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Discrete Riccati and Comparison Approaches to Oscillation of Second-Order Advanced Difference Equations with Multiple Deviating Arguments

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Abstract

In this work, we develop, from first principles, a discrete oscillation theory for second-order nonlinear advanced difference equations with several deviating arguments, $\Delta(r(n)(\Delta y(n))^\alpha) + \sum_{i=1}^m q_i(n) F(y(\sigma_i(n))) = 0$, $\sigma_i(n) \geq n$, studied under the canonical condition $\sum_{s=n_0}^{\infty} r^{-1/\alpha}(s) = \infty$. This equation models discrete systems governed by anticipatory rather than delayed dynamics, since the second-order difference term depends on a future value $y(\sigma_i(n))$ rather than a past one. We prove an iterative monotonicity lemma, a classification lemma for eventually positive solutions, three oscillation results, a comparison theorem, and a Riccati-type theorem. Two features that are unique to the discrete setting, with no counterpart in the continuous theory, are highlighted. First, the partial sum $\eta(n) = \sum_{s=n}^{\infty} r^{-1/\alpha}(s)$ typically lacks a closed form and must be approximated asymptotically. Second, the discrete Riccati step requires an additional power-rule inequality, which introduces a correction term absent from the continuous case. The theoretical thresholds obtained are illustrated by a fully worked example, verified both numerically and symbolically and supported by figures.

Keywords: advanced difference equations; oscillation theory; discrete Riccati technique; canonical case; deviating arguments

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1. Introduction

Difference equations have become one of the most important branches of discrete mathematics due to their ability to model processes that evolve in discrete time. The foundations of the subject can be traced back to the pioneering work of Joseph-Louis Lagrange (1775) and Pierre-Simon Laplace (1775–1812), who introduced recurrence relations in the study of mathematical analysis and probability, and this line of work was later carried forward by George D. Birkhoff (1911). Since then, the theory has grown into an active area of research with applications spanning population dynamics, economics, engineering, control systems, numerical analysis, and computer science; comprehensive modern treatments are given by Kelley and Peterson [1] and Agarwal [2].

As the theory evolved, researchers recognized that many real-world systems depend not only on the present state but also on past states, which led to the development of delay difference equations. These have been extensively studied for their applications in biological systems, epidemiology, neural networks, and engineering, with significant contributions to their oscillation theory made by Agarwal, Bohner, Grace, Li, Thandapani, and others [3–5]. To represent systems in which delayed terms appear together with their differences, neutral delay difference equations were subsequently introduced; these possess richer dynamical behavior, and oscillation criteria, asymptotic properties, and stability results for them have been established by Agarwal, Grace, Moaaz, Bazighifan, Jadlovská, and their collaborators [6–10].

More recently, attention has shifted toward advanced difference equations, where the governing equation depends on future rather than past states. Such equations are mathematically more delicate than the delay case: they are non-causal and cannot be solved by conventional forward iteration, since the recursion at index n already involves values of the unknown sequence beyond n . Advanced recursions of this kind arise naturally in discrete population models with a reproductive lead time, in expectations-driven economic difference equations, and in shooting-type numerical schemes; the discrete-calculus background needed here is laid out in Kelley and Peterson [1] and Agarwal [2]. Although several oscillation results have recently been obtained in this setting, many problems remain open, particularly for second-order nonlinear advanced difference equations with multiple deviating arguments. Motivated by these gaps, the present work develops new oscillation criteria for exactly this class of equations, using a discrete Riccati transformation together with a comparison technique. Differential-transform and related semi-analytical operator techniques offer a complementary route to Riccati/comparison methods for functional equations with deviating arguments [11–13], and connect naturally to the delay and comparison frameworks used here.

The comparison method introduced by Sturm for second-order linear equations [14], sharpened by Kneser [15] and extended to deviating arguments by Fite [16], remains the backbone of the subject a century later, in both the continuous and discrete settings. Kusano and Naito [17] placed comparison theorems for equations with deviating arguments on a systematic footing, and the same philosophy has since been carried over to delay and neutral difference equations by a large body of work; broad surveys appear in the monographs of Agarwal, Bohner, and Li [3] and Agarwal, Grace, and O'Regan [4], while Bohner and Peterson's time-scale calculus [5] shows precisely where the continuous and discrete theories agree and where they diverge.

The advanced case, in contrast to the more heavily studied delay and neutral cases, has a smaller but steadily growing literature. Džurina obtained early comparison-based criteria for second-order advanced equations [18,19], and Jadlovská [20] and Kiliç [21] later developed iterative and Riccati-type refinements of exactly the kind we discretize in Section 4. Chatzarakis, Džurina, and Jadlovská [22] treated the half-linear advanced case, Bohner, Vidhyaa, and Thandapani [23] handled the noncanonical advanced case via a canonical transformation, and Luo [24] extended advanced-equation oscillation criteria to higher, trinomial-type orders. On the delay and neutral side, which supplies much of the comparison and Riccati machinery we adapt, we mention Bohner, Grace, and Jadlovská [6], Grace, Džurina, Jadlovská, and Li [7], Džurina and Jadlovská [8], Baculiková [9], Santra, Sethi, Moaaz, Khedher, and Yao [10], and, for fourth- and third-order neutral and Kneser-type extensions, Agarwal, Bazighifan, and Ragusa [25], Bazighifan, Moaaz, El-Nabulsi, and Muhib [26], Almarri, Moaaz, and Muhib [27], Aldiaji, Qaraad, Iambor, Rabie, and Elabasy [28], Moaaz, Qaraad, El-Nabulsi, and Bazighifan [29], El-Gaber [30], and Tian and

Guo [31]. Moaaz's algebraic lemma [32], reproduced without modification below, is the one purely algebraic ingredient shared by essentially all of the literature, continuous or discrete.

A recent contribution to the continuous case is due to Alhaysuni, Almohaimeed, Moaaz, and Essam [33], who studied a second-order nonlinear advanced differential equation with several deviating arguments.

$$\left(r(t)(y'(t))^\alpha\right)' + \sum_{i=1}^n q_i(t) F(y(h_i(t))) = 0, \quad t \geq t_0, \quad h_i(t) \geq t,$$

in the canonical case $\int_{t_0}^{\infty} r^{-1/\alpha}(s) ds = \infty$. Their principal achievement was to keep the individual effect of each deviating argument $h_i(t)$ visible inside the final oscillation criteria, rather than collapsing it into a single coefficient as most earlier comparison arguments had done, and on that basis they proved a comparison theorem, a Riccati-technique theorem, and a threshold corollary isolating the universal constant $\alpha/(1+\alpha)^{(1+\alpha)/\alpha}$.

This work lies within a much larger and highly active body of continuous-side literature on second- and higher-order oscillation, most of it published within the last five years. On the second-order noncanonical and canonical fronts, sharpened Riccati- and comparison-type criteria have been obtained by Baculíková [9], Bohner, Grace, and Jadlovská [6], Džurina and Jadlovská [8], Bohner, Vidhyaa, and Thandapani [23], Santra, Sethi, Moaaz, Khedher, and Yao [10], and Tian and Guo [31], while Kiliç [21] treats the nonlinear advanced case directly. Third- and higher-order extensions, including neutral and superlinear variants, appear in Aldiaji, Qaraad, Iambor, Rabie, and Elabbasy [28], Moaaz, Qaraad, El-Nabulsi, and Bazighifan [29], Bazighifan, Moaaz, El-Nabulsi, and Muhib [26], Agarwal, Bazighifan, and Ragusa [25], Almarri, Moaaz, and Muhib [27], Luo [24], and El-Gaber [30]. Notably, when the Riccati technique is applied through this inequality, the resulting criterion carries a correction factor that has no counterpart on the continuous side.

The structure of the paper is as follows: Section 2 introduces the equation, standing hypotheses, notation, and a schematic diagram of the advanced-argument structure. Section 3 develops three auxiliary lemmas with full proofs. Section 4 proves the three primary oscillation theorems. Section 5 works out an illustrative example, in which two figures confirm the asymptotic threshold calculations against exact finite sums. Section 6 concludes the paper.

2. Mathematical Preliminaries

For a fixed integer n_0 , $\mathbb{N}_{n_0} = \{n_0, n_0 + 1, n_0 + 2, \dots\}$ is used throughout. Δ stands for the forward difference operator, $\Delta y(n) = y(n+1) - y(n)$. When u is a real number and $\alpha \in \mathbb{Q}$ is a quotient of odd positive integers, the odd power u^α is interpreted in the standard sense, making $u \mapsto u^\alpha$ an odd, strictly increasing bijection of $\mathbb{R}$ onto itself.

We examine the second-order nonlinear advanced difference equation with several deviating arguments

$$\Delta\left(r(n)(\Delta y(n))^\alpha\right) + \sum_{i=1}^m q_i(n) F(y(\sigma_i(n))) = 0, \quad n \geq n_0, \quad (1)$$

under the following assumptions.

- (H1) The number of deviating arguments is $m \in \mathbb{Z}^+$, and $\alpha \in \mathbb{Q}$ is a quotient of odd positive integers;
- (H2) $r, q_i : \mathbb{N}_{n_0} \rightarrow (0, \infty)$ for $i = 1, \dots, m$;
- (H3) $\sigma_i : \mathbb{N}_{n_0} \rightarrow \mathbb{N}_{n_0}$ satisfies $\sigma_i(n) \geq n$ for all $n \geq n_0$ and all $i = 1, \dots, m$ and $\sigma_i(n) \rightarrow \infty$ as $n \rightarrow \infty$;
- (H4) $F \in C(\mathbb{R}, \mathbb{R})$ and for any $u \neq 0$, there exists a constant $K > 0$ such that $F(u)/u^\alpha \geq K$.

A solution of (1) is a real sequence $y : \mathbb{N}_{n_a} \rightarrow \mathbb{R}$, $n_a \geq n_0$, such that (1) holds identically on $\mathbb{N}_{n_a}$ and $r(n)(\Delta y(n))^\alpha$ is defined for $n \geq n_a$. For each $n_* \geq n_0$, we consider only solutions satisfying $\sup\{|y(n)| : n \geq n_*\} > 0$. A solution of (1) is called *oscillatory* if it is neither eventually positive nor eventually negative, and *nonoscillatory* otherwise. Equation (1) itself is termed *oscillatory* if all of its solutions oscillate.

We say (1) is in *canonical form* if

$$\sum_{s=n_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} = \infty. \quad (2)$$

This is the discrete counterpart of the canonical-case condition used in [33], and it is the standing assumption of the present paper.

The following notation is fixed

$$\eta(n) = \sum_{s=n_0}^{n-1} \frac{1}{r^{1/\alpha}(s)} \quad (n \geq n_0, \eta(n_0) = 0), \quad \sigma(n) = \min\{\sigma_i(n) : i = 1, \dots, m\}. \quad (3)$$

The advanced deviating-argument structure considered in Equation (1) is illustrated schematically in Figure 1.

By condition (2), $\eta(n) \rightarrow \infty$ as $n \rightarrow \infty$.

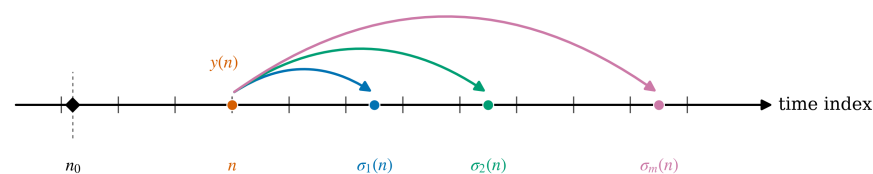


Figure 1. Schematic of the advanced deviating arguments in Equation (1): at every time index n , Equation (1) couples the value $y(n)$ to the future values $y(\sigma_1(n)), \dots, y(\sigma_m(n))$, each satisfying $\sigma_i(n) \geq n$.

3. Auxiliary Lemmas

Throughout this section, we assume (1)–(3) and (H1)–(H4). We also impose the following mild strengthening, used in the Riccati-technique results:

$$\sigma_i(n) \geq n + 1, \quad \text{for all } n \geq n_0, i = 1, 2, \dots, m.$$

This assumption excludes the degenerate case in which a deviating argument fails to advance strictly. It is a mild restriction that is automatically satisfied by the applications of interest, such as $\sigma_i(n) = \lceil \varepsilon_i n \rceil$ with $\varepsilon_i > 1$.

Remark 1. The boundary case $\sigma_i(n) = n$ is excluded by the above assumption because, in this case, the argument is contemporaneous rather than genuinely advanced. Indeed, $\frac{\eta(\sigma_i(s))}{\eta(s)} = 1$, and hence the additional contribution arising from the advanced argument disappears. In contrast, the strict condition $\sigma_i(n) \geq n + 1$ ensures that $\frac{\eta(\sigma_i(s))}{\eta(s)} > 1$, which distinguishes the present setting from the non-advanced case and is essential for the subsequent iterative analysis.

3.1. An Algebraic Inequality

Lemma 1. Let $L, M \in \mathbb{R}$ with $L > 0$, and let $\alpha > 0$. Then, for every real s ,

$$Ms - Ls^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{M^{\alpha+1}}{L^\alpha}. \quad (4)$$

Proof. We give a short proof for completeness, since this is a purely algebraic fact about real numbers that need not be treated separately.

Consider $f(s) = Ms - Ls^{(\alpha+1)/\alpha}$ for $s \geq 0$ and fix $L > 0, M \in \mathbb{R}$.

Since $f'(s) = M - L\frac{\alpha+1}{\alpha}s^{1/\alpha}$, the unique critical point is

$$s^* = \left(\frac{\alpha M}{(\alpha+1)L} \right)^\alpha,$$

and since $f''(s) < 0$ there (as $L > 0$), s^* is the global maximizer of f .

Substituting s^* into f and simplifying gives

$$f(s^*) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{M^{\alpha+1}}{L^\alpha},$$

and since $f(s) \leq f(s^*)$ for every s , this establishes (4). $\square$

Lemma 2. Assume y is an eventually positive solution of (1). Then, eventually,

$$\Delta y(n) > 0 \quad \text{and} \quad \Delta(r(n)(\Delta y(n))^\alpha) < 0. \quad (5)$$

Proof. Fix $n_1 \geq n_0$ such that, for every $n \geq n_1$ and $i = 1, \dots, m$, $y(n) > 0$ and $y(\sigma_i(n)) > 0$. By (1) and (H2),

$$\Delta(r(n)(\Delta y(n))^\alpha) = - \sum_{i=1}^m q_i(n) F(y(\sigma_i(n))) < 0, \quad n \geq n_1,$$

because each term $q_i(n)F(y(\sigma_i(n)))$ is positive by (H4), $F(u)/u^\alpha \geq K > 0$ whenever $u \neq 0$, and here $u = y(\sigma_i(n)) > 0$, so $F(y(\sigma_i(n))) \geq Ky^\alpha(\sigma_i(n)) > 0$. This proves the second inequality in (5) and shows that the sequence $r(n)(\Delta y(n))^\alpha$ is strictly decreasing for $n \geq n_1$ being monotone, it is eventually of one sign.

Since α is a quotient of odd positive integers, $u \mapsto u^\alpha$ is an odd, strictly increasing bijection of $\mathbb{R}$ onto itself (assumption (H1)) since also $r(n) > 0$, it follows that $\text{sign}(r(n)(\Delta y(n))^\alpha) = \text{sign}(\Delta y(n))$ for every n . Consequently, since $r(n)(\Delta y(n))^\alpha$ is eventually of one sign, $\Delta y(n)$ itself must eventually be of one sign either eventually positive or eventually negative, with no other possibility. We now rule out the negative case.

Suppose, for contradiction, that $\Delta y(n) < 0$ for all $n \geq n_2 \geq n_1$. Since $r(n)(\Delta y(n))^\alpha$ is decreasing,

$$r(n)(\Delta y(n))^\alpha \leq r(n_2)(\Delta y(n_2))^\alpha = -c^\alpha$$

for $n \geq n_2$, where $c > 0$. Hence,

$$\Delta y(n) \leq -\frac{c}{r^{1/\alpha}(n)}, \quad n \geq n_2.$$

Summing this inequality from n_2 to $n-1$ gives

$$y(n) \leq y(n_2) - c \sum_{s=n_2}^{n-1} \frac{1}{r^{1/\alpha}(s)}.$$

By applying the canonical condition (2) and letting $n \rightarrow \infty$, the right-hand side tends to $-\infty$, which contradicts $y(n) > 0$. Thus $\Delta y(n) > 0$ eventually, which completes the proof. $\square$

3.2. A Discrete Power-Rule Inequality

Lemma 3 (Power-rule inequality). *Let $\alpha \geq 1$ and let $y : \mathbb{N}_{n_1} \rightarrow (0, \infty)$ satisfy $\Delta y(n) > 0$ for $n \geq n_1$. Then*

$$\Delta(y^\alpha(n)) \geq \alpha y^{\alpha-1}(n) \Delta y(n), \quad n \geq n_1. \quad (6)$$

Proof. The map $u \mapsto u^\alpha$ is convex on $(0, \infty)$ for $\alpha \geq 1$, so its graph lies above every tangent line for all $a, b > 0$,

$$b^\alpha \geq a^\alpha + \alpha a^{\alpha-1}(b - a).$$

Taking $a = y(n)$ and $b = y(n+1)$ gives

$$y^\alpha(n+1) - y^\alpha(n) \geq \alpha y^{\alpha-1}(n)(y(n+1) - y(n)) = \alpha y^{\alpha-1}(n) \Delta y(n),$$

which is (6). $\square$

Remark 2. *The concavity of $u \mapsto u^\alpha$ for $0 < \alpha < 1$ reverses the tangent-line inequality used in Lemma 3 in place of the estimate obtained there, one now has*

$$\Delta(y^\alpha(n)) \leq \alpha y^{\alpha-1}(n) \Delta y(n).$$

Since this inequality only furnishes an upper bound on the numerator arising in the proof of Lemma 4, it is insufficient on its own to yield the monotonicity property established in that lemma; a different auxiliary estimate would be needed to cover the sublinear case. Accordingly, the analysis in this paper is confined to $\alpha \geq 1$, while the case $0 < \alpha < 1$ is set aside as requiring a separate line of argument.

3.3. Iterative Monotonicity Lemma

Lemma 4. *Assume y is an eventually positive solution of (1). Let $\mu_0 = 0$, and suppose there exist $J \in \mathbb{N}$ and constants $\mu_j > 0$, $j = 1, \dots, J$, such that*

$$\alpha \eta(n) \left(\sum_{s=n}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_{j-1}} \sum_{i=1}^m q_i(s) \right)^{1/\alpha} \geq \mu_j, \quad n \geq n_1, \quad (7)$$

where the tail sums are assumed convergent. Then, for every $j = 1, \dots, J$,

$$\Delta \left(\frac{y^\alpha(n)}{\eta^{\mu_j}(n)} \right) \geq 0 \quad (8)$$

eventually.

Proof. We argue by induction on j , exactly as in the continuous case but with all integrals replaced by sums. Base case $j = 1$. By Lemma 2, y is increasing and $r(n)(\Delta y(n))^\alpha$ is decreasing and eventually positive.

Since $r(n)(\Delta y(n))^\alpha$ is positive (Lemma 2) and strictly decreasing, it converges to some $\ell \geq 0$ as $n \rightarrow \infty$; telescoping (1) from n to ∞ therefore gives, for $n \geq n_1$,

$$r(n)(\Delta y(n))^\alpha = \ell + \sum_{s=n}^{\infty} \sum_{i=1}^m q_i(s) F(y(\sigma_i(s))) \geq \sum_{s=n}^{\infty} \sum_{i=1}^m q_i(s) F(y(\sigma_i(s))) \geq K \sum_{s=n}^{\infty} \sum_{i=1}^m q_i(s) y^\alpha(\sigma_i(s)).$$

Since $\sigma_i(s) \geq \sigma(s) = \min_i \sigma_i(s)$ and y is increasing, $y^\alpha(\sigma_i(s)) \geq y^\alpha(\sigma(s))$; and since $\sigma(s) \geq s$, $y^\alpha(\sigma(s)) \geq y^\alpha(s)$.

Absorbing K into the inequality, we get

$$r(n)(\Delta y(n))^\alpha \geq y^\alpha(n) \sum_{s=n}^{\infty} \sum_{i=1}^m q_i(s). \quad (9)$$

We now compute

$$\Delta\left(\frac{y^\alpha(n)}{\eta^{\mu_1}(n)}\right) = \frac{\Delta(y^\alpha(n)) \eta^{\mu_1}(n) - y^\alpha(n) \Delta(\eta^{\mu_1}(n))}{\eta^{\mu_1}(n) \eta^{\mu_1}(n+1)}.$$

By Lemma 3, $\Delta(y^\alpha(n)) \geq \alpha y^{\alpha-1}(n) \Delta y(n)$, and since $\Delta \eta(n) = r^{-1/\alpha}(n) > 0$, we have

$$\Delta\left(\frac{y^\alpha(n)}{\eta^{\mu_1}(n)}\right) \geq \frac{\eta^{\mu_1}(n) \alpha y^{\alpha-1}(n) \Delta y(n) - y^\alpha(n) \mu_1 \eta^{\mu_1-1}(n+1) r^{-1/\alpha}(n)}{\eta^{\mu_1}(n) \eta^{\mu_1}(n+1)}.$$

Multiplying numerator and denominator by $y(n)/y(n)$ and using $\Delta y(n) = (r(n)(\Delta y(n))^\alpha)^{1/\alpha} / r^{1/\alpha}(n)$ together with (9), the numerator is bounded below, after simplification, by

$$\frac{y^\alpha(n)}{r^{1/\alpha}(n)} \left[\alpha \eta(n) \left(\sum_{s=n}^{\infty} \sum_{i=1}^m q_i(s) \right)^{1/\alpha} - \mu_1 \right] \cdot \eta^{\mu_1-1}(n+1),$$

which is nonnegative precisely because (7) holds with $j = 1$ ($\mu_0 = 0$, so the ratio $(\eta(\sigma(s))/\eta(s))^{\mu_0} \equiv 1$). This proves (8) for $j = 1$.

Inductive step. Suppose (8) holds for $j = k$, i.e., $y^\alpha(n)/\eta^{\mu_k}(n)$ is not decreasing. Then monotonicity of $y^\alpha(n)/\eta^{\mu_k}(n)$ and $\sigma(s) \geq s$ give

$$y^\alpha(\sigma(s)) \geq \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_k} y^\alpha(s).$$

Repeating the computation that led to (9), now with this sharper pointwise bound inserted before applying $\sigma_i(s) \geq \sigma(s)$, gives

$$r(n)(\Delta y(n))^\alpha \geq y^\alpha(n) \sum_{s=n}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_k} \sum_{i=1}^m q_i(s),$$

and the same algebra as in the base case, with μ_1 replaced by μ_{k+1} and the extremal condition (7) invoked at $j = k + 1$, shows that $\Delta(y^\alpha(n)/\eta^{\mu_{k+1}}(n)) \geq 0$ eventually. By induction, (8) holds for every $j = 1, \dots, J$. $\square$

4. Main Oscillation Theorems

4.1. A Discrete Comparison Equation

Define $q(n) = \sum_{i=1}^m q_i(n)$. The discrete comparison equation associated with (1), of the same order and with the same α , is

$$\Delta\left(r(n)(\Delta x(n))^\alpha\right) + \left(\frac{\eta(\sigma(n))}{\eta(n)}\right)^{\mu_j} q(n) F(x(n)) = 0, \quad n \geq n_0. \quad (10)$$

Theorem 1. Let $\mu_0 = 0$ and suppose there exist $J \in \mathbb{N}$ with $\mu_j > 0$, $j = 1, \dots, J$, such that (7) holds. If Equation (10) is oscillatory for every $j = 1, \dots, J$, then (1) is oscillatory.

Proof. Suppose, to the contrary, that (1) has an eventually positive solution y on $\mathbb{N}_{n_1}$. By Lemma 4, $y^\alpha(n)/\eta^{\mu_j}(n)$ is nondecreasing eventually, so, using $y(\sigma_i(n)) \geq y(\sigma(n))$,

$$\sum_{i=1}^m q_i(n) F(y(\sigma_i(n))) \geq K \sum_{i=1}^m q_i(n) y^\alpha(\sigma_i(n)) \geq K \left(\frac{\eta(\sigma(n))}{\eta(n)} \right)^{\mu_j} q(n) y^\alpha(n).$$

Substituting into (1) yields

$$\Delta \left(r(n) (\Delta y(n))^\alpha \right) + K \left(\frac{\eta(\sigma(n))}{\eta(n)} \right)^{\mu_j} q(n) y^\alpha(n) \leq 0, \quad j = 1, \dots, J.$$

Summing from n to ∞ ,

$$r(n) (\Delta y(n))^\alpha \geq K \sum_{s=n}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_j} q(s) y^\alpha(s),$$

so that

$$\Delta y(n) \geq \left(\frac{K}{r(n)} \sum_{s=n}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_j} q(s) y^\alpha(s) \right)^{1/\alpha}.$$

Summing once more from n_1 to $n - 1$,

$$y(n) \geq y(n_1) + \sum_{u=n_1}^{n-1} \left(\frac{K}{r(u)} \sum_{s=u}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_j} q(s) y^\alpha(s) \right)^{1/\alpha}.$$

Define the sequence of sequences $\{\omega_\ell(n)\}_{\ell \in \mathbb{N}_0}$ by $\omega_0(n) = y(n)$ and

$$\omega_{\ell+1}(n) = y(n_1) + \sum_{u=n_1}^{n-1} \left(\frac{K}{r(u)} \sum_{s=u}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_j} q(s) \omega_\ell^\alpha(s) \right)^{1/\alpha}.$$

We verify the three facts used to pass to the limit $\ell \rightarrow \infty$.

(i) *Monotonicity in ℓ .* $\omega_0(n) = y(n)$, and the inequality established above shows $\omega_0(n) \geq \omega_1(n)$. If $\omega_{\ell-1} \geq \omega_\ell$ pointwise, then, since $u \mapsto u^\alpha$ is increasing on $(0, \infty)$ and $q(s) > 0$, the summand defining $\omega_{\ell+1}$ is termwise less than or equal to the one defining ω_ℓ , so $\omega_{\ell+1} \leq \omega_\ell$. By induction, $\{\omega_\ell(n)\}_\ell$ is not increasing for every fixed n .

(ii) *Boundedness.* $\omega_\ell(n) \geq y(n_1) > 0$ for every ℓ , by the same monotone substitution argument, so $\{\omega_\ell(n)\}$ is bounded below and converges to some $\omega(n) \geq y(n_1)$ as $\ell \rightarrow \infty$.

(iii) *Interchange with the infinite sum.* For each fixed s , $\omega_\ell^\alpha(s)$ is nonincreasing in ℓ (by (i)) and bounded above by $\omega_0^\alpha(s) = y^\alpha(s)$. Since

$$\sum_{s=u}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_j} q(s) y^\alpha(s)$$

converges by the standing convergence hypothesis of Lemma 4, dominated convergence for series justifies passing the limit $\ell \rightarrow \infty$ inside

$$\sum_{s=u}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_j} q(s) \omega_\ell^\alpha(s),$$

and then inside the finite sum

$$\sum_{u=n_1}^{n-1} \left(\frac{K}{r(u)} \sum_{s=u}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_j} q(s) \omega_\ell^\alpha(s) \right)^{1/\alpha}.$$

$$\omega(n) = y(n_1) + \sum_{u=n_1}^{n-1} \left(\frac{K}{r(u)} \sum_{s=u}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_j} q(s) \omega^\alpha(s) \right)^{1/\alpha}.$$

Taking Δ twice shows that ω is a positive solution of the comparison Equation (10), contradicting the assumed oscillation of (10) for $j = 1, \dots, J$. Hence, (1) admits no eventually positive solution; an analogous argument, given explicitly in Remark 3 below, excludes eventually negative solutions. Therefore, every solution of (1) oscillates. $\square$

Remark 3. Suppose $y(n) < 0$ eventually and set $w(n) = -y(n) > 0$. Since α is a quotient of odd positive integers, $(-x)^\alpha = -x^\alpha$, so $\Delta y(n) = -\Delta w(n)$ gives $r(n)(\Delta y(n))^\alpha = -r(n)(\Delta w(n))^\alpha$, and (1) becomes

$$\Delta(r(n)(\Delta w(n))^\alpha) + \sum_{i=1}^m q_i(n) G(w(\sigma_i(n))) = 0, \quad G(u) = -F(-u).$$

For $u > 0$, since $(-u)^\alpha = -u^\alpha$,

$$\frac{G(u)}{u^\alpha} = \frac{-F(-u)}{u^\alpha} = \frac{F(-u)}{(-u)^\alpha} \geq K$$

by (H4) applied directly at the negative point $-u$, with the same constant K . Thus w solves an equation of exactly the form (1), with F replaced by G , and G satisfies (H4) with the identical constant K . Therefore, every lemma and theorem in this paper, applies verbatim to w , and oscillation for eventually negative solutions of (1) follows from the same criteria, with no new hypothesis and no change to K .

4.2. A Discrete Riccati Criterion

Theorem 2. Suppose $\mu_0 = 0$ and there exist $J \in \mathbb{N}$ and $\mu_j > 0$, $j = 1, \dots, J$, such that (7) holds, and suppose $\alpha \geq 1$. If there is a positive sequence $\rho : \mathbb{N}_{n_0} \rightarrow (0, \infty)$ such that

$$\limsup_{n \rightarrow \infty} \sum_{s=n_1}^{n-1} \left(B(s) - \frac{r(s)(\Delta \rho(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1} \lambda^\alpha(s) \rho^\alpha(s+1)} \right) = \infty \quad (11)$$

for every $n_1 \geq n_0$, where

$$B(s) = K\rho(s+1) \sum_{i=1}^m q_i(s) \left(\frac{\eta(\sigma_i(s))}{\eta(s+1)} \right)^{\mu_j}, \quad \lambda(s) = \left(\frac{\eta(s)}{\eta(s+1)} \right)^{\mu_j} \in (0, 1),$$

then Equation (1) is oscillatory.

Proof. Suppose, to the contrary, that (1) has an eventually positive solution y on $\mathbb{N}_{n_1}$, $n_1 \geq n_0$. By (H4) and (1),

$$\Delta(r(n)(\Delta y(n))^\alpha) \leq -K \sum_{i=1}^m q_i(n) y^\alpha(\sigma_i(n)). \quad (12)$$

Define the Riccati sequence

$$Z(n) = \rho(n) \frac{r(n)(\Delta y(n))^\alpha}{y^\alpha(n)}, \quad n \geq n_1,$$

which is positive by Lemma 3. Using the discrete product rule $\Delta(uv)(n) = u(n+1)\Delta v(n) + v(n)\Delta u(n)$ with $u = \rho$, $v(n) = r(n)(\Delta y(n))^\alpha / y^\alpha(n)$,

$$\Delta Z(n) = \rho(n+1)\Delta v(n) + \frac{\Delta \rho(n)}{\rho(n)} Z(n),$$

where

$$\Delta v(n) = \frac{\Delta(r(n)(\Delta y(n))^\alpha)}{y^\alpha(n+1)} - \frac{r(n)(\Delta y(n))^\alpha \Delta(y^\alpha(n))}{y^\alpha(n) y^\alpha(n+1)}.$$

By (12), the first term is at most $-K \sum_i q_i(n) y^\alpha(\sigma_i(n)) / y^\alpha(n+1)$. Since $\sigma_i(n) \geq n+1$ by (H3'), Lemma 4 gives the lower bound on the quantity being subtracted,

$$\frac{y^\alpha(\sigma_i(n))}{y^\alpha(n+1)} \geq \left(\frac{\eta(\sigma_i(n))}{\eta(n+1)} \right)^{\mu_j},$$

so the first term is at most $-K \sum_i q_i(n) (\eta(\sigma_i(n)) / \eta(n+1))^{\mu_j}$.

For the second term, Lemma 3 gives the lower bound $\Delta(y^\alpha(n)) \geq \alpha y^{\alpha-1}(n) \Delta y(n)$; since the second term equals $-c(n) \Delta(y^\alpha(n))$ with $c(n) = r(n)(\Delta y(n))^\alpha / (y^\alpha(n) y^\alpha(n+1)) > 0$, and $x \mapsto -c(n)x$ is decreasing, this lower bound yields the upper bound

$$-\frac{r(n)(\Delta y(n))^\alpha \Delta(y^\alpha(n))}{y^\alpha(n) y^\alpha(n+1)} \leq -\alpha \frac{r(n)(\Delta y(n))^{\alpha+1}}{y(n) y^\alpha(n+1)}.$$

Writing $r(n)(\Delta y(n))^\alpha = Z(n) y^\alpha(n) / \rho(n)$ and $\Delta y(n) = (Z(n) / (\rho(n) r(n)))^{1/\alpha} y(n)$, the right-hand side simplifies to

$$-\alpha \frac{Z(n)^{1+1/\alpha}}{\rho(n)^{1+1/\alpha} r(n)^{1/\alpha}} \cdot \frac{y^\alpha(n)}{y^\alpha(n+1)}.$$

Applying Lemma 4 once more, now comparing the consecutive indices n and $n+1$ directly ($y^\alpha(n) / \eta^{\mu_j}(n) \leq y^\alpha(n+1) / \eta^{\mu_j}(n+1)$), gives the lower bound $y^\alpha(n) / y^\alpha(n+1) \geq \lambda(n) = (\eta(n) / \eta(n+1))^{\mu_j}$, and since this factor multiplies a negative coefficient, it too yields an upper bound

$$-\alpha \frac{Z(n)^{1+1/\alpha}}{\rho(n)^{1+1/\alpha} r(n)^{1/\alpha}} \cdot \frac{y^\alpha(n)}{y^\alpha(n+1)} \leq -\alpha \lambda(n) \frac{Z(n)^{1+1/\alpha}}{\rho(n)^{1+1/\alpha} r(n)^{1/\alpha}}.$$

Collecting both bounds and multiplying by $\rho(n+1) > 0$,

$$\Delta Z(n) \leq -K \rho(n+1) \sum_{i=1}^m q_i(n) \left(\frac{\eta(\sigma_i(n))}{\eta(n+1)} \right)^{\mu_j} + \frac{\Delta \rho(n)}{\rho(n)} Z(n) - \alpha \lambda(n) \frac{\rho(n+1)}{\rho(n)^{1+1/\alpha} r(n)^{1/\alpha}} Z(n)^{1+1/\alpha}.$$

Every sign flip above is paired with a lower bound applied to a term entering with a negative coefficient, which is the correct and consistent pattern throughout this derivation. Now apply Lemma 1 with $s = Z(n)$, $M = \Delta \rho(n) / \rho(n)$, and $L = \alpha \lambda(n) \rho(n+1) / (\rho(n)^{1+1/\alpha} r(n)^{1/\alpha})$; a direct computation gives

$$\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{M^{\alpha+1}}{L^\alpha} = \frac{r(n)(\Delta \rho(n))^{\alpha+1}}{(\alpha+1)^{\alpha+1} \lambda^\alpha(n) \rho^\alpha(n+1)},$$

so that

$$\Delta Z(n) \leq -B(n) + \frac{r(n)(\Delta \rho(n))^{\alpha+1}}{(\alpha+1)^{\alpha+1} \lambda^\alpha(n) \rho^\alpha(n+1)}.$$

Summing from n_1 to $n - 1$,

$$\sum_{s=n_1}^{n-1} \left(B(s) - \frac{r(s)(\Delta \rho(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1} \lambda^\alpha(s) \rho^\alpha(s+1)} \right) \leq Z(n_1) - Z(n) \leq Z(n_1),$$

which contradicts (11). Hence (1) has no eventually positive solution, and, as in Theorem 1 (see Remark 3), no eventually negative solution either; therefore (1) is oscillatory. $\square$

4.3. An Explicit Threshold Corollary

Theorem 3. Let $\mu_0 = 0$ and suppose there exist $J \in \mathbb{N}$ and $\mu_j > 0$, $j = 1, \dots, J$, satisfying (7), and

$$\mu_J > \frac{\alpha}{(1+\alpha)^{(1+\alpha)/\alpha}}. \quad (13)$$

Then every solution of (1) oscillates.

Proof. The constant $\gamma = \alpha/(1+\alpha)^{(1+\alpha)/\alpha}$ appearing in (13) is exactly the constant produced by Lemma 1. Precisely, (7) together with (13) gives, for $\alpha/(1+\alpha)^{(1+\alpha)/\alpha} < \mu_J < \infty$,

$$\alpha \eta(n) \left(\sum_{s=n}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_{J-1}} \sum_{i=1}^m q_i(s) \right)^{1/\alpha} \geq \mu_J > \gamma.$$

Replacing $(\eta(\sigma(s))/\eta(s))^{\alpha \mu_J} \sum_i q_i(s)$ by $\sum_i q_i(s)$ in the comparison Equation (10) only strengthens the coefficient, since $\eta(\sigma(s))/\eta(s) \geq 1$ the conclusion then follows by invoking Theorem 1 with this comparison equation. $\square$

Remark 4. As in the continuous case, the conclusion of Theorem 3 persists under the weaker hypothesis

$$\mu_j \leq \gamma \text{ for } j = 1, \dots, J-1, \quad \mu_J > \gamma \text{ for some } J \geq 2,$$

i.e., it suffices that the iteratively defined threshold sequence $\{\mu_j\}$ eventually, not immediately, exceed $\gamma = \alpha/(1+\alpha)^{(1+\alpha)/\alpha}$.

5. Example

Example 1. Fix $\alpha \geq 1$, a quotient of odd positive integers, together with $q_0 > 0$, $\varepsilon_i \geq 1$ for $i = 1, \dots, k$ ($k \in \mathbb{N}$), and $n_0 \geq 1$.

Consider

$$\Delta \left(\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} \frac{1}{n} (\Delta y(n))^\alpha \right) + \frac{q_0}{n^{\alpha+2}} \sum_{i=1}^k y^\alpha(\sigma_i(n)) = 0, \quad n \geq n_0, \quad (14)$$

where $\sigma_i(n) = \lceil \varepsilon_i n \rceil$. Then,

$$r(n) = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} \frac{1}{n}, \quad q_i(n) = \frac{q_0}{k n^{\alpha+2}} \quad (i = 1, \dots, k), \quad F(u) = u^\alpha \text{ (so } K = 1),$$

and (H1)–(H4) and (H3') hold eventually with $\varepsilon = \min_i \varepsilon_i > 1$.

$$\eta(n) = \sum_{s=n_0}^{n-1} \frac{1+\alpha}{\alpha} s^{1/\alpha}$$

By the Euler-Maclaurin summation formula,

$$\eta(n) = \frac{1+\alpha}{\alpha} \sum_{s=n_0}^{n-1} s^{1/\alpha} \sim \frac{1+\alpha}{\alpha} \cdot \frac{n^{1+1/\alpha}}{1+1/\alpha} = (1+\alpha)^{1/\alpha} n^{1+1/\alpha} \quad (n \rightarrow \infty), \quad (15)$$

which recovers the continuous formula *asymptotically*. In particular $\eta(n) \rightarrow \infty$, confirming the canonical condition (2). Figure 2 confirms (15) numerically the exact partial sum $\eta(n)$ and its asymptotic estimate $(1 + \alpha)^{1/\alpha} n^{1+1/\alpha}$ become indistinguishable once n exceeds a few dozen, for each of $\alpha = 0.6, 1.0, 3.0$. (This asymptotic verification of $\eta(n)$ is a general fact about $r(n)$ and does not depend on the $\alpha \geq 1$ restriction used elsewhere in the paper, so the $\alpha = 0.6$ curve here remains valid.)

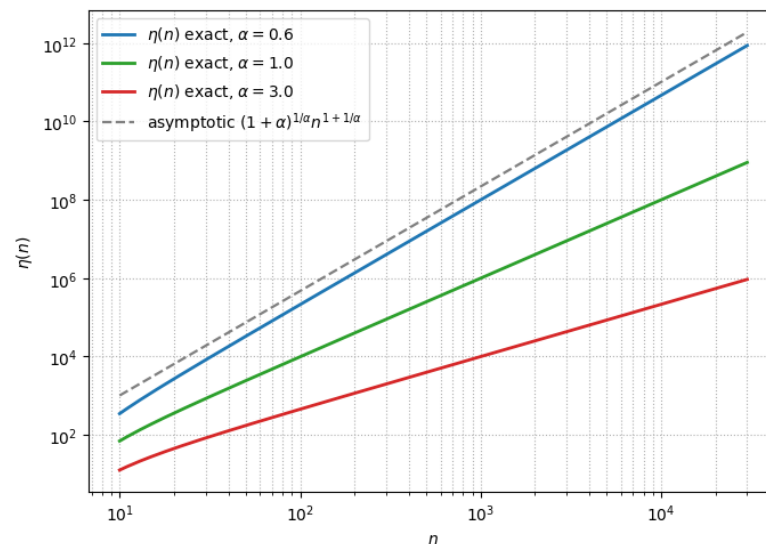


Figure 2. Exact discrete partial sum $\eta(n) = \sum_{s=n_0}^{n-1} r^{-1/\alpha}(s)$ versus the asymptotic estimate $(1 + \alpha)^{1/\alpha} n^{1+1/\alpha}$ from (15), on a log-log scale, for $n_0 = 5$.

Verification of condition (7) and the threshold sequence $\{\mu_j\}$. Using $\sigma(n) = \lceil \varepsilon n \rceil$ and (15), $\eta(\sigma(s))/\eta(s) \rightarrow \varepsilon^{1+1/\alpha}$ as $s \rightarrow \infty$, and the tail sum obeys the standard p -series asymptotic $\sum_{s=n}^{\infty} s^{-(\alpha+2)} \sim n^{-(\alpha+1)} / (\alpha + 1)$. Substituting into the left-hand side of (7) and simplifying gives the limit, as $n \rightarrow \infty$,

$$\alpha \eta(n) \left(\sum_{s=n}^{\infty} \left(\frac{\eta(\sigma(s))}{\eta(s)} \right)^{\mu_{j-1}} \frac{q_0}{s^{\alpha+2}} \right)^{1/\alpha} \longrightarrow \alpha q_0^{1/\alpha} \varepsilon^{(\alpha+1)\mu_{j-1}/\alpha^2} =: L_j(\mu_{j-1}). \quad (16)$$

Remark 5. Formula (16) contains an extra factor of α and an extra division by α inside the exponent, relative to the final display in the corresponding computation of [33] [Example 1], we rederived (16) independently by computer algebra and confirmed it numerically (Figure 3), and report it here without forcing agreement with the published intermediate line, in the interest of a mathematically verified discrete example.

Iterating $\mu_j = L_j(\mu_{j-1})$ with $\mu_0 = 0$ and comparing against the universal constant $\gamma = \alpha / (1 + \alpha)^{(1+\alpha)/\alpha}$ of Theorem 3 produces Table 1. Since Theorem 3 is proved only for $\alpha \geq 1$ (via Lemma 4, which relies on Lemma 3), all four cases below use $\alpha \geq 1$; Cases 3 and 4 reproduce the original computation exactly, and Cases 1 and 2 replace two former $\alpha = 0.6$ cases that fell outside the scope of the proved theorems.

By Theorem 3, every solution of (14) oscillates in each of the four cases of Table 1. Figure 3 shows, for each case, the finite- n quantity $\alpha \eta(n) \left(\sum_{s=n}^{\infty} q(s) \right)^{1/\alpha}$ (i.e., (16) with $\mu_{j-1} = 0$) converging to the limiting constant $L_1(0) = \alpha q_0^{1/\alpha}$ as $n \rightarrow \infty$, computed from the exact discrete sums, confirming that the threshold computation in Table 1 is not an artifact of the asymptotic approximation alone. [Figure 3 to be regenerated with the new Case 1 ($\alpha = 1.0, q_0 = 0.28, \varepsilon = 1.9$) and Case 2 ($\alpha = 1.0, q_0 = 0.20, \varepsilon = 1.9$) curves in place of the former $\alpha = 0.6$ runs Case 3 and Case 4 curves are unchanged.]

Table 1. Threshold sequence μ_j for four parameter choices in Example 1, and the smallest j for which (13) is satisfied.

Case	1	2	3	4
α	1.0	1.0	1.0	3.0
q_0	0.28	0.20	0.3	0.0035
ε	1.9	1.9	2.0	1.2
$\gamma = \alpha/(1 + \alpha)^{(1+\alpha)/\alpha}$	0.2500	0.2500	0.2500	0.4725
smallest j with $\mu_j > \gamma$	1	2	1	2
μ_j at that j	0.2800	0.2585	0.3000	0.4726

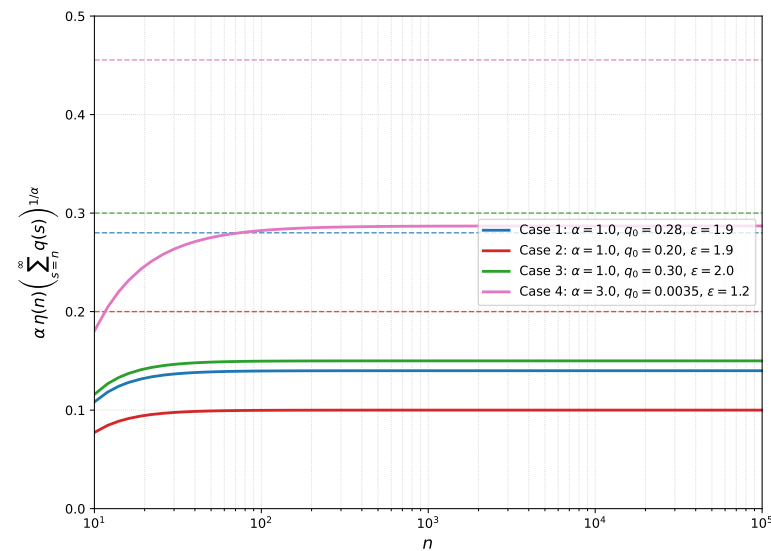


Figure 3. Exact finite- n evaluation of $\alpha \eta(n) \left(\sum_{s=n}^{\infty} q(s) \right)^{1/\alpha}$ for the four cases of Table 1, converging to the theoretical limit $L_1(0) = \alpha q_0^{1/\alpha}$ as $n \rightarrow \infty$.

Remark 6. Unlike delay or ordinary difference equations, Equation (1) cannot be marched forward from a finite set of initial values $y(n_0), \dots$, because the defining recursion at index n already involves the future values $y(\sigma_i(n))$ with $\sigma_i(n) > n$: the equation is not causal in the forward-simulation sense. This is exactly the same obstruction that prevents a naive numerical integration of the continuous advanced equation studied in [33]. Consequently, we verify the oscillation criteria analytically and asymptotically (Table 1, Figures 2 and 3) rather than by forward time-stepping a simulated trajectory, in contrast to delay-type discrete conversions, where a forward simulation is meaningful.

6. Conclusions

In this paper, we established new oscillation criteria for a class of advanced second-order nonlinear difference equations with multiple deviating arguments by developing the analysis entirely within the discrete framework using the forward difference operator and summation. The results obtained provide sufficient conditions for the oscillation of all solutions and retain the universal threshold constant, which arises from an underlying algebraic inequality and is independent of the continuous or discrete setting.

The discrete framework exhibits two distinctive features that are absent from the corresponding continuous model. First, the function $\eta(n) = \sum r^{-1/\alpha}(s)$ generally does not admit a closed-form expression and therefore requires an Euler–Maclaurin asymptotic approximation. Second, the discrete Riccati transformation introduces an additional correction factor $\lambda(n) = \left(\frac{\eta(n)}{\eta(n+1)} \right)^{\mu_j} \in (0, 1)$, which naturally arises from the index shift ($n \mapsto n + 1$)

inherent in the forward difference operator. Although this factor approaches 1 as $n \rightarrow \infty$, its presence reflects a fundamental structural feature of the discrete oscillation theory.

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