Entanglement Detection from Syndrome Data

Institute for Software Technology (DLR)

Drashti Dhoriyani[1,2] Michael Epping[2] Otfried Gühne[1]

[1]Universität Siegen [2]Institute of Software Technology, German Aerospace Center (DLR)

Universität Siegen

Abstract

Entanglement in an error-correcting code is a property of a subspace, not of a single state. Local non-commutation of stabiliser generators decides when a stabilised subspace is completely or genuinely entangled and the conditions turn directly into witnesses for the $[[7,1,3]]$ Steane code. We characterise their detection regions and noise tolerance, extend the entanglement detection beyond the codespace.

Entanglement witnesses

For an n-qubit stabiliser state $|\psi\rangle$ with $\mathcal{S} = \langle g_1, g_2, \ldots, g_n\rangle$ [1],

$$\mathcal{W}_a = a\,\mathbb{1} - \sum_{g\in\mathcal{S}} g, \qquad a \in \mathbb{R} \text{ constant.}$$

$\langle \mathcal{W}_a\rangle < 0 \Rightarrow$ entangled state/states,
$\langle \mathcal{W}_a\rangle \geq 0 \Rightarrow$ all separable states.

From states to subspaces

- Fewer than n generators stabilise a subspace not a single state.
- In Completely Entangled Subspace (**CES**): every state is entangled.
- In Genuinely Entangled Subspace (**GES**): every state is GME(Genuinely Multipartite Entangled) [2].

Necessary and sufficient conditions

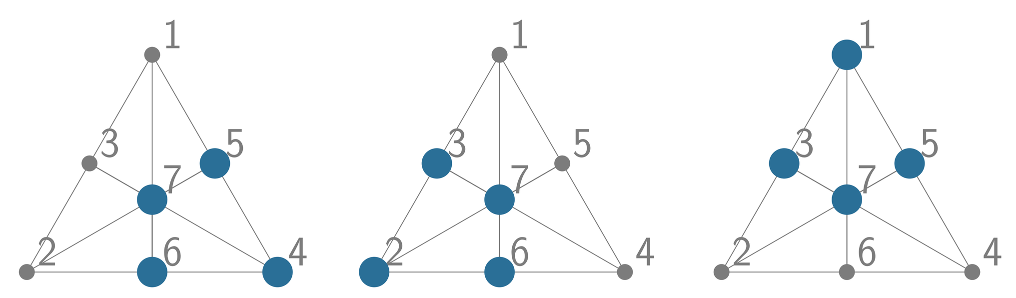
$$\begin{array}{c|cccc:cccc}
\text{qubits} & q^1 & q^2 & \cdots & q^k & q^{k+1} & \cdots & q^n \\
g_1 = & P_1^1 & P_1^2 & \cdots & P_1^k & P_1^{k+1} & \cdots & P_1^n \\
g_2 = & P_2^1 & P_2^2 & \cdots & P_2^k & P_2^{k+1} & \cdots & P_2^n \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
g_r = & P_r^1 & P_r^2 & \cdots & P_r^k & P_r^{k+1} & \cdots & P_r^n \\
 & & & Q_k & & & \bar{Q}_k &
\end{array}$$

CES $\exists k \;\; \exists i \neq j : \; \{g_i^{Q_k}, g_j^{Q_k}\} = 0$

GES $\forall k \;\; \exists i \neq j : \; \{g_i^{Q_k}, g_j^{Q_k}\} = 0$ [3].

The $[[7,1,3]]$ Steane code

2-D codespace: $|\psi\rangle = \alpha\,|0_L\rangle + \beta\,|1_L\rangle$.



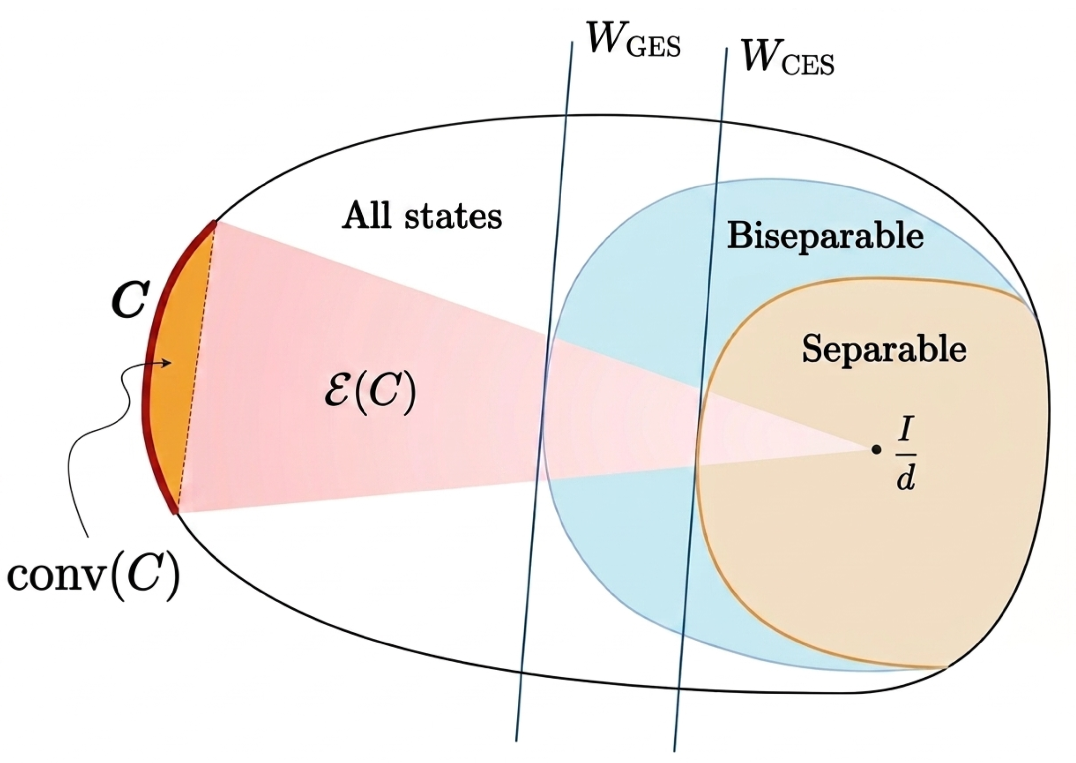
$g_1 = IIIXXXX$, $g_4 = IIIZZZZ$ $\quad$ $g_2 = IXXIIXX$, $g_5 = IZZIIZZ$ $\quad$ $g_3 = XIXIXIX$, $g_6 = ZIZIZIZ$

The Steane codespace is a GES, and therefore also a CES.

Witnesses to detect codespace

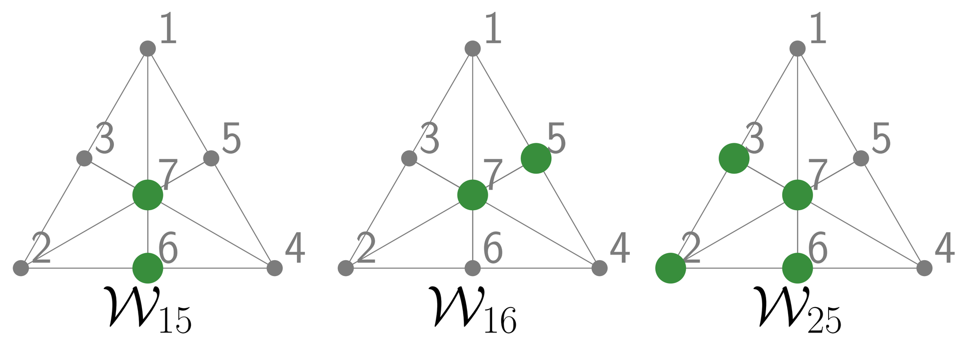
$$\mathcal{W}_{\mathrm{GES}} = \tfrac{1}{2}\mathbb{1} - \prod_{i=1}^{6} \frac{\mathbb{1} + g_i}{2}$$

$$\mathcal{W}_{\mathrm{CES}} = 3\mathbb{1} - \sum_{i=1}^{6} g_i$$



$\langle \mathcal{W}_{\mathrm{GES}}\rangle < 0$ and $\langle \mathcal{W}_{\mathrm{CES}}\rangle < 0$ on the codespace and in its neighbourhood.

Localising entanglement:



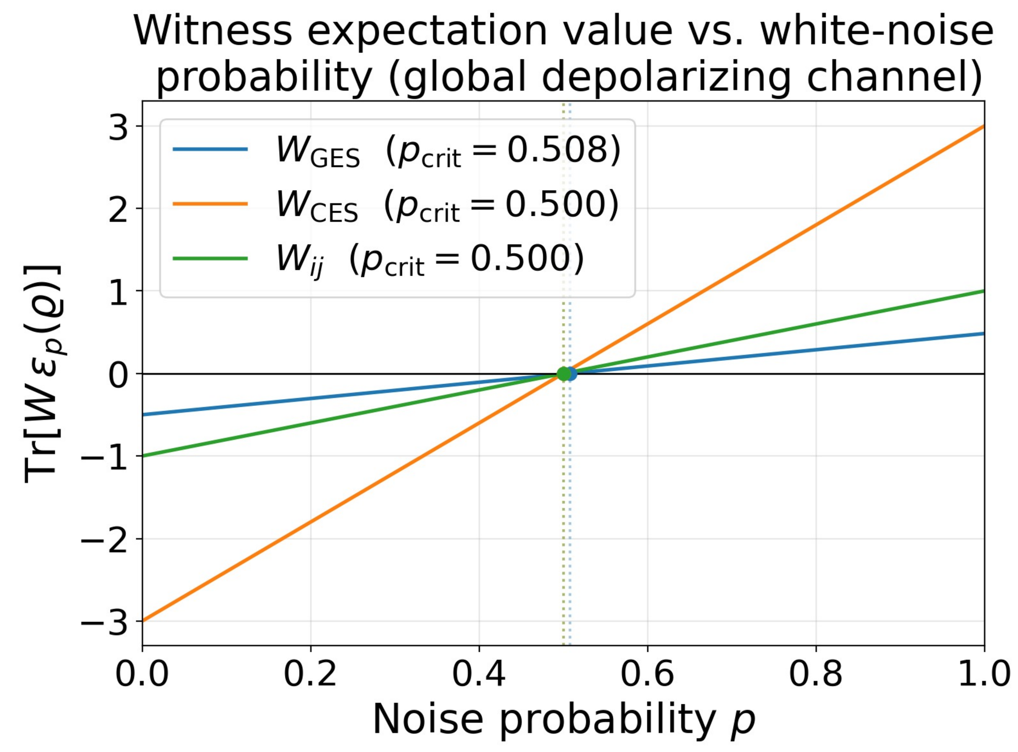
$$\mathcal{W}_{ij} = \mathbb{1} - g_i^X - g_j^Z.$$

W_{ij} witnesses the entanglement among the qubits shared by g_i^X and g_j^Z. [4].

Noise tolerance

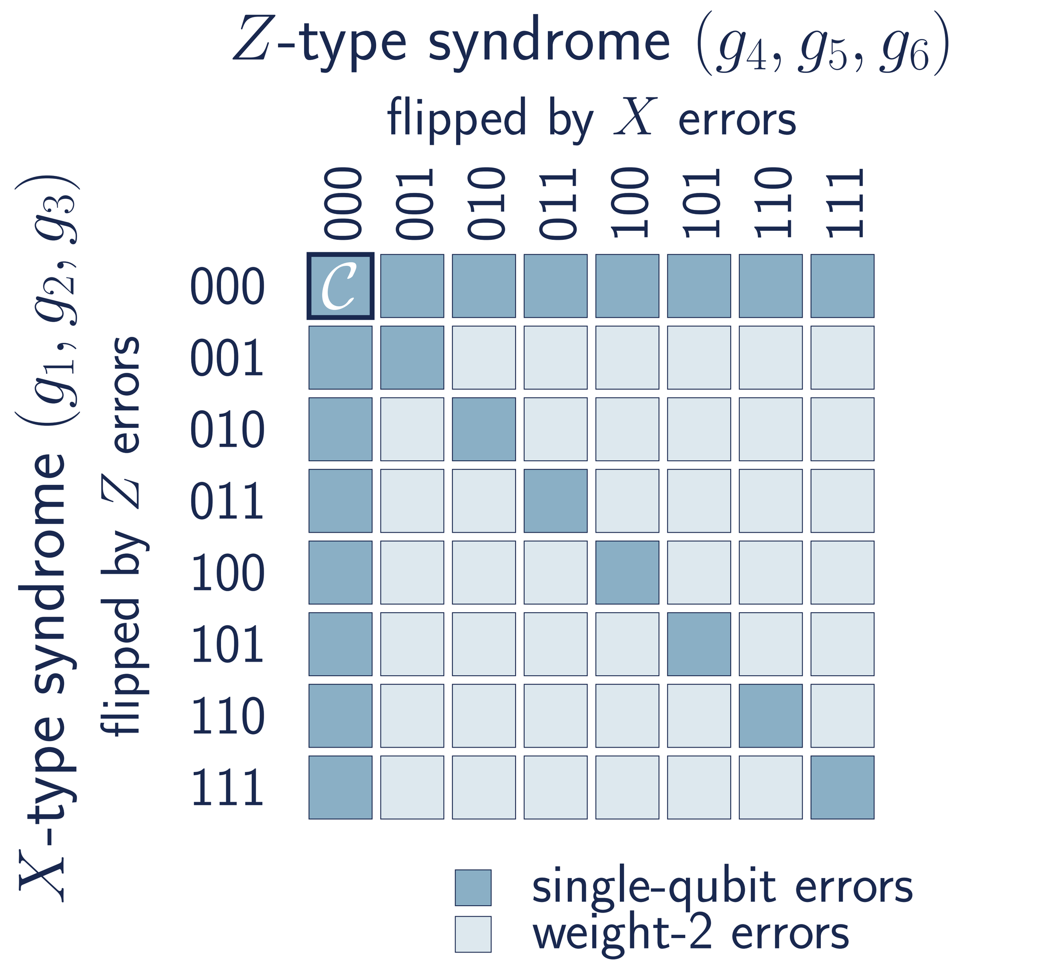
$$\varepsilon_p(\varrho) = p\,\frac{\mathbb{1}}{d} + (1-p)\,\varrho, \qquad \varrho = |\psi\rangle\langle\psi|.$$

$p_{\mathrm{crit}} = 0.508$ for $\mathcal{W}_{\mathrm{GES}}$, and $p_{\mathrm{crit}} = 0.5$ for $\mathcal{W}_{\mathrm{CES}}$ and $\mathcal{W}_{ij}$.



Beyond the codespace

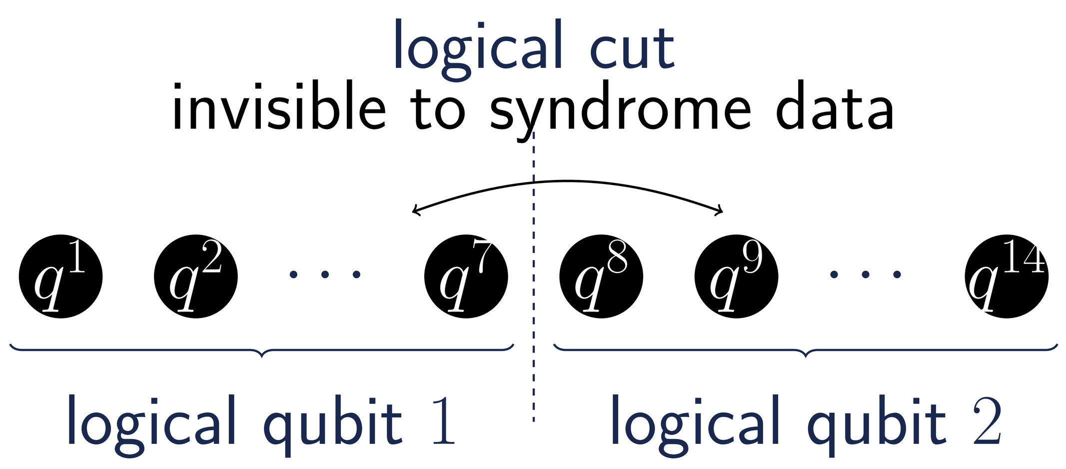
Each syndrome $\vec{s} \in \{\pm 1\}^6$ labels a sector.



Redefining generators $\vec{g}_{\vec{s}} = \vec{s} * \vec{g}$ for all witnesses, detects entanglement in its own syndrome sector.

Entanglement between logical qubits

The logical bipartition violates the GES condition.



Logical product and logical Bell states give identical syndromes.

References

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drashtiben.dhoriyani@student.uni-siegen.de