

7.1.2. Parametric expression of the HoA

Coherent with the rest of the work, the height of ambiguity is considered as the principal figure of merit to optimise. Specifically, for a bistatic acquisition (see Chapter 2) it is defined as:

$$\text{HoA} = \frac{\lambda r(\theta_i) \sin \theta_i}{B_{\perp}(\theta_l, u)}, \quad (7.4)$$

where λ indicates the wavelength, θ_i the angle of incidence (or incidence angle), r_0 the slant range, and $B_{\perp}$ the effective baseline. To understand how θ_i influences the HoA, it is necessary to explicitly state the relations between all the quantities in eq. 7.4.

Referring to the geometry in Figure 7.2, the position of a generic point P on the ground can be defined by means of either the incidence angle θ_i or by the look angle θ_l , whose relation is given by the law of sines (Oliver and Quegan [137]):

$$\sin \theta_i = \frac{R_{\oplus} + h}{R_{\oplus}} \sin \theta_l, \quad (7.5)$$

where $R_{\oplus}$ is the radius of the Earth, or through the angle γ measured from the centre of the Earth, as:

$$\theta_l = \theta_i - \gamma. \quad (7.6)$$

The ground range coordinate of the generic point P is indicated as s and can be related to the corresponding look angle θ_l using eqs. 7.5 and 7.6:

$$\tan \theta_l = \frac{\sin \gamma}{\frac{R_{\oplus} + h}{R_{\oplus}} - \cos \gamma} = \frac{\sin \left(\frac{s}{R_{\oplus}} \right)}{\frac{R_{\oplus} + h}{R_{\oplus}} - \cos \left(\frac{s}{R_{\oplus}} \right)}. \quad (7.7)$$

The total ground width W_g to be imaged for a given latitude ϕ reads:

$$W_g(\phi) = s_B - s_A = R_{\oplus}(\gamma_B - \gamma_A) = W_{g0} \cos \phi, \quad (7.8)$$

where W_{g0} is the ground access range at the equator.

The expression of the slant range r_0 as a function of the look angle can be computed by applying the law of cosines to the triangle OCP , from which the following expression is

For given orbital parameters of the formation $\{a_c, \delta e, \delta i, \varphi, \vartheta\}$, the HoA therefore becomes

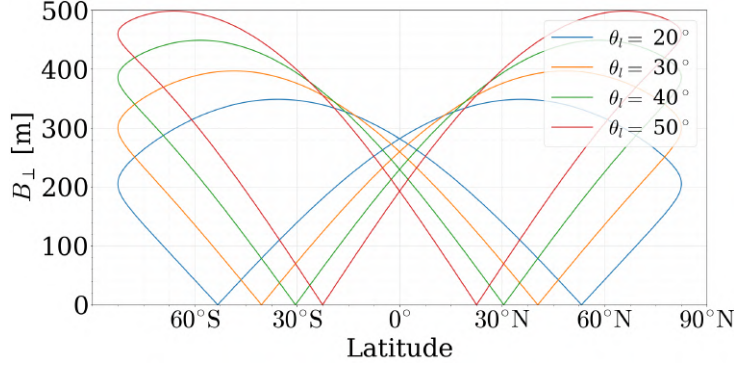


Figure 7.3: Sensitivity of the effective baseline to a change of the look angle.

a function of the two variables u and $\theta_l(s)$ given by:

$$\text{HoA}(\theta_l(s), u) = \lambda \frac{(R_\oplus + h) \cos \theta_l(s) - \sqrt{(R_\oplus + h)^2 \cos^2 \theta_l(s) - (h^2 + 2R_\oplus h)}}{a_c |\delta e \cos(u - \varphi) \sin \theta_l(s) + (-1)^k \delta i \sin(u - \varphi) \cos \theta_l(s)|} \frac{R_\oplus + h}{R_\oplus} \sin \theta_l(s). \quad (7.12)$$

7.2. Optimisation strategy

The purpose of the optimisation is to minimise the variability of the HoA with respect to a given reference value, $\text{HoA}^\star$. Denoting with $s_A(u)$ and $s_B(u)$ the boundaries of the total ground width to be imaged as functions of the argument of latitude, the aim is to define them to minimise the deviation between the HoA and $\text{HoA}^\star$, according to a selected metric. In the following, it is omitted, making explicit the dependency on u to facilitate readability. Thanks to eq. 7.8, s_B is known from s_A and the width of the swath, which in turn depends only on the latitude. Formally, it must be determined the function s_A that minimises the functional J in the following equation:

$$\arg \min_{s_A} J := \int_{s_A}^{s_A + W_g} |\text{HoA}(s) - \text{HoA}^\star|^2 ds. \quad (7.13)$$

To find the minimum of J , eq. 7.13 has to be evaluated in correspondence with its stationary points, which satisfy:

$$\frac{\partial}{\partial s_A} \int_{s_A}^{s_A + W_g} |\text{HoA}(s) - \text{HoA}^\star|^2 ds = 0. \quad (7.14)$$

Eq. 7.14 could be solved with Leibniz integration rule provided that the integrand is continuous over the interval $[s_A, s_A + W_g]$. However, this is not guaranteed for every latitude, since beyond a certain value of u there is always a certain value of the look-angle θ_l (and so of s) for which $B_\perp$ goes to zero. The HoA reaches an asymptote, whose position is computed as in Chapter 4:

$$\tan(u - \varphi) = (-1)^{k+1} \frac{\delta e}{\delta i} \tan \theta_l. \quad (7.15)$$

However, by fixing $\varphi = \pi/2$ and $\vartheta = \varphi + \pi$ (as it is the case for TanDEM-X), one can ensure that the function in the integral is continuous for $u \in [0, \pi/2)$ because the left and right-hand terms in eq. 7.15 would always have opposite signs, and $B_\perp$ could nullify only when $u = \pi/2$ and $\theta_l = 0^\circ$, *i.e.*, at the poles with the radar nadir-pointing. Therefore, eq. 7.14 can be easily solved for u belonging to the first quadrant. As detailed in Chapter 4, since the HoA has a period of π in the mean argument of latitude, it is enough to restrict the optimisation to a single hemisphere. Thanks to the polar orbit hypothesis, a value of u belonging to a given quadrant uniquely defines a pair hemisphere-orbit's half. For example, the points where $u \in (0, \pi/2)$ are located in the northern hemisphere and are imaged in the ascending half of the orbit. This is crucial because it allows ensuring the continuity of the integrand for global coverage. Table 7.1 lists the intervals of u where the integrand is continuous depending on the parity of k (the factor deciding the mutual orientation of $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ introduced in eq. 7.11) and the look direction, for $\varphi = \pi/2$.

Applying Leibniz rule, the solution to eq. 7.14 is given by one of the following two:

$$\text{HoA}(s_A) = \text{HoA}(s_A + W_g); \quad (7.16)$$

$$\frac{\text{HoA}(s_A) + \text{HoA}(s_A + W_g)}{2} = \text{HoA}^*. \quad (7.17)$$

Notably, eq. 7.16 does not admit a solution in any of the cases reported in Table 7.1. This is because the function $\text{HoA}(s)$ is monotonic with respect to s except when the look angle reaches the critical value, which the intervals in Table 7.1 are designed to avoid.

Look direction	k parity	Continuity interval	North hemisphere	South hemisphere
right	odd	$u \in [0, \pi/2) \bmod \pi$	ascending	descending
right	even	$u \in (3\pi/2, \pi] \bmod \pi$	descending	ascending
left	odd	$u \in (3\pi/2, \pi] \bmod \pi$	descending	ascending
left	even	$u \in (0, \pi/2) \bmod \pi$	ascending	descending

Table 7.1: Acquisitions ensuring continuity for $\varphi = \pi/2, \vartheta = \varphi + k\pi$

Substituting the expression of the HoA from eq. 7.4 into eq. 7.17, a non-differential scalar problem is obtained, which cannot be solved in closed form. Nevertheless, an analytical solution can still be found by introducing the following two, whose limitations are better detailed below: 1) the slant range is expressed according to a flat-Earth geometry, and 2) the HoA is linear with respect to s :

$$\begin{cases} r_0 \approx \frac{h}{\cos \theta_l} \\ \frac{\text{HoA}(s_A) + \text{HoA}(s_A + W_g)}{2} \approx \text{HoA}\left(s_A + \frac{W_g}{2}\right) \end{cases} \quad (7.18)$$

The idea is therefore to determine an approximation $\tilde{\theta}_l$ of the look angle $\theta_l^*(s^*)$, solution of $\text{HoA}(\theta_l^*) = \text{HoA}^*$, and assume a swath perfectly centred around the corresponding point on the ground: $\tilde{s}$. The resulting simplified formulation requires the computation of the roots of the four-degree polynomial:

$$\begin{aligned} & t^4 - 2(-1)^k \left(\frac{\delta e}{\delta i \tan(u - \varphi)} + (-1)^j \frac{\lambda}{\text{HoA}^*} \frac{R_\oplus + h}{R_\oplus} \frac{h}{\delta i \sin(u - \varphi)} \right) t^3 - 2t^2 \\ & + 2(-1)^k \left(\frac{\delta e}{\delta i \tan(u - \varphi)} - (-1)^j \frac{\lambda}{\text{HoA}^*} \frac{R_\oplus + h}{R_\oplus} \frac{h}{\delta i \sin(u - \varphi)} \right) t + 1 = 0, \end{aligned} \quad (7.19)$$

where $t = \tan(\tilde{\theta}_l/2)$, and j is even or odd, if the argument of the absolute value in eq. 7.11 is positive or negative, respectively.

Concerning the limitations caused by the hypotheses in eqs. 7.18, they introduce an error that shifts the resulting swath away from the nadir compared to the exact one. To compensate for it, the following corrections to be applied ex-post, once eq. 7.19 has been solved, are proposed. First, consider that the first assumption underestimates the actual value of the slant range, leading to an overestimation of the angle θ_l^* . Since the introduced error is relatively small, a first-order correction is sufficient to correct it. Specifically, this consists of performing a single step of the Newton method:

$$\theta_l^* \approx \tilde{\theta}_l + \frac{\text{HoA}^* - \text{HoA}(\tilde{\theta}_l)}{\left. \frac{\partial \text{HoA}}{\partial \theta_l} \right|_{\tilde{\theta}}} . \quad (7.20)$$

The linear hypothesis, on the other hand, assumes that the point $\tilde{s}$ to be kept at the centre of the swath to respect eq. 7.17 coincides with s^* . This introduces an error that depends on how much the function $\text{HoA}(s)$ differs from a straight line. To compute the shift between $\tilde{s}$ and s^* , it is therefore necessary to consider the non-linear behaviour of $\text{HoA}(s)$, which is accounted for by substituting its Taylor expansion truncated to the

second order into eq. 7.17. The required derivatives of the HoA with respect to s can be computed with the chain rule:

$$\text{HoA}' = \frac{\partial \text{HoA}}{\partial s} = \frac{\partial \text{HoA}}{\partial \theta_l} \frac{d\theta_l}{ds}; \quad (7.21)$$

$$\text{HoA}'' = \frac{\partial^2 \text{HoA}}{\partial s^2} = \frac{\partial^2 \text{HoA}}{\partial \theta_l^2} \left(\frac{d\theta_l}{ds} \right)^2 + \frac{\partial \text{HoA}}{\partial \theta_l} \frac{d^2 \theta_l}{ds^2} \quad (7.22)$$

where the expressions of all factors are obtainable leveraging eq. 7.7 and eq. 7.12 explicitly given in appendix B. The shift between the two points, therefore, reads:

$$\delta = \tilde{s} - s^* = -\frac{\text{HoA}'(s^*)}{\text{HoA}''(s^*)} + \sqrt{\left(\frac{\text{HoA}'(s^*)}{\text{HoA}''(s^*)} \right)^2 - \frac{W_g^2}{4}}. \quad (7.23)$$

7.3. TanDEM-X Example

Some examples of the application of the proposed method are shown in the following, using as a reference the TanDEM-X mission. The two distinct scenarios reported in Table 7.2 are considered: In the first one, the orbital parameters are taken from D'amico et al. [138]; in the second one, the parameters $\delta e = \delta i$ have been considered in compliance with those used by TanDEM-X to map high latitudes according to Montenbruck et al. [139].

Figure 7.4 shows how the optimisation process shapes the HoA across the swath for different values of the latitude without limitations on the access range in Scenario 1. The proposed solution progressively introduces a shift towards the nadir to keep the HoA always centred around the desired value as the latitude increases. In practice, the radar would select an increasingly smaller part of the ground access range initially covered at the equator when approaching the poles, *i.e.*, it is here assumed that the optimal swath must never exceed the limits imposed at the equator (this can be justified, *e.g.*, by radar observation requirements). Figure 7.4 reports, therefore, the optimal results with and without the equatorial boundaries, with both the numerical and the analytical approximated solution. The maximum error between the two is 0.37 km, or 0.15 % of the access range. Remarkably, the correction factor plays a crucial role, as the solution

	$a_c \delta e$	$a_c \delta i$	φ	ϑ	HoA*	W_{g0}
Scenario 1	300 m	600 m	$\pi/2$	$3\pi/2$	30 m	240 km
Scenario 2	500 m	500 m	$\pi/2$	$3\pi/2$	30 m	240 km

Table 7.2: TanDEM-X parameters in two distinct mission phases

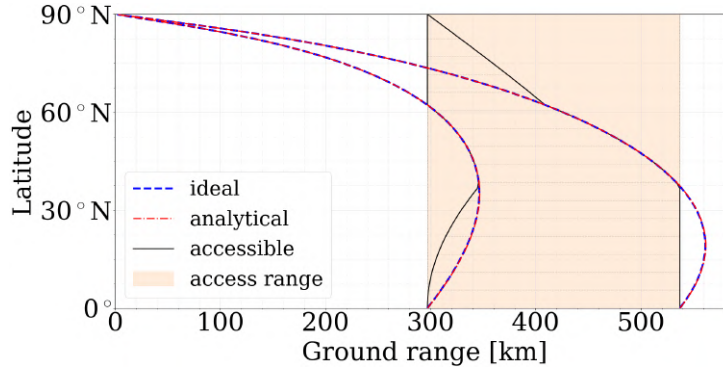


Figure 7.4: Ideal and accessible swaths for Scenario 1 in Table 7.2; the numerical and analytical solutions are represented in blue and red, respectively

of eq. 7.19 would result in an error of 20.27 km, equal to 8.45% of the entire access interval. Focusing on Scenario 1, the shape in Figure 7.4 shows that the optimal look angle increases to a maximum and then decreases with the latitude. Specifically, the coloured area represents the ground access range, *i.e.*, the ground strip that the radar has access to as the latitude varies. The dashed curves give the theoretically optimal portion of the ground range to be acquired; more precisely, the one found with a numerical optimisation and with the proposed approach are displayed in blue and red, respectively. Finally, the black line indicates the optimal swath to acquire, considering the limited ground access range. This resulting shape is mainly due to the effective baseline, which increases in the first part of the orbit, reaching its maximum when:

$$\tan u = \frac{\pi}{2} + \frac{\delta e}{\delta i} \tan \theta_l, \quad (7.24)$$

and then decreases, as the satellites move towards the pole. For $\varphi = \pi/2$ or $3\pi/2$, the influence of δe and δi is well separable. As pointed out in Chapter 4, they respectively determine the HoA at the poles ($u = \pi/2$) and at the equator ($u = 0$):

$$\text{HoA}|_{u=\pi/2} = \frac{\lambda r_0(\theta_l)}{a_c \delta e} \frac{R_\oplus + h}{R_\oplus}; \quad (7.25)$$

$$\text{HoA}|_{u=0} = \frac{\lambda r_0(\theta_l)}{a_c \delta i} \frac{R_\oplus + h}{R_\oplus} \tan \theta_l. \quad (7.26)$$

This is a direct consequence of having chosen $\varphi = \pi/2$, for which the separation between satellites is exclusively in the across-track direction at the equator and in the vertical direction at the poles, as in TanDEM-X. Remarkably, these last two equations explain the optimal swath in Figure 7.4 to move towards the nadir as the latitude increases. In

fact, to keep a constant HoA, it must hold:

$$\frac{r(\theta_i)|_{u=\pi/2}}{\delta e} = \frac{r(\theta_i) \tan(\theta_l)|_{u=0}}{\delta i}, \quad (7.27)$$

from which it follows that $\theta_i|_{u=\pi/2} < \theta_i|_{u=0}$ (*i.e.*, the ideal swath is closer to the nadir at the poles than at the equator) when:

$$\frac{r(\theta_i)|_{u=\pi/2}}{r(\theta_i)|_{u=0}} < 1 \Leftrightarrow \tan \theta_l < \frac{\delta i}{\delta e}. \quad (7.28)$$

Since the nominal parameters reported in Table 7.2 are such that $\delta i = 2\delta e$, the last equation becomes:

$$\theta_l < \arctan(2) \approx 63.4^\circ, \quad (7.29)$$

which is likely to be verified in realistic scenarios. Indeed, a smaller look-angle is needed to prevent the loss from shadowing and geometric distortion such as layover (see Chapter 2). As an example, consider that TanDEM-X can operate in the look-angle range: $\theta_l \in [20^\circ, 45^\circ]$ [6].

7.3.1. Comparison with alternative strategies

To evaluate the benefit of the proposed strategy, it is compared with two other straightforward cases. In the first one, the swath is always taken starting from the near range point of the access range (near range acquisition); in the second one, the swath is taken at the centre of the access range. Figure 7.5 illustrates the part of the access range that is retained in each of the three strategies: near range (a), centre of the access range (b), and optimal (c) acquisitions as the satellites move along the orbit.

In Table 7.3, the reported root mean square error (RMSE) with respect to the target value $\text{HoA}^* = 30$ m for all three strategies in the two scenarios listed in Table 7.2, compared to the optimal solution. With respect to the worst-performing alternative, the optimised solution provides an absolute improvement of 11.8 m, corresponding to a relative error reduction of approximately 40%. An explicative representation of the improvement provided by the optimal choice of the incidence angle is given in Figure 7.6. Here, the histograms show the frequency with which a value of the HoA is achieved, as the satellites map all the latitudes of one hemisphere. The goal is to minimise the variance: in agreement with the results in Table 7.3, this is indeed achieved in Figure 7.6c, which corresponds to the proposed optimisation strategy.

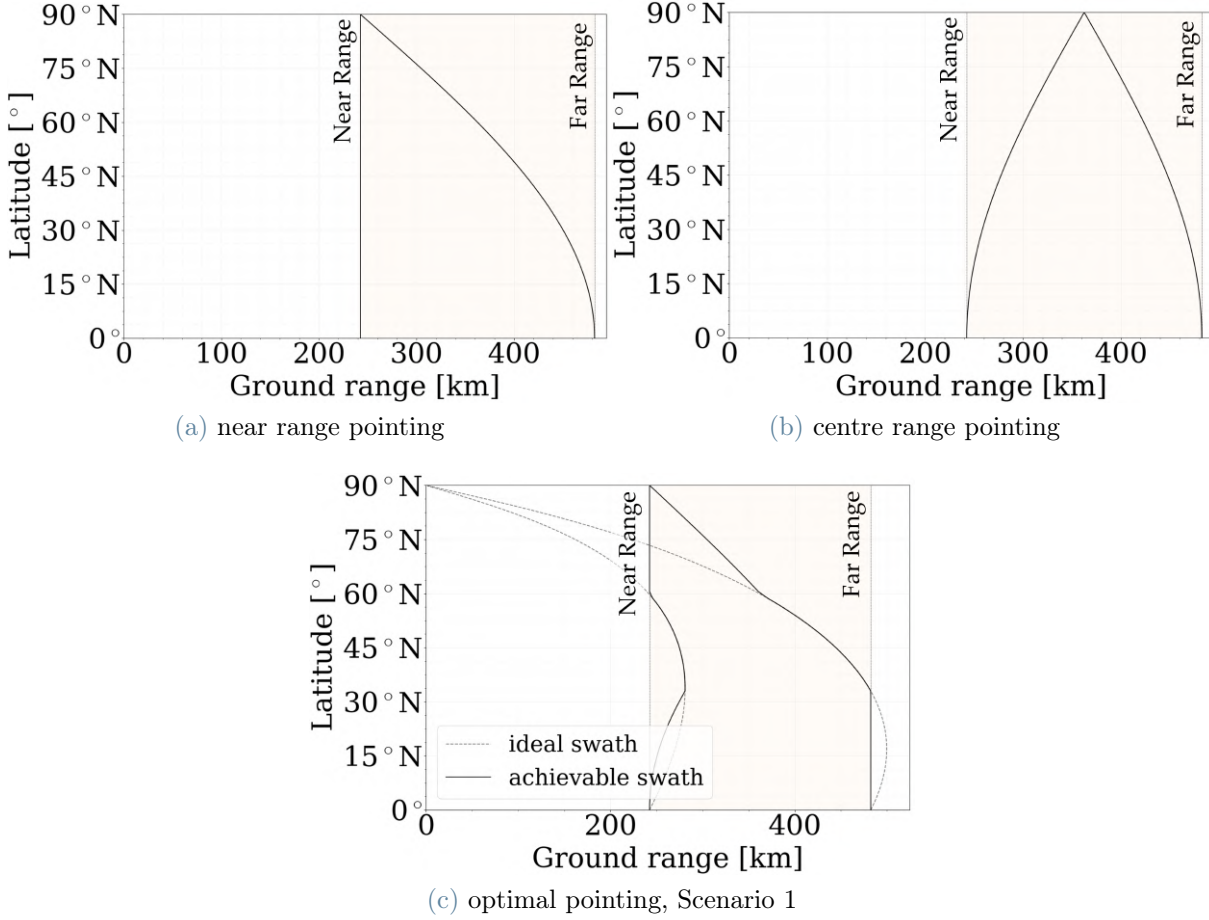


Figure 7.5: Swath acquired at each latitude with different pointing strategies.

Notably, the choice of the nominal values δe and δi deeply influences the optimal solution. This explains why each of the strategies (a) and (b) outperforms the other in different scenarios. A clear interpretation of how those two parameters shape the optimal solution is provided in Figure 7.7, where the optimal swath is computed for different values of δe and δi . Accordingly with eq. 7.25 and 7.26, δe alone determines the optimal pointing in correspondence with the pole, without affecting the swath at the equator. On the other hand, the choice of δi alone determines the result at the equator. For all other latitudes, both δe and δi are relevant.

7.3.2. Combination with the optimised ROEs

The proposed strategy to optimise the incidence angle considers a fixed set of ROEs, specifically, those of TanDEM-X reported in Table 7.2. However, in an ideal scenario, an improvement can be provided by combining the method with the optimal ROEs determined following the optimisation in Chapter 5. It has been demonstrated that, for

Strategy	Scenario 1		Scenario 2	
	absolute RMSE	relative RMSE	absolute RMSE	relative RMSE
Near range	8.96 m	29.8 %	7.23 m	24.1 %
Centre of ground range	12.25 m	40.8 %	6.25 m	20.8 %
Optimal incidence angle	8.70 m	29.0 %	5.09 m	16.9 %

Table 7.3: RMSE with respect to $\text{HoA}^* = 30$ m for sub-optimal vs optimal strategies.

Strategy	absolute RMSE	relative RMSE
Optimal ROEs + incidence angle	5.00 m	16.6%

Table 7.4: RMSE with the fully optimised strategy.

a two-satellite formation, the magnitudes of the $\Delta \mathbf{e}$ and $\Delta \mathbf{i}$ satisfy at optimality the following relation:

$$a_c \sqrt{\delta e^2 \sin^2 \theta_l + \delta i^2 \cos^2 \theta_l} = \frac{2}{\log(3 + 2\sqrt{2})} \frac{\lambda r_0}{\text{HoA}^*} \frac{R_\oplus + h}{R_\oplus} \sin \theta_l. \quad (7.30)$$

Given complete freedom in the choice of the ROEs, it is convenient to increase the value of the look angle if swaths of considerable width (in the order of hundred metres) are acquired. In fact, for a higher θ_l , the same ground range width would correspond to a narrower angular interval. Since the HoA ultimately depends on the look angle, restricting its range would prevent it from varying significantly. It is not reasonable, however, to arbitrarily increment this angle, as the farther the acquired points are from the nadir, the more the signal would be affected by atmospheric and spatial losses (Larson and Wertz [140]). As the maximum angle of incidence used by TanDEM-X is $\sim 48^\circ$ (Krieger et al. [72]), an angle of 50° is taken as the allowed upper limit. With this choice, after determining the corresponding set of ROEs compliant with an objective $\text{HoA}^* = 30$ m, the fully optimised results are reported in Table 7.4. Figure 7.8 reports the frequency with which the different values of the HoA are achieved: by comparing it with the histograms in Figure 7.6 referring to Scenario 1, the histogram shows a lower variance of the attained HoAs. On the other hand, the similarity with the results of Scenario 2 is readily explained by the fact that the δe and δi parameters are, in that case, not far from the optimal ones.

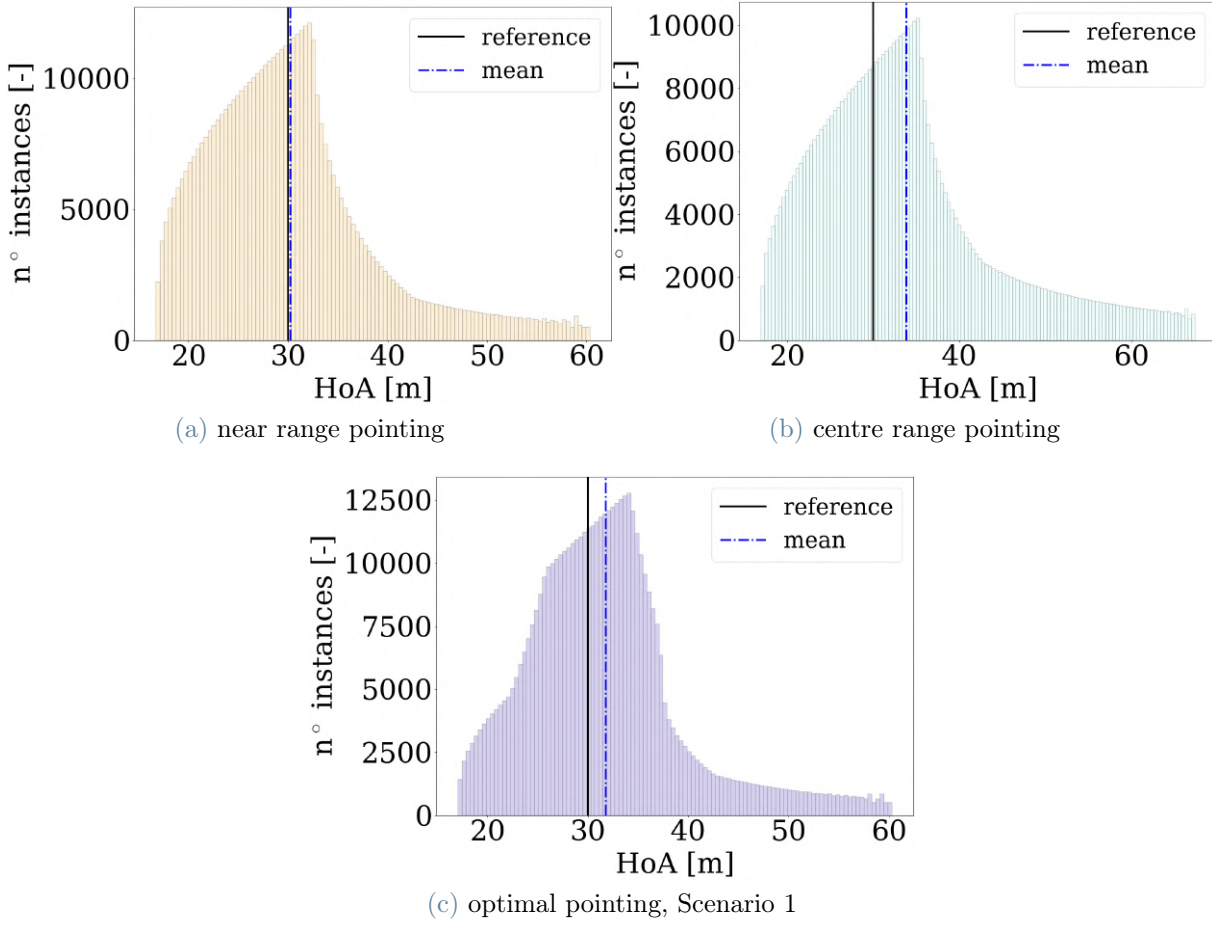


Figure 7.6: Distribution of HoA attained in a hemisphere.

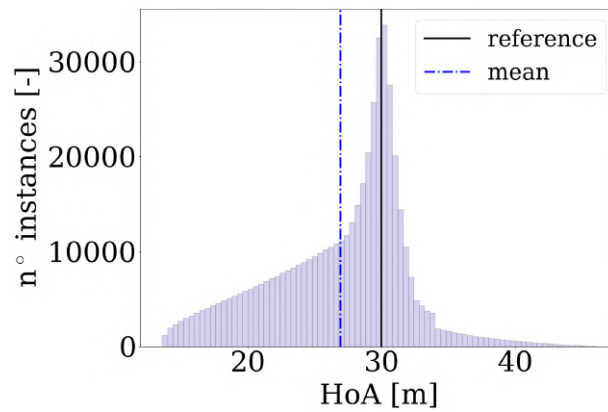


Figure 7.8: Distribution of HoA, fully optimised configuration.

The results presented in this section demonstrate two key points. First, the proposed method for selecting the angle of incidence can substantially improve the performance of systems that support its modification, even for a predetermined formation type. Second,

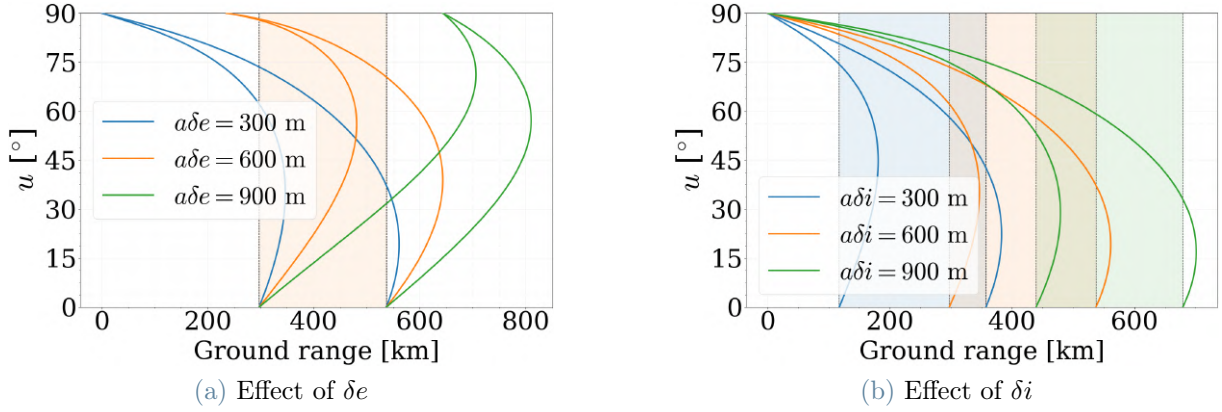


Figure 7.7: Sensitivity of the optimal solution to a variation of δe and δi .

combining this strategy with the optimal formation-flying categories identified in Chapter 5 results in an impressive reduction in HoA variability, on the order of 37% for the examples considered.

7.4. Extension to Formations with $\varphi \neq \pi/2$

This section examines possible ways to extend the proposed methodology to formations in which the condition $\varphi = \pi/2$ is not imposed from the beginning. In this case, complications mainly arise from the fact that, for $u > \varphi$, there is always a certain incidence angle in correspondence with which there is an asymptote for the HoA, as derived in 7.15. Figure 7.9 represents the behaviour of the HoA along the swath in the case of $\varphi = 60^\circ$ at four different values of u . Specifically, the plots of the first row show the monotonic behaviour of the function $\text{HoA}(\theta_i)$ if no asymptote is present. On the other hand, the asymptote appears when u exceeds φ and causes the curves in the figures of the second row not to be monotonic.

A key observation is that the position of the asymptote tends to move towards the farthest limit of the ground range as the latitude increases. As a consequence, eq. 7.17 provides the optimal solution only near the equator and the poles, where the HoA remains centred around the target value HoA^* , whereas outside these regions it deviates significantly. Figure 7.10 shows the HoA along the optimal swath for $\varphi = 60^\circ$ across different values of u . Each coloured line corresponds to the function $\text{HoA}(s)$ for a specific mean argument of latitude. The dashed black line corresponds to the target value $\text{HoA}^* = 30$ m. The objective of achieving a HoA distribution centred around the target value is achieved for latitudes close to the poles and to the equator. The curves deviate from the target value in correspondence with the latitudes between 60° and 70° . The inferior and subpar results

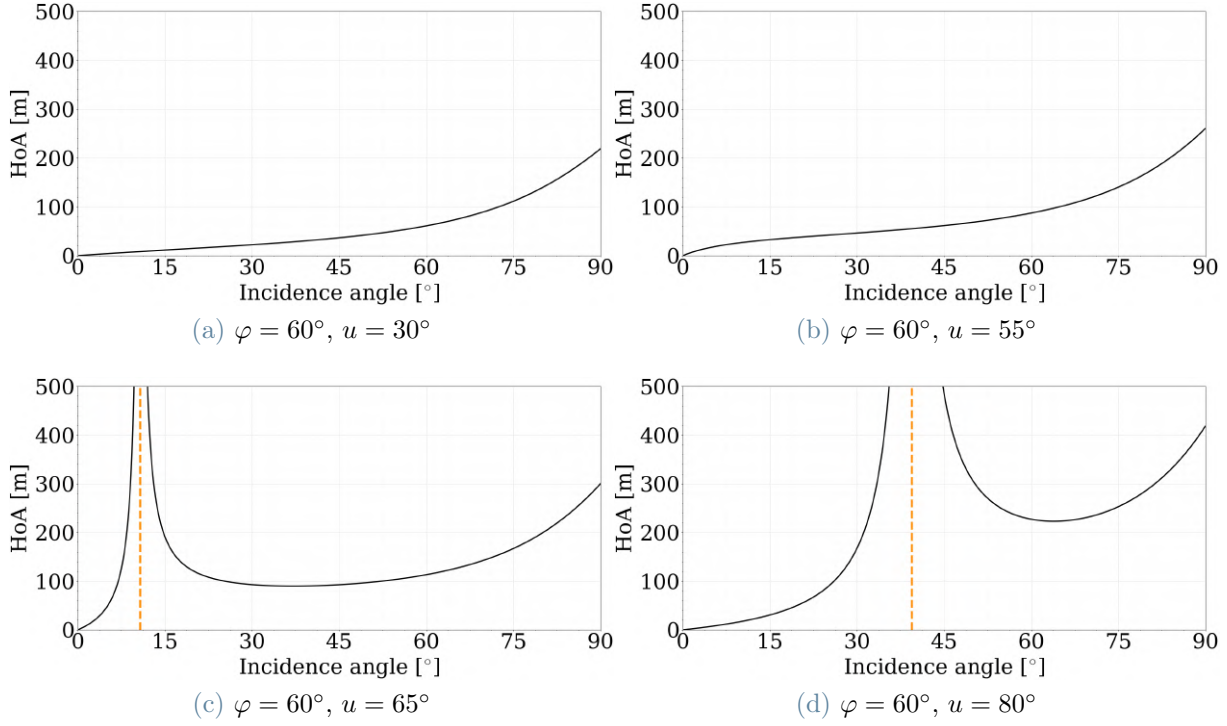


Figure 7.9: HoA along the swath for different mean arguments of latitude, the position of the asymptote is indicated in orange.

correspond to latitudes at which eq. 7.17 does not have a solution, as explained below. Indeed, to identify the latitudes where eq. 7.17 yields the optimal solution; two conditions must be verified.

1. The asymptote must not prevent the acquisition of a continuous swath, with the near range in correspondence with the nadir. In other words, it must not be in correspondence with a look angle lower than $\theta_{l,min}$, which is the minimum incidence angle allowing for acquiring the required width. The curves for which this condition is not verified lead to the inferior results in terms of closeness to the nominal HoA^* , and are identified by the red colour in Figure 7.10.
2. The constraints on the look angle given by eq. 7.10 must not prevent eq. 7.17 from having a solution. This is guaranteed if the following condition is respected:

$$\text{HoA}(0 + W_g) \leq 2 \text{HoA}^* \quad \forall u \in (0, \pi/2). \quad (7.31)$$

If eq. 7.31 does not hold, subpar results in terms of closeness to the nominal HoA^* are obtained. These are highlighted in yellow in Figure 7.10.

Figure 7.11 displays the values of u at which these two conditions are met in the example

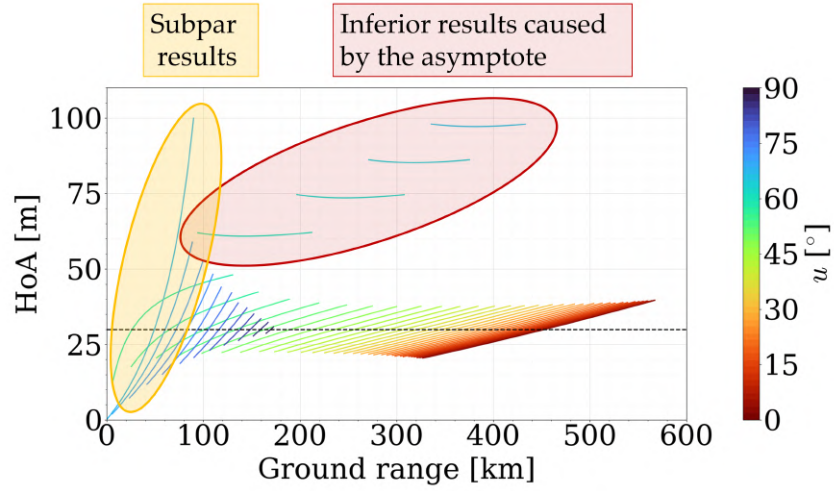


Figure 7.10: HoA at different u , optimised result for $\varphi = 60^\circ$.

case where $\varphi = 60^\circ$. The green line represents the swath with the minimum incidence angle $\theta_{i,min}$, attained when the acquisition starts from the nadir. The black line indicates the angle at which the asymptote is located for a given u . The red hatched area highlights the mean arguments of latitude at which the acquisition results would be penalised by the presence of the asymptote. Additionally, the region hatched in yellow identifies the mean arguments of latitude at which the asymptote does not represent a problem. However, it is still not possible to acquire a swath centred at the target HoA^* because of the condition in eq. 7.31. Overall, the two coloured areas represent the mean arguments of latitude at which the results are substantially inferior in terms of HoA homogeneity with respect to the rest of the orbit. All the proposed considerations suggest that choosing $\varphi \neq \pi/2$ is not recommended for systems, such as TanDEM-X, that are based on the acquisition of considerably wide swaths. However, the issues arise mainly because an

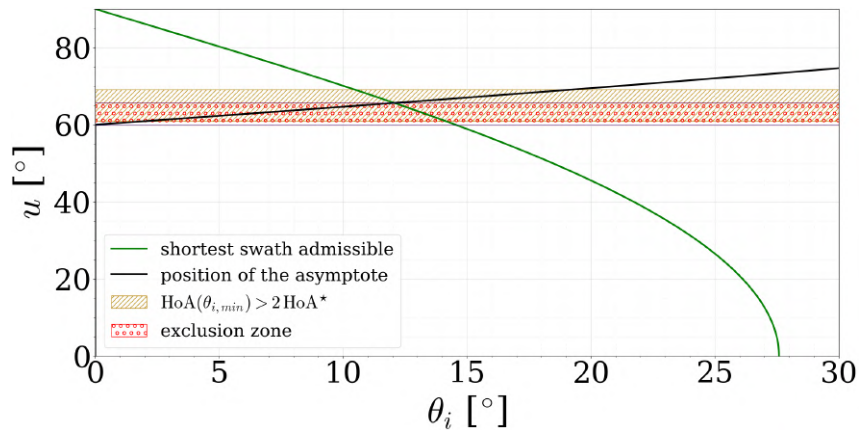


Figure 7.11: Regions with inferior interferometric performance: example for $\varphi = 60^\circ$.

imaging policy based on mapping the northern hemisphere during the orbit's ascending phase and the southern hemisphere during the descending one (as done by TanDEM-X) is being considered. Removing this limitation, it would be possible to adopt a more convenient split of the acquisitions between ascending/descending, to make the case $\varphi \neq \pi/2$ equally viable.

Notably, choosing to acquire a point in a certain part of the orbit can be equivalently expressed as the choice between a right/left looking acquisition, in agreement with Table 7.1. This holds because a change in the orbit's quadrant yields the same modification to the effective baseline as a variation in the line of sight direction. More precisely, by changing the sign of θ_l , one obtains an identical result for a point in the opposite hemisphere, symmetric with respect to the centre of the Earth. This can be readily shown by substituting (u) with $(\pi - u)$ and (θ_l) with $(-\theta_l)$ in the effective baseline defined in eq. 7.11. The resulting modified effective baseline $\tilde{B}_\perp$ reads:

$$\begin{aligned}\tilde{B}_\perp &= |\delta e \cos(\pi - u - \varphi) \sin(-\theta_l) + (-1)^k \delta i \sin(\pi - u - \vartheta) \cos(-\theta_l)| = \\ &= |\delta e(-\cos(u - \varphi))(-\sin \theta_l) + (-1)^k \delta i \sin(u - \vartheta) \cos \theta_l| = \\ &= |\delta e \cos(u - \varphi) \sin \theta_l + (-1)^k \delta i \sin(u - \vartheta) \cos \theta_l| = B_\perp,\end{aligned}\tag{7.32}$$

so, equal to the original one.

Eq. 7.32 is relevant to the purpose of this study, as it provides two different but equally viable ways of obtaining the same desired acquisition geometry. A first one based on chaining the quadrant in which the acquisitions take place, and a second one that entails switching the look direction of the radar.

In the following, the problem of selecting the best orbit phase (ascending or descending) that minimises the HoA error for each latitude is thus assessed. The idea driving the following considerations is illustrated by the explanatory Figure 7.12. Instead of mapping each hemisphere continuously, the acquisitions would be split optimally by changing either the look direction or the mapped quadrant of u . Note that the two are equivalent by virtue of eq. 7.32.

The objective is hence to find the optimal set of arguments of latitude U_φ that specifies whether each point is best imaged in the ascending or descending orbit phase. Mathematically, U_φ contains the element:

$$\begin{cases} \hat{u} & \text{if } \int_{W_g^*(\hat{u})} |\text{HoA}(s, \hat{u}) - \text{HoA}^*|^2 ds - \int_{W_g^*(\pi - \hat{u})} |\text{HoA}(s, \pi - \hat{u}) - \text{HoA}^*|^2 ds \leq 0 \\ \pi - \hat{u} & \text{if } \int_{W_g^*(\hat{u})} |\text{HoA}(s, \hat{u}) - \text{HoA}^*|^2 ds - \int_{W_g^*(\pi - \hat{u})} |\text{HoA}(s, \pi - \hat{u}) - \text{HoA}^*|^2 ds > 0 \end{cases} \quad \forall \hat{u} \in [0, \pi/2]$$

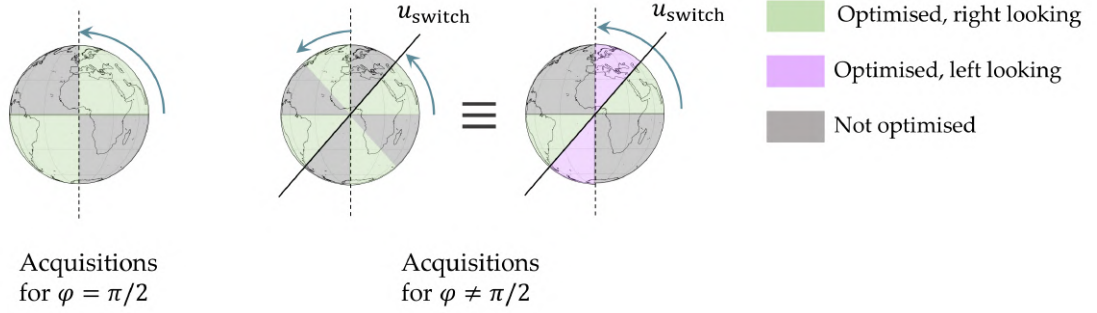


Figure 7.12: Hypothetical new acquisition policies: change direction/quadrant at u_{switch} .

where the integrals are computed on the ideal ground range interval $W_g^*(\hat{u})$, which is the ideal swath found with the optimisation illustrated in Section 7.2 at the mean argument of latitude $\hat{u}$.

A crucial observation is that the error $|\text{HoA}(s, u) - \text{HoA}^*|^2$ can be considered continuous with u , because no asymptote is present in W_g^* . This means that it is enough to compute the values of u for which there is a switch of the optimal configuration between ascending and descending. In other words, the set U_φ is the union of n intervals of the kind:

$$U_\varphi = \bigcup_{i=1}^n [u_{i-1}, u_i), \quad (7.33)$$

where $u_0 = 0$, and u_i is any value of u that satisfies the condition:

$$\int_{W_g^*(u)} |\text{HoA}(s, u) - \text{HoA}^*|^2 ds - \int_{W_g^*(\pi-u)} |\text{HoA}(s, \pi - u) - \text{HoA}^*|^2 ds = 0. \quad (7.34)$$

Remarkably, the solution to eq. 7.34 is unique regardless of the value of φ . This means that the optimal imaging policy would be to perform the mapping in the ascending (descending) part of the orbit from the equator up to a certain limit indicated as u_{switch} , and in the descending (ascending) part in the interval $u \in (u_{\text{switch}}, \pi/2]$.

This is a result that is worth emphasising: the determination of the optimal set U_φ can indeed be reduced to the computation of a unique value $u_{\text{switch}}(\varphi) \forall \varphi \in [0, \pi/2]$, which can be readily done numerically. Figure 7.13 represents the optimal solution for $\varphi = 60^\circ$ having implemented this optimal switching policy. By comparison with previous results in Figure 7.10, the new strategy leads to achieved HoAs always centred around the nominal HoA^* , eliminating the presence of inferior and subpar results. In particular, the solid lines are used to represent the HoA for the acquisitions in the ascending part of the orbit,

while the dashed lines for the descending ones. Note that the HoA is, at each u , centred at the target value HoA^* indicated by the black dashed line, as requested.

Having shown that a thoughtful selection of the best part of the orbit in which to acquire any latitude, it is worth considering if also the case with $\varphi = \pi/2$ could benefit from this strategy. However, by numerically solving eq. 7.34, no solution is found in the interval $u \in (0, \pi/2)$, meaning that a switch is not convenient in this case. This result arises because the HoA difference between a given mean argument of latitude $\hat{u}$ and $\pi - \hat{u}$ is negligible near the poles, when $u \approx \pi/2$. Moreover, the most problematic latitudes are the ones close to the point $u = \varphi$, as it is clear from the exclusion zones displayed in Figure 7.11. Therefore, in the case with $\varphi = \pi/2$, the critical latitudes near $u = \varphi = \pi/2$ do not benefit from descending acquisitions, as the HoA remains nearly identical to that achieved in the ascending quadrant.

In summary, this Chapter aimed to derive an analytical method for selecting the incidence angle in single-pass across-track SAR interferometry, assuming a Helix formation. The proposed solution is the result of a two-step process, in which an initial estimate, obtained by solving a fourth-degree polynomial equation, is refined with the introduction of suitable correction factors. Remarkably, the examples based on TanDEM-X orbital parameters show that the results match almost perfectly those obtained numerically, with a maximum percentage error $< 0.1\%$. The presented strategy grants an improvement in the homogeneity of the height of ambiguity with respect to fixed-angle approaches, which can be pushed even further by combining it with the optimal formations identified in Chapter 5. Moreover, even if most of the results are referred to the case with $\varphi = \pi/2$, the proposed method, which envisions a switch in acquisition strategy at a specific point of the orbit, represents a more than valid tool for generalising them. Overall, the proposed methodology has this merit: it provides an analytical tool capable, by itself, of optimis-

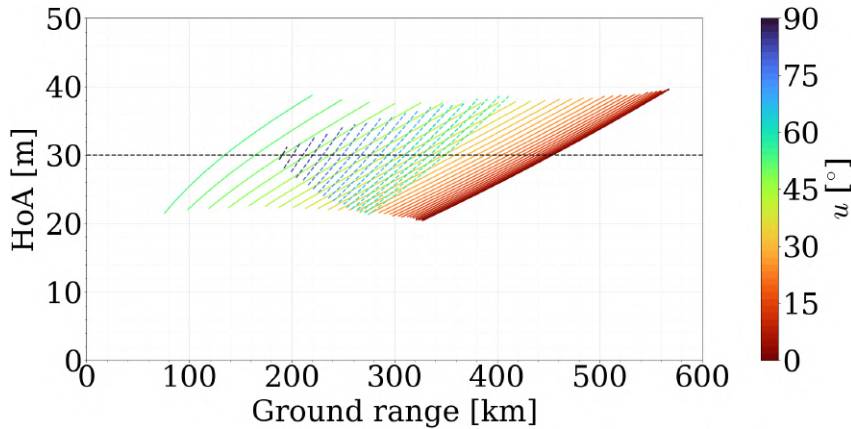


Figure 7.13: HoA at different u with switching policy for $\varphi = 60^\circ$ -

ing the acquisition geometry of specific systems, similar to TanDEM-X. Furthermore, the latter can be used in combination with the strategies introduced in Chapter 5 for the baseline optimisation, contributing to a substantial enhancement of the interferometric performance.

8 | Conclusions and Future Developments

In this Chapter the conclusions of the entire work are drawn. First, a summary of the methodology employed and the consequent findings is provided in Section 8.1. Section 8.2 outlines possible future research lines emerging as a natural continuation of the study conducted in this thesis. Finally, in Section 8.3, the value and relevance of this work are discussed.

8.1. Summary

This thesis has explored the design of optimal satellite formations for multi-baseline InSAR, specifically considering the generation of high-quality DEMs in single-pass acquisitions with LEO satellites. The main challenge driving such research was the compromise InSAR is subject to between accuracy and robustness to phase unwrapping errors. The objective was therefore to investigate feasible formation-flying solutions to overcome such limitations, also ensuring viability from the operational standpoint.

8.1.1. Methodology

The work followed a progressive methodology, in which exact solutions to simpler problems were subsequently generalised, addressing the requirements of a real mission. The fundamental steps are summarised in the following paragraphs.

Beginning with a two-satellite case, an analytical mapping connecting the ROEs and the HoA was derived, providing a theoretical foundation regarding the impact that the relative geometry has on interferometric performance.

The problem was extended to larger formations, namely, with three and four satellites. Initially, a heuristic solution was examined, in which for each desired HoA value, one and only one pair of satellites was designed to handle it. This method does not account for all the baselines that are created among satellite pairs, so the solution was refined using

numerical methods. In particular, genetic algorithms were employed, also providing a comparison between those more suitable for solving the problem in question, from which the Self-Adaptive Evolutionary Algorithm emerged as the most adequate. The result was the characterisation of interferometrically optimal configurations that ensure maximum passive safety at the same time.

Subsequently, the effects of orbital perturbations were considered, providing an estimate of the costs for LEO formation maintenance. The most promising results were obtained for nested Helix configurations, which combine passive safety, robustness to geopotential perturbations, and reduced Δv costs.

Finally, an optimisation of the angle of incidence was conducted, considering that it represents, for systems with wide access ranges (in the order of hundreds of metres), an additional degree of freedom. This demonstrated that, even for predefined relative formations, interferometric performance can be further improved by carefully adjusting the acquisition geometry.

8.1.2. Findings

The following paragraphs summarise the key results of the proposed investigation:

1. The analytical solution for the two-satellite case contributes to the understanding of the HoA behaviour depending on the category of formation adopted.
2. Multi-baseline constellations can provide accurate DEMs while at the same time avoiding phase unwrapping errors by leveraging different interferograms generated in a single acquisition. This allows for improving the quality of the DEM without the coherence loss that characterises repeat-pass solutions.
3. Passive safety and interferometric optimality are not mutually exclusive.
4. Notable solutions envision nested Helix configurations, as they offer either maximum safety and robustness against perturbations, leading to the minimisation of maintenance costs.
5. Systems like TanDEM-X that feature high flexibility in the acquisition geometry can leverage the derived latitude-varying angle of incidence to improve HoA homogeneity further.

Overall, the results presented in this thesis provide a starting point for the design of future multi-baseline InSAR missions aimed at DEM generation with a single-pass acquisition strategy. The intrinsic limitations of single-baseline systems like TanDEM-X and Lutan-

1 can be overcome by exploiting a framework that integrates general harmonic relative dynamics, not limited to the cartwheel formation used in Hongtu-1. This widens the spectrum of possible acquisition geometries for InSAR, since after deciding the desired HoA, the most suitable set of ROEs for achieving it can be identified through the proposed mapping.

8.2. Future Works

The work done in this thesis leaves several research directions open for further exploration.

Improved realism: in the study only the J_2 and atmospheric drag perturbations were considered. Future works could envision the integration of higher-order harmonics and other disturbances, such as the SRP and third-body perturbations, into the optimisation framework. Moreover, a thorough analysis of a possible semi-analytical approach, including the coupling between along-track and across-track motion caused by Earth rotation, is missing, as this effect was only considered in the numerical optimisation.

Operational strategies: the optimal formations identified shall be supported by efficient guidance, navigation and control laws. The development of suitable strategies, tailored to such specific configurations, is therefore the natural next step.

Cohesive optimisation framework: only the effective baseline following natural relative motion was considered in the optimisation. On the one hand, the impact of other parameters relevant to interferometry, such as bandwidth, wavelength, and the entire mission structure, should be investigated. On the other, the benefits provided by flexible strategies based on the balancing between fixed and time-varying baselines are worth exploring.

Performance metrics: more ambitious goals could be considered in the optimisation, consistently integrating other objectives as coherence, spatial coverage, and sensitivity to topography beyond HoA homogeneity.

Applications beyond DEMs: the strategies developed in this thesis could be extended to applications other than interferometry. An initial study of analogous solutions for SAR tomography was conducted numerically; more comprehensive investigations should provide a deeper understanding of how the choice of relative geometry influences tomographic results. On a broader level, the potential of multi-baseline SAR for deformation monitoring, biomass estimation or even planetary exploration is still an open question.

8.3. Relevance of This Thesis

The contributions of this thesis are twofold. From the scientific standpoint it deepens the theoretical understanding of how interferometric performance and relative dynamics can be optimised simultaneously. From a more technical perspective, it shows that novel, safe, and effective formation-flying solutions can meet stringent requirements on the uniformity of the generated DEMs. Overall, evidence is provided of the enormous potential of multi-baseline InSAR; a tool that future missions shall employ to provide DEMs of unprecedented quality at low formation control effort.

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A | Appendix A

A.1. Optimal effective baseline

From the Cramer-Rao Bound, denoting with $K = \frac{\lambda r_0 \sin \theta_i}{\sqrt{2N}}$, it yields:

$$\sigma_z^2(B_\perp) = \frac{K^2}{B_\perp^2} \frac{1}{\left(1 - \frac{B_\perp}{B_{\perp crit}}\right)^2} \left(1 - \left(1 - \frac{B_\perp}{B_{\perp crit}}\right)^2\right) \quad (\text{A.1})$$

Calling $\frac{B_\perp}{B_{\perp crit}} = x$, the height variance reads:

$$\sigma_z^2 = \frac{K^2}{B_{\perp crit}^2} \frac{1 - (1 - x)^2}{x^2 (1 - x)^2}. \quad (\text{A.2})$$

Imposing its derivative with respect to x to nullify, it yields:

$$-x^3 + 4x^2 - 4x + 1 = 0. \quad (\text{A.3})$$

The roots of the polynomial are 1 and $\frac{3 \pm \sqrt{5}}{2}$. Since the effective baseline cannot exceed the critical baseline, the only viable solution is:

$$B_\perp = \frac{3 - \sqrt{5}}{2} B_{\perp crit}. \quad (\text{A.4})$$

□

A.2. Relationship between mean and eccentric arguments of latitude for quasi-circular orbits

Starting from the eccentric anomaly:

$$\tan \frac{E}{2} = \beta \tau, \quad \tau = \tan \frac{\nu}{2}, \quad \beta = \sqrt{\frac{1 - e}{1 + e}} = 1 - e + \frac{1}{2}e^2 + \mathcal{O}(e^3),$$

expand $e(\nu)$ about $\beta = 1$:

$$\arctan(\beta\tau) = \arctan \tau + \frac{(\beta - 1)\tau}{1 + \tau^2} - \frac{\tau(\beta - 1)^2\tau^2}{(1 + \tau^2)^2} + \mathcal{O}(e^3).$$

Consider that $\beta - 1 = -e + \frac{1}{2}e^2$. Leveraging the following identities:

$$\begin{cases} \frac{\tau}{1 + \tau^2} = \frac{1}{2} \sin \nu \\ \frac{\tau^3}{(1 + \tau^2)^2} = \sin^3 \frac{\nu}{2} \cos \frac{\nu}{2} = \frac{1}{4} \sin \nu (1 - \cos \nu) \end{cases}$$

it yields get:

$$\begin{aligned} E &= \nu + 2 \left(-e + \frac{e^2}{2} \right) \frac{\tau}{1 + \tau^2} - 2e^2 \frac{\tau^3}{(1 + \tau^2)^2} + \mathcal{O}(e^3) \\ &= \nu + \left(-e + \frac{e^2}{2} \right) \sin \nu - \frac{e^2}{2} \sin \nu (1 - \cos \nu) + \mathcal{O}(e^3) \\ &= \nu - e \sin \nu + \frac{e^2}{2} \sin \nu \cos \nu + \mathcal{O}(e^3) \\ &= \nu - e \sin \nu + \frac{e^2}{4} \sin(2\nu) + \mathcal{O}(e^3). \end{aligned}$$

Now, to exploit Kepler time equation $M = E - e \sin E$, expand $\sin E$:

$$\sin E = \sin \nu + (E - \nu) \cos \nu + \mathcal{O}((E - \nu)^2)$$

Since $E - \nu = \mathcal{O}(e)$, then $(E - \nu)^2 = \mathcal{O}(e^3)$ and thus this term can be dropped. From the previous formula substitute $E - \nu = -e \sin \nu + \frac{e^2}{4} \sin(2\nu)$ to find:

$$e \sin E = e \sin \nu - e^2 \sin \nu \cos \nu + \mathcal{O}(e^3).$$

Finally:

$$\begin{aligned} M &= E - e \sin E = \left(\nu - e \sin \nu + \frac{e^2}{4} \sin(2\nu) \right) - \left(e \sin \nu - e^2 \sin \nu \cos \nu \right) + \mathcal{O}(e^3) \\ &= \nu - 2e \sin \nu + \left(\frac{e^2}{4} \sin(2\nu) + e^2 \sin \nu \cos \nu \right) + \mathcal{O}(e^3) \\ &= \nu - 2e \sin \nu + \frac{3}{4}e^2 \sin(2\nu) + \mathcal{O}(e^3). \end{aligned}$$

The maximum latitude is achieved at $\nu = \pi/2 - \omega$, thus $\sin \nu = \cos \omega$ and $\sin(2\nu) = \sin(2\omega)$. Therefore the mean argument of latitude reads:

$$\begin{aligned} M &\approx \pi/2 - \omega - 2e \cos \omega + \frac{3}{4}e^2 \sin(2\omega) \\ \Leftrightarrow \bar{u} &\approx \pi/2 - 2e \cos \omega + \frac{3}{4}e^2 \sin(2\omega) + \mathcal{O}(e^3). \end{aligned}$$

□

A.3. Derivation of the optimal ROEs for a two-satellite formation

Suitable auxiliary variables are introduced: $E = \delta e \sin \theta_l$ and $I = \delta i \cos \theta_l$, while α indicates the argument of latitude in correspondence with which an asymptote arises. As such, it is defined as:

$$\alpha = \arctan\left(-\frac{E}{I}\right) + k\pi. \quad (\text{A.5})$$

Applying the rectangular window of width $\pi - 2\varepsilon$ the objective function reads:

$$J = \int_{\alpha+\varepsilon}^{\alpha+\pi-\varepsilon} \left(\frac{K}{|E \cos \beta + I \sin \beta|} - \text{HoA}^\star \right)^2 d\beta \quad (\text{A.6})$$

Expanding the product and applying trigonometric identities it reads:

$$J = \int_{\alpha+\varepsilon}^{\alpha+\pi-\varepsilon} \left(\frac{1}{(\delta e \cos(\beta) \sin(\theta_l) + \delta i \sin(\beta) \cos(\theta_l))^2} - \text{HoA}^\star \right)^2 d\beta + \left(\frac{\text{HoA}^\star}{K} \right)^2 \left(\frac{\pi}{2} - \varepsilon \right) - 2 \frac{\text{HoA}^\star}{K} \int_{\alpha+\varepsilon}^{\alpha+\pi/2} \frac{d\beta}{\left| \frac{\delta e \delta i}{4} \cos(2\beta) \cos(2\theta_l) \right|} \quad (\text{A.7})$$

Note that the effect of φ is to simply shift the HoA function with respect to u , without affecting its magnitude in any way. The optimal solution can therefore be expressed in terms of δe and δi or, equivalently, of E and I . Since the interval of integration stays inside $[\alpha, \alpha+\pi]$, one can safely neglect the absolute value in the denominator, Its argument would be either always positive or always negative. Without loss of generality, one can assume it as always positive, since its sign could be conveniently adjusted ex-post by changing φ .

The integral can then be expanded and solved explicitly, yielding the following result:

$$J = \frac{\sin(\beta)}{E(E \cos \beta + I \sin \beta)} \Big|_{\beta=\alpha+\varepsilon}^{\alpha+\pi-\varepsilon} + \left(\frac{\text{HoA}^*}{K} \right)^2 (\pi - 2\varepsilon) - \frac{2\text{HoA}^*}{K\sqrt{E^2 + I^2}} \log \left(\frac{\sqrt{E^2 + I^2} - I + E \tan(\beta/2)}{\sqrt{E^2 + I^2} + I - E \tan(\beta/2)} \right) \Big|_{\beta=\alpha+\varepsilon}^{\alpha+\pi-\varepsilon}. \quad (\text{A.8})$$

The optimal solution is found by imposing its gradient to nullify. Remarkably, this can be done explicitly by leveraging the specific choice of the integration interval $\mathcal{U} = [0, \pi]$, because the last term cancels out as shown in the following.

Proof that the last term does not depend on α , nor on E or I

First, consider the following equalities:

$$\begin{cases} \tan \left(\frac{\alpha + \varepsilon}{2} = \frac{\tan(\frac{\alpha}{2}) \pm \tan(\frac{\varepsilon}{2})}{1 \mp \tan(\frac{\alpha}{2}) \tan(\frac{\varepsilon}{2})} \right) = \frac{E - \sqrt{E^2 + I^2} + E \tan(\frac{\varepsilon}{2})}{E \mp (I - \sqrt{E^2 + I^2}) \tan(\frac{\varepsilon}{2})} \\ \tan \left(\frac{\alpha}{2} \right) = \frac{I - \sqrt{E^2 + I^2}}{E} \end{cases} \quad (\text{A.9})$$

Call $S = \sqrt{E^2 + I^2}$, $T(+)=\tan(\frac{\alpha+\varepsilon}{2})$, $T(-)=\tan(\frac{\alpha-\varepsilon}{2})$, and focus on the argument of the logarithm:

$$\frac{N}{D} = \frac{E - ST(+) + IT(+) - \frac{S}{T(-)} - \frac{I}{T(-)} + E \frac{T(+)}{T(-)}}{E + ST(+) + IT(+) + \frac{S}{T(-)} - \frac{I}{T(-)} + E \frac{T(+)}{T(-)}}. \quad (\text{A.10})$$

Concerning the the numerator N , introducing $\gamma = \tan(\frac{\varepsilon}{2})$ it yields:

$$N = E - S \frac{I - S + E\gamma}{E - (I - S)\gamma} + I \frac{I - S + E\gamma}{E - (I - S)\gamma} - S \frac{E + (I - S)\gamma}{E - S - E\gamma} - I \frac{E + (I - S)\gamma}{I - S - E\gamma} + E \frac{I - S + E\gamma}{E - (I - S)\gamma} \frac{E + (I - S)\gamma}{I - S - E\gamma} \quad (\text{A.11})$$

An analogous equation holds for the denominator. After some straightforward (albeit tedious) algebra it yields:

$$\frac{N}{D} = \frac{1}{\gamma^2} = \frac{1}{\tan^2 \frac{\varepsilon}{2}}. \quad (\text{A.12})$$

$$\log \left(\frac{\sqrt{E^2 + I^2} - I + E \tan(\beta/2)}{\sqrt{E^2 + I^2} + I - E \tan(\beta/2)} \right) \Big|_{\beta=\alpha+\varepsilon}^{\alpha+\pi-\varepsilon} = -\log \left(\tan^2 \left(\frac{\varepsilon}{2} \right) \right). \quad (\text{A.13})$$

Therefore the last term in eq.A.8 cancels out when the gradient is computed

□

Reduction of the cost J

Thus, the cost function simplifies to:

$$J = \frac{\sin(\alpha + \pi - \varepsilon)}{E(E \cos(\alpha + \pi - \varepsilon)) + I \sin(\alpha + \pi - \varepsilon)} - \frac{\sin(\alpha + \varepsilon)}{E(E \cos(\alpha + \varepsilon) + I \sin(\alpha + \varepsilon))} - \frac{2\text{HoA}^*}{K} \frac{1}{\log\left(\tan^2\left(\frac{\varepsilon}{2}\right)\right) \sqrt{E^2 + I^2}}. \quad (\text{A.14})$$

The first two terms of J can be simplified as follows:

$$\begin{aligned} J &= \frac{-\sin\alpha \cos\varepsilon + \cos\alpha \sin\varepsilon}{-E^2 \cos\alpha \cos\varepsilon - E^2 \sin\alpha \sin\varepsilon - EI \sin\alpha \cos\varepsilon + EI \cos\alpha \sin\varepsilon} + \frac{-\sin\alpha \cos\varepsilon - \cos\alpha \sin\varepsilon}{E^2 \alpha \cos\varepsilon - E^2 \sin\alpha \sin\varepsilon + EI \sin\alpha \cos\varepsilon + EI \cos\alpha \sin\varepsilon} + \frac{2\text{HoA}^*}{L} \frac{F}{\sqrt{E^2 + I^2}} = \\ &= \frac{-2 \sin\alpha \cos\varepsilon (-E^2 \sin\alpha \sin\varepsilon + EI \cos\alpha \sin\varepsilon) + 2 \cos\alpha \sin\varepsilon (E^2 \cos\alpha \cos\varepsilon + EI \sin\alpha \cos\varepsilon)}{E[(I \cos\alpha \sin\varepsilon - E \sin\alpha \sin\varepsilon)^2 - (E \cos\alpha \cos\varepsilon + I \sin\alpha \cos\varepsilon)^2]} - \frac{2\text{HoA}^*}{K} \frac{F}{\sqrt{E^2 + I^2}} = \\ &= \frac{2E \sin\varepsilon \cos\varepsilon}{E(I^2 \cos^2\alpha \sin^2\varepsilon + E^2 \sin^2\alpha \sin^2\varepsilon - 2EI \cos\alpha \sin\alpha \sin^2\varepsilon - E^2 \cos^2\alpha \cos^2\varepsilon - I^2 \sin^2\alpha \cos^2\varepsilon - 2EI \cos\alpha \sin\alpha \cos^2\varepsilon)} = \\ &= \frac{2 \sin\varepsilon \cos\varepsilon}{-E^2 \cos(\alpha + \varepsilon) \cos(\alpha - \varepsilon) - I^2 \sin(\alpha + \varepsilon) \sin(\alpha - \varepsilon) - 2EI \cos\alpha \sin\alpha} = \\ &= \frac{2 \sin(2\varepsilon)}{(-I^2 - E^2) \cos(2\varepsilon) - (E^2 - I^2) \cos(2\alpha) - 2EI \sin(2\alpha)}. \end{aligned}$$

Additionally, any dependence on the position of the asymptote α can be removed exploiting the following identities:

$$\begin{cases} \sin(2\alpha) = \frac{2 \tan \alpha}{1 + \tan^2 \alpha} = \frac{2EI}{E^2 + I^2} \\ \cos(2\alpha) = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{I^2 - E^2}{I^2 + E^2} \end{cases} \quad (\text{A.15})$$

Substituting into the expression of the first term of J it yields:

$$\begin{aligned} &\frac{2 \sin(2\varepsilon)}{-(E^2 + I^2) \cos(2\varepsilon) - (E^2 - I^2) \frac{I^2 - E^2}{I^2 + E^2} - 2EI \frac{-2EI}{E^2 + I^2}} = \\ &\frac{2 \sin(2\varepsilon)(E^2 + I^2)}{-(E^2 + I^2)^2 \cos(2\varepsilon) + E^4 + I^4 - 2E^2 I^2 + 4E^2 I^2} = \\ &\frac{2 \sin(2\varepsilon)(E^2 + I^2)}{(E^2 + I^2)^2 (1 - \cos(2\varepsilon))} = \frac{1}{E^2 + I^2} \frac{2 \sin(2\varepsilon)}{1 - \cos(2\varepsilon)}. \end{aligned} \quad (\text{A.16})$$

The cost function can finally be reduced to the following purely algebraic expression:

$$J = \frac{1}{E^2 + I^2} \frac{2 \sin(2\varepsilon)}{1 - \cos(2\varepsilon)} + \frac{2\text{HoA}^*}{K} \frac{\log\left(\tan^2\left(\frac{\varepsilon}{2}\right)\right)}{\sqrt{E^2 + I^2}} + \left(\frac{\text{HoA}^*}{K}\right)^2 (\pi - 2\varepsilon), \quad (\text{A.17})$$

□

Gradient and final solution

The zeros of the gradient are computed as follows:

$$\nabla J = \begin{bmatrix} \frac{2 \sin(2\varepsilon)}{1 - \cos(2\varepsilon)} \frac{-2E}{(E^2 + I^2)^2} - \frac{\text{HoA}^*}{K} \log\left(\tan^2\left(\frac{\varepsilon}{2}\right)\right) \frac{2E}{(E^2 + I^2)^{3/2}} \\ \frac{2 \sin(2\varepsilon)}{1 - \cos(2\varepsilon)} \frac{-2I}{(E^2 + I^2)^2} - \frac{\text{HoA}^*}{K} \log\left(\tan^2\left(\frac{\varepsilon}{2}\right)\right) \frac{2I}{(E^2 + I^2)^{3/2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (\text{A.18})$$

Neither E nor I can be cancelled out, as they cannot be null for a Helix formation. This makes the two equations identical, indicating the presence of an infinite set of possible solutions. Therefore, the locus of points (E, I) satisfying eq. A.18 reads:

$$\frac{2 \sin(2\varepsilon)}{1 - \cos(2\varepsilon)} = \frac{\text{HoA}^*}{K \log((\tan^2(\varepsilon/2)))} \sqrt{E^2 + I^2} \quad (\text{A.19})$$

$$\Leftrightarrow \frac{\delta e^2}{\left(\frac{K}{\text{HoA}^* \sin \theta_l} \frac{\log(\tan^2(\varepsilon/2))}{2 \tan \varepsilon}\right)^2} + \frac{\delta i^2}{\left(\frac{K}{\text{HoA}^* \cos \theta_l} \frac{\log(\tan^2(\varepsilon/2))}{2 \tan \varepsilon}\right)^2} = 1, \quad (\text{A.20})$$

which consists of an ellipse in the δe - δi plane. $\square$

A.4. Hohmann transfer between close circular orbits

The following derivation considers just the first burn with intensity Δv_1 , however, the second one can be treated analogously. For a Hohmann transfer between two orbits with radii r_1 and r_2 it yields:

$$\Delta v_1 = \sqrt{\mu \left(\frac{2}{r_1} - \frac{1}{a_t} \right)} - \sqrt{\frac{\mu}{r_1}} \quad (\text{A.21})$$

where $a_t = \frac{r_1 + r_2}{2}$ is the semi-major axis of the transfer orbit. If the variation in the semi-major axis is Δa : $a_t = r_1 + \frac{\Delta a}{2}$. Moreover, given $\Delta a \ll r_1$, one can approximate the corresponding term in eq. A.21 with a first-order Taylor expansion:

$$\frac{1}{r_1 + \Delta a} \approx \frac{1}{r_1} \left(1 - \frac{\Delta a}{2r_1} \right) \quad (\text{A.22})$$

Collecting the similar terms and factoring out the r_1 factor, the following is obtained:

$$\Delta v_1 \approx \sqrt{\frac{\mu}{r_1} \left(1 + \frac{\Delta a}{2r_1} \right)} - \sqrt{\frac{\mu}{r_1}} \quad (\text{A.23})$$

Again, leveraging $\Delta a \ll r_1$:

$$\Delta v_1 \approx \sqrt{\frac{\mu}{r_1} \left(1 + \frac{\Delta a}{4r_1} \right)} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1^3}} \cdot \frac{\Delta a}{4} \quad (\text{A.24})$$

For the second burn the same reasoning applies, so, clearly $\Delta v_{tot} = \Delta v_1 + \Delta v_2 \propto \Delta a$ for small entity transfers. $\square$

B | Appendix B

B.1. Hungarian algorithm for optimal baseline assignment in SAR tomography

Algorithm B.1 Hungarian Algorithm for Assignment Problem

Require: Cost matrix $C \in \mathbb{R}^{k \times m}$, with $C_{ij} = |f_j(x) - t_i|^2$

Ensure: Assignment matrix $A \in \{0, 1\}^{k \times m}$

- 1: Subtract the minimum of each row from all row elements
 - 2: Subtract the minimum of each column from all column elements
 - 3: Cover all zeros in the matrix with a minimum number of horizontal/vertical lines
 - 4: **if** number of covering lines = $\max(k, m)$ **then**
 - 5: An optimal assignment exists among the zeros
 - 6: **else**
 - 7: Find the smallest uncovered value δ
 - 8: Subtract δ from all uncovered elements
 - 9: Add δ to all elements covered twice
 - 10: Go back to step 3
 - 11: **end if**
 - 12: From the zero entries, construct an injective assignment:
 - 13: **for** each target t_i **do**
 - 14: Assign it to a function f_j such that $C_{ij} = 0$ and no other assignment uses f_j
 - 15: Set $A_{ij} \leftarrow 1$
 - 16: **end for**
 - 17: **return** A
-

B.2. Analytical selection of the optimal angle of incidence

Derivatives of $\theta_l(s)$:

$$\begin{aligned}
 K &= \frac{R_E + h}{R_E} \\
 x(s) &= \frac{\sin\left(\frac{s}{R_E}\right)}{K - \cos\left(\frac{s}{R_E}\right)} \\
 x'(s) &= \frac{1}{R_E} \frac{K \cos\left(\frac{s}{R_E}\right)}{\left(\cos\left(\frac{s}{R_E}\right) - K\right)^2} \\
 x''(s) &= \frac{\sin\left(\frac{s}{R_E}\right)}{R_E^2} \frac{2 - K \cos\left(\frac{s}{R_E}\right) - K^2}{\left(K - \cos\left(\frac{s}{R_E}\right)\right)^3} \\
 \frac{d\theta_l}{ds} &= \frac{x'(s)}{1 + x(s)^2} \\
 \frac{d^2\theta_l}{ds^2} &= \frac{x''(s)(x(s)^2 + 1) - 2(x'(s))^2 x(s)}{(x(s)^2 + 1)^2}
 \end{aligned}$$

Derivatives of HoA(θ_l):

$$\begin{aligned}
 r(\theta_l) &= (R_E + h) \cos \theta_l - \sqrt{(R_E + h)^2 \cos^2 \theta_l - 2R_E h - h^2} \\
 r'(\theta_l) &= -(R_E + h) \sin \theta_l + \frac{(R_E + h)^2 \sin(2\theta_l)}{2\sqrt{(R_E + h)^2 \cos^2 \theta_l - 2R_E h - h^2}} \\
 r''(\theta_l) &= -(R_E + h) \cos \theta_l + \frac{(R_E + h)^4 \sin^2(2\theta_l)}{4\sqrt{[(R_E + h)^2 \cos^2 \theta_l - 2R_E h - h^2]^3}} - \frac{-2(R_E + h)^2 \cos(2\theta_l)}{2\sqrt{(R_E + h)^2 \cos^2 \theta_l - 2R_E h - h^2}} \\
 B_{\perp}(\theta_l) &= a|\delta e \cos(u - \varphi) \sin \theta_l + \delta i \sin(u - \vartheta) \cos \theta_l| \\
 B'_{\perp}(\theta_l) &= a(\delta e \cos(u - \varphi) \cos \theta_l - \delta i \sin(u - \vartheta) \sin \theta_l) \operatorname{sgn}(\delta e \cos(u - \varphi) \sin \theta_l + \delta i \sin(u - \vartheta) \cos \theta_l) \\
 B''_{\perp}(\theta_l) &= -B_{\perp}(\theta_l) \\
 N'(\theta_l) &= [r'(\theta_l)B_{\perp}(\theta_l) - r(\theta_l)B'_{\perp}(\theta_l)] \sin \theta_l + r(\theta_l)B_{\perp}(\theta_l) \cos \theta_l \\
 N''(\theta_l) &= (r''(\theta_l) \sin \theta_l + 2r'(\theta_l) \cos \theta_l)B_{\perp}(\theta_l)
 \end{aligned}$$

$$D'(\theta_l) = (B_\perp(\theta_l))^2$$

$$D''(\theta_l) = 2B_\perp(\theta_l)B'_\perp(\theta_l)$$

$$\frac{\partial \text{HoA}}{\partial \theta_l} = \lambda K \frac{N'(\theta_l)}{D'(\theta_l)}$$

$$\frac{\partial^2 \text{HoA}}{\partial \theta_l^2} = \lambda K \frac{N''(\theta_l)D'(\theta_l) - N'(\theta_l)D''(\theta_l)}{(D'(\theta_l))^2}$$

Chain rule:

$$\text{HoA}' = \frac{\partial \text{HoA}}{\partial s} = \frac{\partial \text{HoA}}{\partial \theta_l} \frac{d\theta_l}{ds}$$

$$\text{HoA}'' = \frac{\partial^2 \text{HoA}}{\partial s^2} = \frac{\partial^2 \text{HoA}}{\partial \theta_l^2} \left(\frac{d\theta_l}{ds} \right)^2 + \frac{\partial \text{HoA}}{\partial \theta_l} \frac{d^2 \theta_l}{ds^2}$$

Proof that the second assumption in Equation 7.14 is equivalent to a 1st order Taylor expansion:

$$\text{HoA}(s_A) \approx \text{HoA}(s^*) + \text{HoA}'(s^*)(s_A - s^*)$$

$$\text{HoA}(s_B) \approx \text{HoA}(s^*) + \text{HoA}'(s^*)(s_A + W_g - s^*)$$

Substituting into Equation 7.17:

$$\text{HoA}(s^*) + \text{HoA}'(s^*) \frac{(s_A - s^*) + (s_A + W_g - s^*)}{2} = \text{HoA}^*$$

By definition, $\text{HoA}(s^*) = \text{HoA}^*$; moreover, the center of the swath is at $\bar{s} = s_A + W_g/2$. Therefore:

$$\delta = \bar{s} - s^* = s_A + \frac{W_g}{2} - s^* = 0.$$

□

Derivation of Equation 7.23:

$$\text{HoA}(s_A) \approx \text{HoA}(s^*) + \text{HoA}'(s^*)(s_A - s^*) + \frac{1}{2}\text{HoA}''(s^*)(s_A - s^*)^2$$

$$\text{HoA}(s_B) \approx \text{HoA}(s^*) + \text{HoA}'(s^*)(s_A + W_g - s^*) + \frac{1}{2}\text{HoA}''(s^*)(s_A + W_g - s^*)^2$$

Substituting into Equation 7.17:

$$\begin{aligned}
& \text{HoA}(s^*) + \frac{\text{HoA}'(s^*)[(s_A - s^*) + (s_A + W_g - s^*)] + \frac{1}{2}\text{HoA}''(s^*)[(s_A - s^*)^2 + (s_A + W_g - s^*)^2]}{2} = \text{HoA}^* \\
& \Leftrightarrow \text{HoA}'(s^*)(s_A + \frac{W_g}{2} - s^*) + \frac{1}{2}\text{HoA}''(s^*)(s_A^2 + s^{*2} - 2s_A s^* + W_g(s_A + \frac{W_g}{2} - s^*)) = 0 \\
& \Leftrightarrow \text{HoA}'(s^*)(\bar{s} - s^*) + \frac{1}{2}\text{HoA}''(s^*)[(\bar{s} - s^*)^2 + \frac{W_g^2}{4}] = 0 \\
& \Leftrightarrow \frac{1}{2}\text{HoA}''(s^*)\delta^2 + \text{HoA}'(s^*)\delta + \frac{1}{2}\text{HoA}''(s^*)\frac{W_g^2}{4} = 0 \\
& \Leftrightarrow \delta_{1,2} = -\frac{\text{HoA}'(s^*)}{\text{HoA}''(s^*)} \pm \sqrt{\left(\frac{\text{HoA}'(s^*)}{\text{HoA}''(s^*)}\right)^2 - \frac{W_g^2}{4}}
\end{aligned}$$

Notably, the correction is expected to be small ($\delta \ll s^*$); additionally, it holds:

$$\text{HoA}'(s^*) \gg \text{HoA}''(s^*).$$

Therefore, the only reasonable solution is:

$$\delta_{1,2} = -\frac{\text{HoA}'(s^*)}{\text{HoA}''(s^*)} + \sqrt{\left(\frac{\text{HoA}'(s^*)}{\text{HoA}''(s^*)}\right)^2 - \frac{W_g^2}{4}}.$$

□

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Acknowledgements

First of all, I would like to express my gratitude to my advisor, Professor Gabriella Gaias, not only for giving me the opportunity to investigate such a multifaceted and stimulating topic, but also for her constant guidance and valuable advice. Furthermore, I am deeply grateful to my co-advisors, Dr. Francesca Scala and Dr. Michelangelo Villano: thank you for having offered so much of your time and expertise.

I would like to thank all the people who accompanied me during my time at DLR and who were so generous in sharing knowledge and ideas, especially Antonio, Arthur, Maxwell, Nertjana, Olena, and Victor: friends, before colleagues. Thanks also to my friends from Milano and Sergnano, for having made my years as a student such a defining, intense and meaningful time. It would take too long to mention you all, but I cannot leave out Laura and Roby, Armand, Gio and Edo, Jannice, Jin and Linda. I also acknowledge the people who instilled in me curiosity and love of knowledge, science, art, and music: Professors Amighetti, Ferrari, and Piantelli.

At last, thanks to my family, to dad and mum, without whose unconditional love and support, not a single word of this thesis would have been written.

