

Fast Decoding of Lifted Interleaved Linearized Reed–Solomon Codes for Multishot Network Coding

Hannes Bartz^{1*} and Sven Puchinger²

^{1*} Institute of Communications and Navigation,
German Aerospace Center (DLR), D-82234, Oberpfaffenhofen, Germany.

² Hensoldt Sensors GmbH,
D-89077, Ulm, Germany.

*Corresponding author(s). E-mail(s): hannes.bartz@dlr.de;
Contributing authors: sven.puchinger@tum.de;

Abstract

Martínez-Peñas and Kschischang (IEEE Trans. Inf. Theory, 2019) proposed lifted linearized Reed–Solomon codes as suitable codes for error control in multishot network coding. We show how to construct and decode lifted interleaved linearized Reed–Solomon (LILRS) codes. Compared to the construction by Martínez-Peñas–Kschischang, interleaving allows to increase the decoding region significantly and decreases the overhead due to the lifting (i.e., increases the code rate), at the cost of an increased packet size. We propose two decoding schemes for LILRS that are both capable of correcting insertions and deletions beyond half the minimum distance of the code by either allowing a list or a small decoding failure probability. We propose a probabilistic unique Loidreau–Overbeck-like decoder for LILRS codes and an efficient interpolation-based decoding scheme that can be either used as a list decoder (with exponential worst-case list size) or as a probabilistic unique decoder. We derive upper bounds on the decoding failure probability of the probabilistic-unique decoders which show that the decoding failure probability is very small for most channel realizations up to the maximal decoding radius. The tightness of the bounds is verified by Monte Carlo simulations.

Keywords: Multishot network coding, subspace codes, sum-subspace metric, multishot operator channel, interleaving

MSC Classification: 94B35 , 94B05

1 Introduction

Network coding [1] is a powerful approach to achieve the capacity of multicast networks. Unlike the classical routing schemes, network coding allows to mix (e.g. linearly combine) incoming packets at intermediate nodes. Kötter and Kschischang proposed codes in the subspace metric as a suitable tool for error correction in (random) linear network coding [2] and defined the corresponding operator channel model. In an operator channel two type of errors, namely *insertions* and *deletions*, can occur. Insertions correspond to additional dimensions that are not contained in the transmitted space whereas deletions correspond to dimensions that are removed from the transmitted space.

The single-shot scenario from [2] was extended to the multishot case, i.e. the transmission over several instances of the operator channel, in [3]. It was shown in [4] that lifted linearized Reed–Solomon (LLRS) codes provide reliable and secure coding schemes for noncoherent multishot network coding, a scenario where the in-network linear combinations are not known or used at the receiver, under an adversarial model in the sum-subspace metric.

An s -interleaved code is a direct sum of s codes of the same length (called constituent codes). This means that if the constituent codes are over \mathbb{F}_q , then the interleaved code can be viewed as a (not necessarily linear) code over \mathbb{F}_{q^s} . In the Hamming and rank metric, there are various decoders that can significantly increase the decoding radius of a constituent code by collaboratively decoding in an interleaved variant thereof. Such decoders are known in the Hamming metric for Reed–Solomon [5–17] and in general algebraic geometry codes [18–20], and in the rank metric for Gabidulin codes [21–28]. All of these decoders have in common that they are either list decoders with exponential worst-case and small average-case list size, or probabilistic unique decoders that fail with a very small probability.

Interleaving was suggested in [29] as a method to decrease the overhead in lifted Gabidulin codes for error correction in noncoherent (single-shot) network coding, at the cost of a larger packet size while preserving a low decoding complexity. It was later shown [23, 25, 30] that it can also increase the error-correction capability of the code using suitable decoders for interleaved Gabidulin codes.

1.1 Our Contributions

In this paper, we define and analyze LILRS codes that are obtained by lifting interleaved linearized Reed–Solomon (ILRS) codes as considered in [31]. We propose and analyze two decoding schemes that both allow for decoding insertions and deletions beyond the unique decoding region by allowing a (potentially exponential) list or a small decoding failure probability.

First, we propose a Loidreau–Overbeck-like probabilistic unique decoder and derive an upper bound on the decoding failure probability. The upper bound shows the the decoder succeeds with an overwhelming probability close to one for random realizations of the multishot operator channel that stay within the decoding region.

The second decoding approach is a novel interpolation-based list decoder that is based on the list decoder by Wachter-Zeh and Zeh [25] for interleaved Gabidulin

codes. We derive a decoding region for the codes in the sum-subspace metric, analyze the complexity of the decoder and give an exponential upper bound on the list size. We derive an upper bound on the decoding failure probability of the interpretation as a probabilistic-unique decoder by relating the success conditions to the Loidreau–Overbeck-like decoder.

Compared to [4], we decrease the relative overhead introduced by lifting (or equivalently, increase the rate for the same code length and block size) and at the same time extend the decoding region, especially for insertions, significantly. These advantages come at the cost of a larger packet size of the packets within the network and a supposedly small failure probability. By considering the decoding problem in the complementary code, we show how the proposed LILRS coding schemes can be used to improve the decoding region w.r.t. *deletions* significantly.

Moreover, for the case $s = 1$ (no interleaving), our algorithm does not require the assumption from [4, Sec. V.H] that $n_r \leq n_t$, where n_t and n_r denotes the sum of the dimensions of the transmitted and received subspaces, respectively. Hence, the proposed decoder works in cases in which [4] does not work.

Compared to the conference version [32], which mainly considers interpolation-based decoding of LILRS codes, this work contains several new results. Apart from providing proofs that were omitted in [32] due to space restrictions, we propose and analyze a new Loidreau–Overbeck-like decoder for LILRS codes that allows for deriving a strict upper bound on the decoding failure probability. Compared to the heuristic upper bound provided in [32] the new upper bound takes into account the distribution of insertions in the multishot operator channel. Another novelty compared to the conference version is the definition and analysis of complementary LILRS codes which allows for an improved deletion-correction capability in the interleaved setting. We also provide tools that allow for an efficient implementation of the multishot operator channel that were also not considered in [32].

The main results of this paper, in particular the improvements upon the existing noninterleaved variants, are illustrated in Table 1.

1.2 Structure of the paper:

In Section 2 the notation as well as basic definitions on vector spaces and skew polynomials are introduced. Section 3 is dedicated to a brief introduction to multishot network coding. In Section 4 we consider decoding of LILRS codes in the sum-subspace metric for error-control in noncoherent multishot network coding. In particular, we derive a Loidreau–Overbeck-like and an interpolation-based decoding scheme for LILRS codes, that can correct errors beyond the unique decoding radius in the sum-subspace metric. Section 5 concludes the paper.

2 Preliminaries

We now give some definitions and notation related to multishot network coding and LLRS codes. Since the notation for multishot network coding is quite involved, we tried to stick to common notation as much as possible such that readers familiar with the topic can skip parts of this section.

Table 1 Overview of new decoding regions and computational complexities. Parameters: interleaving parameter s (usually $s \ll n_t$), n_t resp. n_r is the dimension of the transmitted resp. received subspace, γ and δ the overall number of insertions resp. deletions. $\mathcal{M}(n) \in O(n^{1.635})$ is the cost (in operations in \mathbb{F}_{q^m}) of multiplying two skew-polynomials of degree at most n and $\omega < 2.373$ is the matrix multiplication exponent.

Code Class / (Decoder)	Decoding Region	List Size $ \mathcal{L} $ Failure Prob. P_f	Complexity (over \mathbb{F}_{q^m})	Reference(s)
LLRS Codes ^a (unique)	$\gamma + \delta < n_t - k + 1$	—	$O(n_r^2)$	[4]
LILRS Codes (list)	$\gamma + s\delta < s(n_t - k + 1)$	$ \mathcal{L} \leq q^{m(k(s-1))}$	$\tilde{O}(s^\omega \mathcal{M}(n_r))$	Thm. 5 Sec. 4.3.3
LILRS Codes (prob. unique)	$\gamma + s\delta \leq s(n_t - k)$	$P_f \leq 3.5^{\ell+1} q^{-m}$	$\tilde{O}(s^\omega \mathcal{M}(n_r))$	Thms. 3 & 6 Sec. 4.2 / 4.3

^aThe decoder from [4] has the restriction that $n_t = n_r$. Therefore, the proposed decoder for $s = 1$ is a more general decoder for LLRS codes compared to the decoder in [4].

2.1 Notation

A *set* is a collection of distinct elements and is denoted by $\mathcal{S} = \{s_1, s_2, \dots, s_r\}$. The cardinality of \mathcal{S} , i.e. the number of elements in \mathcal{S} , is denoted by $|\mathcal{S}|$. By $[i, j]$ with $i < j$ we denote the set of integers $\{i, i+1, \dots, j\}$. We denote the set of nonnegative integers by $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$.

Let \mathbb{F}_q be a finite field of order q and denote by \mathbb{F}_{q^m} the extension field of \mathbb{F}_q of degree m with primitive element α . The multiplicative group $\mathbb{F}_{q^m} \setminus \{0\}$ of \mathbb{F}_{q^m} is denoted by $\mathbb{F}_{q^m}^*$. Matrices and vectors are denoted by bold uppercase and lowercase letters like \mathbf{A} and \mathbf{a} , respectively, and indexed starting from one. Under a fixed basis of \mathbb{F}_{q^m} over \mathbb{F}_q any element $a \in \mathbb{F}_{q^m}$ can be represented by a corresponding column vector $\mathbf{a} \in \mathbb{F}_q^{m \times 1}$. For a matrix $\mathbf{A} \in \mathbb{F}_{q^m}^{M \times N}$ the \mathbb{F}_q -rank $\text{rk}_q(\mathbf{A})$ of \mathbf{A} is defined as the maximum number of \mathbb{F}_q -linearly independent columns of \mathbf{A} . Let $\sigma : \mathbb{F}_{q^m} \rightarrow \mathbb{F}_{q^m}$ be a finite field automorphism given by $\sigma(a) = a^{q^r}$ for all $a \in \mathbb{F}_{q^m}$, where we assume that $1 \leq r \leq m$ and $\text{gcd}(r, m) = 1$. For a matrix \mathbf{A} and a vector \mathbf{a} we use the notation $\sigma(\mathbf{A})$ and $\sigma(\mathbf{a})$ to denote the element-wise application of the automorphism σ , respectively. For $\mathbf{A} \in \mathbb{F}_{q^m}^{M \times N}$ we denote by $\langle \mathbf{A} \rangle_q$ the \mathbb{F}_q -linear rowspace of the matrix $\mathbf{A}_q \in \mathbb{F}_q^{M \times Nm}$ obtained by row-wise expanding the elements in \mathbf{A} over \mathbb{F}_q . The *left* and *right* kernel of a matrix $\mathbf{A} \in \mathbb{F}_{q^m}^{M \times N}$ is denoted by $\ker_l(\mathbf{A})$ and $\ker_r(\mathbf{A})$, respectively.

For a set $\mathcal{I} \subset \mathbb{Z}_{>0}$ we denote by $[\mathbf{A}]_{\mathcal{I}}$ (respectively $[\mathbf{a}]_{\mathcal{I}}$) the matrix (vector) consisting of the columns (entries) of the matrix \mathbf{A} (vector \mathbf{a}) indexed by \mathcal{I} .

2.2 Vector Spaces & Subspace Metrics

Vector spaces are denoted by calligraphic letters such as e.g. \mathcal{V} . By $\mathcal{P}_q(N)$ we denote the set of all subspaces of \mathbb{F}_q^N . The *Grassmannian* is the set of all l -dimensional subspaces in $\mathcal{P}_q(N)$ and is denoted by $\mathcal{G}_q(N, l)$. The cardinality of the Grassmannian $|\mathcal{G}_q(N, l)|$, i.e. the number of l -dimensional subspace of \mathbb{F}_q^N is given by the Gaussian

binomial $\begin{bmatrix} N \\ l \end{bmatrix}_q$ which is defined as

$$\begin{bmatrix} N \\ l \end{bmatrix}_q = \prod_{i=1}^l \frac{q^{N-l+i} - 1}{q^i - 1}.$$

The Gaussian binomial satisfies (see e.g. [2])

$$q^{(N-l)l} \leq \begin{bmatrix} N \\ l \end{bmatrix}_q \leq \kappa_q q^{(N-l)l},$$

where $\kappa_q := \prod_{i=1}^{\infty} (1 - q^{-i})^{-1}$. Note that κ_q is monotonically decreasing in q with a limit of 1, and e.g. $\kappa_2 \approx 3.463$, $\kappa_3 \approx 1.785$, and $\kappa_4 \approx 1.452$.

The *subspace distance* between two subspaces $\mathcal{U}, \mathcal{V} \in \mathcal{P}_q(N_i)$ is defined as (see [2])

$$d_S(\mathcal{U}, \mathcal{V}) := \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}).$$

For $\mathbf{N} = (N_1, N_2, \dots, N_\ell)$ and $\mathbf{l} = (l_1, l_2, \dots, l_\ell)$ we define the ℓ -fold Cartesian products

$$\mathcal{P}_q(\mathbf{N}) := \prod_{i=1}^{\ell} \mathcal{P}_q(N_i) = \mathcal{P}_q(N_1) \times \dots \times \mathcal{P}_q(N_\ell)$$

and

$$\mathcal{G}_q(\mathbf{N}, \mathbf{l}) := \prod_{i=1}^{\ell} \mathcal{G}_q(N_i, l_i) = \mathcal{G}_q(N_1, l_1) \times \dots \times \mathcal{G}_q(N_\ell, l_\ell).$$

For any tuple of subspaces $\mathcal{V} \in \mathcal{P}_q(\mathbf{N})$ we define its *sum-dimension* as

$$\dim_{\Sigma}(\mathcal{V}) := \sum_{i=1}^{\ell} \dim(\mathcal{V}^{(i)})$$

and call the tuple

$$\left(\dim(\mathcal{V}^{(1)}), \dim(\mathcal{V}^{(2)}), \dots, \dim(\mathcal{V}^{(\ell)}) \right)$$

the *sum-dimension partition* of \mathcal{V} .

We extend fundamental operators on subspaces to tuples of subspaces by considering their application in an element-wise manner, i.e. for $\mathcal{V}, \mathcal{U} \in \mathcal{P}_q(\mathbf{N}, \mathbf{l})$ we define

$$\begin{aligned} \mathcal{V}^{\perp} &:= \left(\mathcal{V}^{(1)\perp}, \mathcal{V}^{(2)\perp}, \dots, \mathcal{V}^{(\ell)\perp} \right), \\ \mathcal{V} \cap \mathcal{U} &:= \left(\mathcal{V}^{(1)} \cap \mathcal{U}^{(1)}, \mathcal{V}^{(2)} \cap \mathcal{U}^{(2)}, \dots, \mathcal{V}^{(\ell)} \cap \mathcal{U}^{(\ell)} \right), \\ \mathcal{V} \oplus \mathcal{U} &:= \left(\mathcal{V}^{(1)} \oplus \mathcal{U}^{(1)}, \mathcal{V}^{(2)} \oplus \mathcal{U}^{(2)}, \dots, \mathcal{V}^{(\ell)} \oplus \mathcal{U}^{(\ell)} \right). \end{aligned}$$

We conclude this subsection considering the extension of the subspace distance to the multishot case [3].

Definition 1 (Sum-Subspace Distance [3]). Given $\mathbf{U} = (\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \dots, \mathcal{U}^{(\ell)}) \in \mathcal{P}_q(\mathbf{N})$ and $\mathbf{V} = (\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \dots, \mathcal{V}^{(\ell)}) \in \mathcal{P}_q(\mathbf{N})$ the sum-subspace distance between \mathbf{U} and \mathbf{V} is defined as

$$d_{\Sigma S}(\mathbf{U}, \mathbf{V}) := \sum_{i=1}^{\ell} d_S(\mathcal{U}^{(i)}, \mathcal{V}^{(i)}) = \sum_{i=1}^{\ell} \left(\dim(\mathcal{U}^{(i)} + \mathcal{V}^{(i)}) - \dim(\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}) \right). \quad (1)$$

In [3, 33] the sum-subspace distance is also called the *extended* subspace distance. Note, that there exists other metrics for multishot network coding like e.g. the sum-injection distance (see [4, Definition 7]) which are related to the sum-subspace distance and are not considered in this work.

2.3 Skew Polynomials

Skew polynomials are a special class of non-commutative polynomials that were introduced by Ore [34]. A *skew polynomial* is a polynomial of the form

$$f(x) = \sum_i f_i x^i$$

with a finite number of coefficients $f_i \in \mathbb{F}_{q^m}$ being nonzero. The degree $\deg(f)$ of a skew polynomial f is defined as $\max\{i : f_i \neq 0\}$ if $f \neq 0$ and $-\infty$ otherwise.

The set of skew polynomials with coefficients in \mathbb{F}_{q^m} together with ordinary polynomial addition and the multiplication rule

$$xa = \sigma(a)x, \quad a \in \mathbb{F}_{q^m}$$

forms a non-commutative ring denoted by $\mathbb{F}_{q^m}[x; \sigma]$.

The set of skew polynomials in $\mathbb{F}_{q^m}[x; \sigma]$ of degree less than k is denoted by $\mathbb{F}_{q^m}[x; \sigma]_{<k}$. For any $a, b \in \mathbb{F}_{q^m}$ we define the operator

$$\mathcal{D}_a(b) := \sigma(b)a.$$

For an integer $i \geq 0$, we define (see [35, Proposition 32])

$$\mathcal{D}_a^{i+1}(b) = \mathcal{D}_a(\mathcal{D}_a^i(b)) = \sigma^{i+1}(b)\mathcal{N}_{i+1}(a)$$

where $\mathcal{D}_a^0(b) = b$ and $\mathcal{N}_i(a) = \sigma^{i-1}(a)\sigma^{i-2}(a) \dots \sigma(a)a$ is the generalized power function (see [36]). The *generalized operator evaluation* of a skew polynomial $f \in \mathbb{F}_{q^m}[x; \sigma]$ at an element b w.r.t. a , where $a, b \in \mathbb{F}_{q^m}$, is defined as (see [35, 37])

$$f(b)_a = \sum_i f_i \mathcal{D}_a^i(b).$$

For any fixed evaluation parameter $a \in \mathbb{F}_{q^m}$ the generalized operator evaluation forms an \mathbb{F}_q -linear map, i.e. for any $f \in \mathbb{F}_{q^m}[x; \sigma]$, $\beta, \gamma \in \mathbb{F}_q$ and $b, c \in \mathbb{F}_{q^m}$ we have that

$$f(\beta b + \gamma c)_a = \beta f(b)_a + \gamma f(c)_a.$$

For two skew polynomials $f, g \in \mathbb{F}_{q^m}[x; \sigma]$ and elements $a, b \in \mathbb{F}_{q^m}$ the generalized operator evaluation of the product $f \cdot g$ at b w.r.t a is given by (see [38])

$$(f \cdot g)(b)_a = f(g(b)_a).$$

An important notion for the generalized operator evaluation is the concept of conjugacy, which is defined as follows.

Definition 2 (Conjugacy [39]). *For any two elements $a \in \mathbb{F}_{q^m}$ and $c \in \mathbb{F}_{q^m}^*$ define*

$$a^c := \sigma(c)ac^{-1}.$$

- Two elements $a, b \in \mathbb{F}_{q^m}$ are called σ -conjugates, if there exists an element $c \in \mathbb{F}_{q^m}^*$ such that $b = a^c$.
- Two elements that are not σ -conjugates are called σ -distinct.

The notion of σ -conjugacy defines an equivalence relation on \mathbb{F}_{q^m} and thus a partition of \mathbb{F}_{q^m} into conjugacy classes [36]. The set

$$\mathcal{C}(a) := \{a^c : c \in \mathbb{F}_{q^m}^*\}$$

is called *conjugacy class* of a . A finite field \mathbb{F}_{q^m} has at most $\ell \leq q - 1$ distinct conjugacy classes. For $\ell \leq q - 1$ the elements $1, \alpha, \alpha^2, \dots, \alpha^{\ell-2}$ are representatives of all (nontrivial) disjoint conjugacy classes of \mathbb{F}_{q^m} .

Let a_1, \dots, a_ℓ be representatives from different conjugacy classes of \mathbb{F}_{q^m} and let $b_1^{(i)}, \dots, b_{n_i}^{(i)}$ be elements from \mathbb{F}_{q^m} for all $i = 1, \dots, \ell$. Then for any nonzero $f \in \mathbb{F}_{q^m}[x; \sigma]$ satisfying

$$f(b_j^{(i)})_{a_i} = 0, \quad \forall i = 1, \dots, \ell, j = 1, \dots, n_i$$

we have that $\deg(f) \leq \sum_{i=1}^{\ell} n_i$ where equality holds if and only if the $b_1^{(i)}, \dots, b_{n_i}^{(i)}$ are \mathbb{F}_q -linearly independent for each $i = 1, \dots, \ell$ (i.e. for each evaluation parameter a_i , see [40]).

The existence of a (generalized operator evaluation) interpolation polynomial is considered in Lemma 1 (see e.g. [40]).

Lemma 1 (Lagrange Interpolation (Generalized Operator Evaluation)). *Let $b_1^{(i)}, \dots, b_{n_i}^{(i)}$ be \mathbb{F}_q -linearly independent elements from \mathbb{F}_{q^m} for all $i = 1, \dots, \ell$. Let $c_1^{(i)}, \dots, c_{n_i}^{(i)}$ be elements from \mathbb{F}_{q^m} and let a_1, \dots, a_ℓ be representatives for different conjugacy classes of \mathbb{F}_{q^m} . Define the set of tuples $\mathcal{B} := \{(b_j^{(i)}, c_j^{(i)}, a_i) : i = 1, \dots, \ell, j = 1, \dots, n_i\}$. Then there exists a unique interpolation polynomial $\mathcal{I}_{\mathcal{B}}^{\text{op}} \in \mathbb{F}_{q^m}[x; \sigma]$ such that*

$$\mathcal{I}_{\mathcal{B}}^{\text{op}}(b_j^{(i)})_{a_i} = c_j^{(i)}, \quad \forall i = 1, \dots, \ell, \forall j = 1, \dots, n_i,$$

and $\deg(\mathcal{I}_{\mathcal{B}}^{\text{op}}) < \sum_{i=1}^{\ell} n_i$.

For an element $a \in \mathbb{F}_{q^m}$, a vector $\mathbf{b} \in \mathbb{F}_{q^m}^n$ and a skew polynomial $f \in \mathbb{F}_{q^m}[x; \sigma]$ we define

$$f(\mathbf{b})_a := (f(b_1)_a, f(b_2)_a, \dots, f(b_n)_a).$$

The set of all skew polynomials of the form

$$Q(x, y_1, \dots, y_s) = Q_0(x) + Q_1(x)y_1 + \dots + Q_s(x)y_s,$$

where $Q_j \in \mathbb{F}_{q^m}[x; \sigma]$ for all $j = 0, \dots, s$ is denoted by $\mathbb{F}_{q^m}[x, y_1, \dots, y_s; \sigma]$.

Definition 3 (*w*-weighted Degree). *Given a vector $\mathbf{w} \in \mathbb{Z}_{\geq 0}^{s+1}$, the \mathbf{w} -weighted degree of a multivariate skew polynomial $Q \in \mathbb{F}_{q^m}[x, y_1, \dots, y_s; \sigma]$ is defined as*

$$\deg_{\mathbf{w}}(Q) = \max_j \{\deg(Q_j) + w_j\}.$$

For an element $a \in \mathbb{F}_{q^m}$ and a vector $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{F}_{q^m}^n$ we define the vector

$$\mathcal{D}_a^j(\mathbf{b}) := (\mathcal{D}_a^j(b_1), \mathcal{D}_a^j(b_2), \dots, \mathcal{D}_a^j(b_n))$$

and the matrix

$$\mathbf{V}_d(\mathbf{b})_a := \begin{pmatrix} \mathbf{b} \\ \mathcal{D}_a^1(\mathbf{b}) \\ \mathcal{D}_a^2(\mathbf{b}) \\ \vdots \\ \mathcal{D}_a^{d-1}(\mathbf{b}) \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ \mathcal{D}_a^1(b_1) & \mathcal{D}_a^1(b_2) & \dots & \mathcal{D}_a^1(b_n) \\ \mathcal{D}_a^2(b_1) & \mathcal{D}_a^2(b_2) & \dots & \mathcal{D}_a^2(b_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_a^{d-1}(b_1) & \mathcal{D}_a^{d-1}(b_2) & \dots & \mathcal{D}_a^{d-1}(b_n) \end{pmatrix} \in \mathbb{F}_{q^m}^{d \times n}.$$

For a vector $\mathbf{x} = (\mathbf{x}^{(1)} \mid \mathbf{x}^{(2)} \mid \dots \mid \mathbf{x}^{(\ell)}) \in \mathbb{F}_{q^m}^n$ with $\mathbf{x}^{(i)} \in \mathbb{F}_{q^m}^{n_i}$ for all $i = 1, \dots, \ell$, a length partition $\mathbf{n} = (n_1, n_2, \dots, n_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ such that $\sum_{i=1}^\ell n_i = n$ and a vector $\mathbf{a} = (a_1, a_2, \dots, a_\ell) \in \mathbb{F}_{q^m}^\ell$ we define the vector¹

$$\mathcal{D}_a^i(\mathbf{x}) := (\mathcal{D}_{a_1}^i(\mathbf{x}^{(1)}) \mid \mathcal{D}_{a_2}^i(\mathbf{x}^{(2)}) \mid \dots \mid \mathcal{D}_{a_\ell}^i(\mathbf{x}^{(\ell)})) \in \mathbb{F}_{q^m}^n.$$

By the properties of the operator $\mathcal{D}_a^i(\cdot)$, we have that

$$\mathcal{D}_a^{i+j}(\mathbf{x}) = \mathcal{D}_a^j(\mathcal{D}_a^i(\mathbf{x}))$$

and

$$\mathcal{D}_a^i(\xi \mathbf{x}) = \sigma^i(\xi) \mathcal{D}_a^i(\mathbf{x}) \quad \forall \xi \in \mathbb{F}_{q^m}.$$

For a matrix

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_d \end{pmatrix} \in \mathbb{F}_{q^m}^{d \times n},$$

¹To simplify the notation we omit the length partition \mathbf{n} from the vector operator $\mathcal{D}_a^i(\mathbf{x})$ since it will be always clear from the context (i.e. as the length partition of the vector \mathbf{x}).

and integer j and a vector $\mathbf{a} = (a_1, a_2, \dots, a_\ell) \in \mathbb{F}_{q^m}^\ell$ we define $\mathcal{D}_{\mathbf{a}}^j(\cdot)$ applied to \mathbf{X} as

$$\mathcal{D}_{\mathbf{a}}^j(\mathbf{X}) := \begin{pmatrix} \mathcal{D}_{\mathbf{a}}^j(\mathbf{x}_1) \\ \mathcal{D}_{\mathbf{a}}^j(\mathbf{x}_2) \\ \vdots \\ \mathcal{D}_{\mathbf{a}}^j(\mathbf{x}_d) \end{pmatrix}.$$

The element-wise application of the operator to matrices does not affect the rank, i.e. we have that $\text{rk}_{q^m}(\mathcal{D}_{\mathbf{a}}^j(\mathbf{X})) = \text{rk}_{q^m}(\mathbf{X})$ (see [31, Lemma 3]).

Definition 4 (σ -Generalized Moore Matrix). *For an integer $d \in \mathbb{Z}_{>0}$, a length partition $\mathbf{n} = (n_1, n_2, \dots, n_\ell) \in \mathbb{Z}_{>0}^\ell$ such that $\sum_{i=1}^\ell n_i = n$ and the vectors $\mathbf{a} = (a_1, a_2, \dots, a_\ell) \in \mathbb{F}_{q^m}^\ell$ and $\mathbf{x} = (\mathbf{x}^{(1)} \mid \mathbf{x}^{(2)} \mid \dots \mid \mathbf{x}^{(\ell)}) \in \mathbb{F}_{q^m}^n$ with $\mathbf{x}^{(i)} \in \mathbb{F}_{q^m}^{n_i}$ for all $i = 1, \dots, \ell$, the σ -Generalized Moore matrix is defined as*

$$\Lambda_d(\mathbf{x})_{\mathbf{a}} := \begin{pmatrix} \mathcal{D}_{\mathbf{a}}^0(\mathbf{x}) \\ \mathcal{D}_{\mathbf{a}}^1(\mathbf{x}) \\ \vdots \\ \mathcal{D}_{\mathbf{a}}^{d-1}(\mathbf{x}) \end{pmatrix} = (\mathbf{V}_d(\mathbf{x}^{(1)})_{a_1} \mid \mathbf{V}_d(\mathbf{x}^{(2)})_{a_2} \mid \dots \mid \mathbf{V}_d(\mathbf{x}^{(\ell)})_{a_\ell}) \in \mathbb{F}_{q^m}^{d \times n}.$$

Similar as for ordinary polynomials and Vandermonde matrices, there is a relation between the generalized operator evaluation and the product with a σ -Generalized Moore matrix. In particular, for a skew polynomial $f(x) = \sum_{i=0}^{k-1} f_i x^i \in \mathbb{F}_{q^m}[x; \sigma]_{<k}$ and vectors $\mathbf{a} = (a_1, a_2, \dots, a_\ell) \in \mathbb{F}_{q^m}^\ell$ and $\mathbf{x} = (\mathbf{x}^{(1)} \mid \mathbf{x}^{(2)} \mid \dots \mid \mathbf{x}^{(\ell)}) \in \mathbb{F}_{q^m}^n$ we have that

$$f(\mathbf{a})_{\mathbf{x}} = (f_0, f_1, \dots, f_{k-1}) \cdot \Lambda_k(\mathbf{x})_{\mathbf{a}}.$$

The rank of a σ -Generalized Moore matrix satisfies $\text{rk}_{q^m}(\Lambda_d(\mathbf{x})_{\mathbf{a}}) = \min\{d, n\}$ if and only if $\text{wt}_{\Sigma_R}(\mathbf{x}) = n$ (see e.g. [36, Theorem 4.5]).

Remark 1. *To simplify the notation we omit the rank partition \mathbf{n} in $\Lambda_j(\cdot)_{\mathbf{a}}$ since it will be always clear from the context (i.e. the length partition of the considered vector).*

3 Multishot Network Coding

As a channel model we consider the *multishot operator channel* from [3] which consists of multiple independent channel uses of the operator channel from [2]. The operator channel is a discrete channel that relates the input $\mathcal{V} \in \mathcal{P}_q(N)$ with $n_t := \dim(\mathcal{V})$ to the output $\mathcal{U} \in \mathcal{P}_q(N)$ by

$$\mathcal{U} = \mathcal{H}_{n_t - \delta}(\mathcal{V}) \oplus \mathcal{E} \quad (2)$$

where $\mathcal{H}_{n_t - \delta}(\mathcal{V})$ is an erasure operator that returns an $(n_t - \delta)$ -dimensional subspace of \mathcal{V} and $\mathcal{E} \in \mathcal{G}_q(N, \gamma)$ is a γ -dimensional subspace with $\mathcal{V} \cap \mathcal{E} = \{\mathbf{0}\}$. The dimension of the received space $n_r := \dim(\mathcal{U})$ is then

$$n_r = n_t - \delta + \gamma$$

where δ is called the number of *deletions* and γ is called the number of *insertions*. Observe that the subspace distance between the input \mathcal{V} and the output \mathcal{U} is $d_S(\mathcal{V}, \mathcal{U}) = \gamma + \delta$.

3.1 Multishot Operator Channel

A multishot (or ℓ -shot) operator channel [3] with overall γ insertions and δ deletions is a discrete channel with input and output alphabet $\mathcal{P}_q(\mathbf{N})$. Consider the partitions of insertions $\boldsymbol{\gamma} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(\ell)})$ and deletions $\boldsymbol{\delta} = (\delta^{(1)}, \delta^{(2)}, \dots, \delta^{(\ell)})$ such that $\gamma = \sum_{i=1}^{\ell} \gamma^{(i)}$ and $\delta = \sum_{i=1}^{\ell} \delta^{(i)}$. The input is a tuple of subspaces $\mathcal{V} \in \mathcal{P}_q(\mathbf{N})$ with sum-dimension $\dim_{\Sigma}(\mathcal{V}) = n_t$ and sum-dimension partition \mathbf{n}_t . The output $\mathcal{U} \in \mathcal{P}_q(\mathbf{N})$ is related to the input $\mathcal{V} \in \mathcal{P}_q(\mathbf{N})$ by

$$\mathcal{U} = \mathcal{H}_{n_t - \delta}(\mathcal{V}) \oplus \mathcal{E}, \quad (3)$$

where $\mathcal{H}_{n_t - \delta}(\mathcal{V})$ returns a tuple with sum-dimension $n_t - \delta$ and sum-dimension partition $\mathbf{n}_t - \boldsymbol{\delta}$ and $\mathcal{E} \in \mathcal{G}_q(\mathbf{N}, \boldsymbol{\gamma})$ is a tuple of error spaces with $\dim_{\Sigma}(\mathcal{E}) = \gamma$ and $\mathcal{V} \cap \mathcal{E} = \{\mathbf{0}\}$. The multishot operator channel can be considered as ℓ instances of the (single-shot) operator channel defined in (2) such that an overall number of γ insertions and δ deletions occurs.

The output \mathcal{U} of the multishot operator channel has sum-dimension

$$n_r = n_t - \delta + \gamma$$

with sum-dimension partition

$$\mathbf{n}_r = \mathbf{n}_t - \boldsymbol{\delta} + \boldsymbol{\gamma}. \quad (4)$$

The multishot operator channel is illustrated in Figure 1.

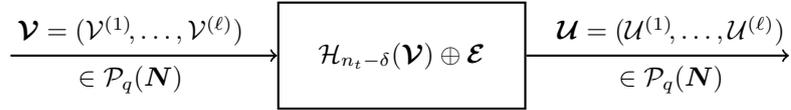


Fig. 1 Illustration of the ℓ -shot operator channel.

In [4, 33] a sum-rank-metric representation of the multishot operator channel in the spirit of [29, 41] was considered. This equivalent channel representation is more suitable for decoding of LILRS codes in the sum-rank metric. In this work, we consider the interpretation as a multishot operator channel that is more closely related to the sum-subspace metric.

Remark 2. By the term “random instance of the ℓ -shot operator channel with overall γ insertions and δ deletions” we mean, that we draw uniformly at random an instance from all instances of the ℓ -shot operator channel (3), i.e. we draw uniformly at random

from all partitions of the insertions γ and deletions δ , and for fixed \mathbf{V} , γ and δ , the error space is chosen uniformly at random from the set

$$\mathcal{E}^{(\gamma)} := \left\{ \mathcal{E} \in \mathcal{G}_q(\mathbf{N}, \gamma) : \mathcal{V}^{(i)} \cap \mathcal{E}^{(i)} = \{\mathbf{0}\}, \forall i \wedge \dim_{\Sigma}(\mathcal{E}) = \gamma \right\}. \quad (5)$$

In Appendix C we propose an efficient procedure to implement random instances of the multishot operator channel for parameters $\bar{\mathbf{N}} = (\bar{N}, \dots, \bar{N})$ and $\bar{\mathbf{n}}_t = (\bar{n}_t, \dots, \bar{n}_t)$ by adapting the dynamic-programming routine in [42, Appendix A] for drawing an error of given sum-rank weight uniformly at random to the sum-subspace case (see Algorithm 5).

We now extend the definition of (γ, δ) reachability for the operator channel [28] to the multishot operator channel (see also [32, Definition 2]).

Definition 5 ((γ, δ) Reachability). *Given two tuples of subspaces $\mathbf{U}, \mathbf{V} \in \mathcal{P}_q(\mathbf{N})$ we say that \mathbf{V} is (γ, δ) -reachable from \mathbf{U} if there exists a realization of the multishot operator channel (3) with γ insertions and δ deletions that transforms the input \mathbf{V} to the output \mathbf{U} .*

Next, we relate the (γ, δ) -reachability with the sum-subspace distance (see also [32, Proposition 2]).

Proposition 1. *Consider $\mathbf{U} \in \mathcal{P}_q(\mathbf{N})$ and $\mathbf{V} \in \mathcal{P}_q(\mathbf{N})$. If \mathbf{V} is (γ, δ) -reachable from \mathbf{U} , then we have that $d_{\Sigma S}(\mathbf{U}, \mathbf{V}) = \gamma + \delta$.*

Similar as in [2] we now define normalized parameters for codes in the sum-subspace metric. The normalized weight λ , the code rate R and the normalized minimum distance η of a sum-subspace code \mathcal{C} with parameters $N = \sum_{i=1}^{\ell} N_i$ and $n_t = \sum_{i=1}^{\ell} n_t^{(i)}$ is defined as

$$\lambda := \sum_{i=1}^{\ell} \frac{n_t^{(i)}}{N} = \frac{n_t}{N}, R := \frac{\log_q(\mathcal{C})}{\sum_{i=1}^{\ell} n_t^{(i)} N_i} \quad \text{and} \quad \eta := \frac{d_{\Sigma S}(\mathcal{C})}{2n_t} = \frac{d_{\Sigma S}(\mathcal{C})}{2\lambda N}, \quad (6)$$

respectively. The normalized parameters λ , R and η defined in (6) lie naturally within the interval $[0, 1]$. Define $\bar{n}_t := n_t/\ell$. For $n_t^{(i)} = \bar{n}_t$ for all $i = 1, \dots, \ell$ we can write the code rate as

$$R = \frac{\log_q(|\mathcal{C}|)}{\sum_{i=1}^{\ell} \bar{n}_t N_i} = \frac{\ell \log_q(|\mathcal{C}|)}{n_t N} = \frac{\ell \log_q(|\mathcal{C}|)}{\lambda N^2}.$$

A *sum-subspace* code \mathcal{C} is a non-empty subset of $\mathcal{P}_q(\mathbf{N})$, and has minimum subspace distance $d_{\Sigma S}(\mathcal{C})$ when all subspaces in the code have distance at least $d_{\Sigma S}(\mathcal{C})$ and there is at least one pair of subspaces with distance exactly $d_{\Sigma S}(\mathcal{C})$. In the following we consider *constant-shot-dimension* codes², i.e. codes that inject the same number of (linearly independent) packets $n_t^{(i)}$ in a given shot. In this setup, we transmit a tuple of subspaces

$$\mathbf{V} = \left(\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \dots, \mathcal{V}^{(\ell)} \right) \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_t)$$

²In [4] these codes are called *sum-constant-dimension* codes.

and receive a tuple of subspaces

$$\mathbf{u} = (\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \dots, \mathcal{U}^{(\ell)}) \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r),$$

where

$$\mathcal{U}^{(i)} = \left\langle \left(\xi^{(i)\top} \mathbf{u}_1^{(i)\top} \mathbf{u}_2^{(i)\top} \dots \mathbf{u}_s^{(i)\top} \right) \right\rangle_q = \left\langle \left(\left(\begin{array}{cccc} \xi_1^{(i)} & \mathbf{u}_{1,1}^{(i)} & \mathbf{u}_{1,2}^{(i)} & \dots & \mathbf{u}_{1,s}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{n_r}^{(i)} & \mathbf{u}_{n_r,1}^{(i)} & \mathbf{u}_{n_r,2}^{(i)} & \dots & \mathbf{u}_{n_r,s}^{(i)} \end{array} \right) \right) \right\rangle_q \quad (7)$$

for all $i = 1, \dots, \ell$.

Similar as for subspace codes in [2] we now define *complementary* sum-subspace codes. For any $\mathcal{V} \in \mathcal{P}_q(N)$ the *dual* space \mathcal{V}^\perp is defined as

$$\mathcal{V}^\perp := \{\mathbf{u} \in \mathbb{F}_q^N : \mathbf{u}\mathbf{v}^\top = 0, \forall \mathbf{v} \in \mathcal{V}\}$$

where $\dim(\mathcal{V}^\perp) = N - \dim(\mathcal{V})$. For a tuple $\mathbf{v} = (\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \dots, \mathcal{V}^{(\ell)}) \in \mathcal{P}_q(\mathbf{N})$ we define the dual tuple as

$$\mathbf{v}^\perp := \left((\mathcal{V}^{(1)})^\perp, (\mathcal{V}^{(2)})^\perp, \dots, (\mathcal{V}^{(\ell)})^\perp \right) \in \mathcal{P}_q(\mathbf{N}).$$

Note, that if $\mathbf{v} \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_t)$, then we have that $\mathbf{v}^\perp \in \mathcal{G}_q(\mathbf{N}, \mathbf{N} - \mathbf{n}_t)$. By applying [2, Equation 4] to the subspace distance between each component space of two tuples $\mathbf{u}, \mathbf{v} \in \mathcal{P}_q(\mathbf{N})$ in (1) we get that

$$d_{\Sigma S}(\mathbf{v}^\perp, \mathbf{u}^\perp) = d_{\Sigma S}(\mathbf{v}, \mathbf{u}).$$

Consider a constant-shot-dimension sum-subspace code $\mathcal{C} \subseteq \mathcal{G}_q(\mathbf{N}, \mathbf{n}_t)$. Then the complementary constant-shot-dimension sum-subspace code \mathcal{C}^\perp is defined as

$$\mathcal{C}^\perp := \{\mathbf{v}^\perp : \mathbf{v} \in \mathcal{C}\} \subseteq \mathcal{G}_q(\mathbf{N}, \mathbf{N} - \mathbf{n}_t).$$

The complementary code \mathcal{C}^\perp has cardinality $|\mathcal{C}^\perp| = |\mathcal{C}|$, minimum sum-subspace distance $d_{\Sigma S}(\mathcal{C}^\perp) = d_{\Sigma S}(\mathcal{C})$ and code rate

$$R^\perp = \frac{\log_q(|\mathcal{C}^\perp|)}{\sum_{i=1}^{\ell} (N_i - n_t^{(i)}) N_i}.$$

3.2 Lifted Linearized Reed–Solomon Codes

Lifted linearized Reed–Solomon (LRS) codes [4] are constant-shot-dimension multi-shot network codes for error-control in noncoherent multishot network coding. The main idea behind the construction of LLRS codes is to lift codewords of an LRS code

in a block-wise manner by augmenting each (transposed) codeword block by the corresponding \mathbb{F}_q -linearly independent code locators. For the special case of $\ell = 1$ the construction coincides with the Kötter–Kschischang subspace codes [2].

Let $\mathbf{a} = (a_1, a_2, \dots, a_\ell)$ be a vector containing representatives from different conjugacy classes of \mathbb{F}_{q^m} . Let the vectors $\boldsymbol{\beta}^{(i)} = (\beta_1^{(i)}, \beta_2^{(i)}, \dots, \beta_{n_t^{(i)}}^{(i)}) \in \mathbb{F}_{q^m}^{n_t^{(i)}}$ contain \mathbb{F}_q -linearly independent elements from \mathbb{F}_{q^m} for all $i = 1, \dots, \ell$ and define $\boldsymbol{\beta} = (\boldsymbol{\beta}^{(1)} \mid \boldsymbol{\beta}^{(2)} \mid \dots \mid \boldsymbol{\beta}^{(\ell)}) \in \mathbb{F}_{q^m}^{n_t}$ and $\mathbf{n}_t = (n_t^{(1)}, n_t^{(2)}, \dots, n_t^{(\ell)})$. Then an LLRS code $\text{LLRS}[\boldsymbol{\beta}, \mathbf{a}, \ell; \mathbf{n}_t, k]$ of sum-subspace dimension $n_t = n_t^{(1)} + n_t^{(2)} + \dots + n_t^{(\ell)}$, sum-dimension partition \mathbf{n}_t and dimension $k \leq n_t$ is defined as

$$\left\{ \mathcal{V}(f) := \left(\mathcal{V}^{(1)}(f), \dots, \mathcal{V}^{(\ell)}(f) \right) : f \in \mathbb{F}_{q^m}[x; \sigma]_{<k} \right\} \subseteq \mathcal{G}_q(\mathbf{N}, \mathbf{n}_t)$$

where $\mathbf{N} = (N_1, \dots, N_\ell)$ with $N_i = n_t^{(i)} + m$, and, for $f \in \mathbb{F}_{q^m}[x; \sigma]_{<k}$, we have

$$\mathcal{V}^{(i)}(f) := \left\langle (\boldsymbol{\beta}^{(i)\top} f (\boldsymbol{\beta}^{(i)})_{a_i}^\top) \right\rangle_q \in \mathcal{G}_q(N_i, n_t^{(i)}), \quad \forall i = 1, \dots, \ell.$$

The *lifting* operation corresponds to augmenting each transposed codeword block of an LRS codeword by the corresponding (transposed) code locators and considering the \mathbb{F}_q -linear rowspace thereof (see [4, 29, 33]). The lifting operation causes a rate-loss since the code locators do not carry information since they are common for all codewords.

The minimum sum-subspace distance of $\text{LLRS}[\boldsymbol{\beta}, \mathbf{a}, \ell; \mathbf{n}_t, k]$ equals (see [4])

$$d_{\Sigma S}(\text{LLRS}[\boldsymbol{\beta}, \mathbf{a}, \ell; \mathbf{n}_t, k]) = 2(n_t - k + 1) \quad (8)$$

and the code rate is

$$R = \frac{mk}{\sum_{i=1}^{\ell} n_t^{(i)} (n_t^{(i)} + m)}.$$

In [4] an efficient interpolation-based decoding algorithm that can correct an overall number of γ insertions and δ deletions up to

$$\gamma + \delta < n_t - k + 1$$

was presented. However, the decoder from [4] has the restriction that the dimension of the received spaces and the dimension of the transmitted spaces must be the same (c.f. [4, Section V.H]).

4 Decoding of Lifted Interleaved LRS Codes for Error-Control in Multishot Network Coding

In this section, we consider the application of *lifted* ILRS codes for error control in multishot network coding. In particular, we focus on noncoherent transmissions, where the

network topology and/or the coefficients of the in-network linear combinations at the intermediate nodes are not known (or used) at the transmitter and the receiver. Therefore, we define and analyze lifted interleaved linearized Reed–Solomon (LILRS) codes. We derive a Loidreau–Overbeck-like decoder [21, 43, 44] for LILRS codes which is capable of correcting insertions and deletions beyond the unique decoding region at the cost of a (very) small decoding failure probability. Although the Loidreau–Overbeck-like decoder is not the most efficient decoder in terms of computational complexity, it allows to analyze the decoding failure probability and gives insights about the decoding procedure. We derive a tight upper bound on the decoding failure probability of the Loidreau–Overbeck-like decoder for LILRS codes, which, unlike simple heuristic bounds, considers the distribution of the error spaces caused by insertions.

We propose an efficient interpolation-based decoding scheme, which can correct insertions and deletions beyond the unique decoding region and **which can be used as** a list decoder or as a probabilistic unique decoder. We derive upper bounds on the worst-case list size and use the relation between the interpolation-based decoder and the Loidreau–Overbeck-like decoder to derive an upper bound on the decoding failure probability for the interpolation-based probabilistic unique decoding approach. Unlike the interpolation-based decoder in [4, Section V.H], the proposed decoding schemes *do not* have the restriction that the dimension of the received spaces and the dimension of the transmitted spaces must be the same.

4.1 Lifted Interleaved Linearized Reed–Solomon Codes

In this section we consider LILRS codes for transmission over a multishot operator channel (3). We generalize the ideas from [4] to obtain multishot subspace codes by *lifting* the ILRS codes defined in [31] (see also [32]).

Definition 6 (Lifted Interleaved Linearized Reed–Solomon Code). *Let $\mathbf{a} = (a_1, a_2, \dots, a_\ell)$ be a vector containing representatives from different conjugacy classes of \mathbb{F}_{q^m} . Let the vectors $\boldsymbol{\beta}^{(i)} = (\beta_1^{(i)}, \beta_2^{(i)}, \dots, \beta_{n_t^{(i)}}^{(i)}) \in \mathbb{F}_{q^m}^{n_t^{(i)}}$ contain \mathbb{F}_q -linearly independent elements from \mathbb{F}_{q^m} for all $i = 1, \dots, \ell$ and define $\boldsymbol{\beta} = (\boldsymbol{\beta}^{(1)} \mid \boldsymbol{\beta}^{(2)} \mid \dots \mid \boldsymbol{\beta}^{(\ell)}) \in \mathbb{F}_{q^m}^{n_t}$ and $\mathbf{n}_t = (n_t^{(1)}, n_t^{(2)}, \dots, n_t^{(\ell)})$. A **homogeneous** lifted s -interleaved linearized Reed–Solomon (LILRS) code $\text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ of sum-subspace dimension $n_t = n_t^{(1)} + n_t^{(2)} + \dots + n_t^{(\ell)}$ and dimension $k \leq n_t$ is defined as*

$$\left\{ \mathcal{V}(\mathbf{f}) := \left(\mathcal{V}^{(1)}(\mathbf{f}), \dots, \mathcal{V}^{(\ell)}(\mathbf{f}) \right) : \mathbf{f} \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s \right\} \subseteq \mathcal{G}_q(\mathbf{N}, \mathbf{n}_t)$$

where $\mathbf{N} = (N_1, \dots, N_\ell)$ with $N_i = n_t^{(i)} + sm$, and, for $\mathbf{f} = (f_1, \dots, f_s)$, we have

$$\mathcal{V}^{(i)}(\mathbf{f}) := \left\langle (\boldsymbol{\beta}^{(i)\top} \ f_1(\boldsymbol{\beta}^{(i)})_{a_i}^\top \ \dots \ f_s(\boldsymbol{\beta}^{(i)})_{a_i}^\top) \right\rangle_q \in \mathcal{G}_q(N_i, n_t^{(i)}).$$

Remark 3. *In order to not further complicate the notation, we consider homogeneous LILRS codes, i.e. codes where each component code has the same code dimension k , only. Note that the proposed results and concepts can be adapted to inhomogeneous*

codes with different code dimension k_1, \dots, k_s in a straight forward manner (analog to single-shot subspace codes in e.g. [30, 45, 46]).

Observe, that compared to LLRS codes the relative overhead due to lifting decreases in s since the evaluations are performed at the same code locators and thus have to be appended only once. The reduction of the relative overhead comes at the cost of an increased packet size N_i for each shot $i = 1, \dots, \ell$.

The definition of LILRS codes generalizes several code families. For $s = 1$ we obtain the lifted linearized Reed–Solomon codes from [4, Section V.III]. For $\ell = 1$ we obtain lifted interleaved Gabidulin codes as considered in e.g. [25, 30] with Kötter–Kschischang codes [2] as special case for $s = 1$.

Proposition 2 shows that interleaving does not increase the minimum sum-subspace distance of the code.

Proposition 2 (Minimum Distance). *The minimum sum-subspace distance of a LILRS code $\text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ as in Definition 6 is*

$$d_{\Sigma S}(\text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]) = 2(n_t - k + 1).$$

Proof. Consider two distinct codewords

$$\mathbf{v}_1 := \mathbf{V}(\mathbf{f}_1), \mathbf{v}_2 := \mathbf{V}(\mathbf{f}_2) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$$

constructed by the polynomial vectors $\mathbf{f}_1 = (f_1, 0, \dots, 0)$ and $\mathbf{f}_2 = (f_2, 0, \dots, 0)$ with $f_1 \neq f_2$. Then the corresponding tuples of subspaces are $\mathbf{v}_j = (\mathcal{V}_j^{(1)}, \dots, \mathcal{V}_j^{(\ell)})$ with

$$\mathcal{V}_j^{(i)} := \langle (\boldsymbol{\beta}^{(i)\top} \ f_j(\boldsymbol{\beta}^{(i)})_{a_i}^\top \ \mathbf{0} \ \dots \ \mathbf{0}) \rangle_q \quad (9)$$

for $j = 1, 2$. Observe that the $\ell - 1$ rightmost zero matrices in the component spaces in (9) do not contribute to the sum-subspace distance between \mathbf{v}_1 and \mathbf{v}_2 . Therefore we have that

$$d_{\Sigma S}(\mathbf{v}_1, \mathbf{v}_2) = d_{\Sigma S}(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2)$$

where $\tilde{\mathbf{v}}_j = (\tilde{\mathcal{V}}_j^{(1)}, \dots, \tilde{\mathcal{V}}_j^{(\ell)})$ with

$$\tilde{\mathcal{V}}_j^{(i)} := \langle (\boldsymbol{\beta}^{(i)\top} \ f_j(\boldsymbol{\beta}^{(i)})_{a_i}^\top) \rangle_q \quad (10)$$

for $j = 1, 2$. Since $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2 \in \text{LLRS}[\boldsymbol{\beta}, \mathbf{a}, \ell; \mathbf{n}_t, k]$ we have that (see (8))

$$d_{\Sigma S}(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2) \geq 2(n_t - k + 1).$$

Now assume that there exist two codewords $\mathbf{v}'_1, \mathbf{v}'_2 \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ with component spaces of the form (9) such that $d_{\Sigma S}(\mathbf{v}'_1, \mathbf{v}'_2) < 2(n_t - k + 1)$. This would imply that the corresponding (truncated) spaces $\tilde{\mathbf{v}}'_1, \tilde{\mathbf{v}}'_2$ of the form (10) which are codewords in $\text{LLRS}[\boldsymbol{\beta}, \mathbf{a}, \ell; \mathbf{n}_t, k]$ also satisfy $d_{\Sigma S}(\tilde{\mathbf{v}}'_1, \tilde{\mathbf{v}}'_2) < 2(n_t - k + 1)$, which

contradicts that LLRS $[\boldsymbol{\beta}, \mathbf{a}, \ell; \mathbf{n}_t, k]$ has minimum sum-subspace distance $2(n_t - k + 1)$. Therefore, we conclude that $d_{\Sigma S}(\text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]) = 2(n_t - k + 1)$. \square

For a LILRS code $\mathcal{C} = \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ we have that $N_i = n_t^{(i)} + sm$ for all $i = 1, \dots, \ell$ and therefore the code rate is

$$R = \frac{\log_q(|\mathcal{C}|)}{\sum_{i=1}^{\ell} n_t^{(i)} N_i} = \frac{smk}{\sum_{i=1}^{\ell} n_t^{(i)} (n_t^{(i)} + sm)}. \quad (11)$$

Note, that there exist other definitions of the code rate for multishot codes that are not considered in this paper (see e.g. [3, Section IV.A]).

The normalized weight λ and the normalized distance η of an LILRS code $\text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ is

$$\lambda = \frac{n_t}{N} \quad \text{and} \quad \eta = \frac{n_t - k + 1}{n_t}.$$

For $n_t^{(i)} = \bar{n}_t$ for all $i = 1, \dots, \ell$ the code rate of an LILRS code $\text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ in (11) becomes

$$R = \frac{\ell smk}{n_t N}.$$

For $n_t^{(i)} = \bar{n}_t$ for all $i = 1, \dots, \ell$ the Singleton-like upper bound for constant-shot-dimension sum-subspace codes [4, Theorem 7] evaluated for the parameters of LILRS codes becomes

$$|\mathcal{C}| \leq \kappa_q^\ell q^{sm(n_t - (d_{\Sigma S}(\mathcal{C})/2 - 1))} = \kappa_q^\ell q^{smk}$$

which shows, that a Singleton-like bound achieving code can be at most $\kappa_q^\ell < 3.5^\ell$ times larger than the corresponding LILRS code. Equivalently, the code rate is therefore upper-bounded by

$$R \leq \frac{\ell(\log_q(\kappa_q) + smk)}{n_t N}.$$

The benefit of the decreased relative overhead due to interleaving is illustrated in Figure 2. The figure shows, that the rate loss due to the overhead introduced by the lifting is reduced significantly, even for small interleaving orders. Further, we see that LILRS codes approach the Singleton-like bound for sum-subspace codes (see [4]) with increasing interleaving order while preserving the extension field degree m^3 .

4.2 Loidreau–Overbeck-like Decoder for LILRS Codes

Loidreau and Overbeck proposed the first efficient decoder for interleaved Gabidulin codes in the rank metric [21, 43, 44]. The main idea behind the Loidreau–Overbeck decoder is to compute an \mathbb{F}_q -linear transformation matrix from a decoding matrix (which depends on the code and the received word) that allows to transform the received word into a *corrupted* part and a *noncorrupted* part. The noncorrupted part is then used to recover the message polynomials e.g. via Lagrange interpolation.

³The Singleton-like depends on the sum-dimension N of the ambient space and thus changes in s .

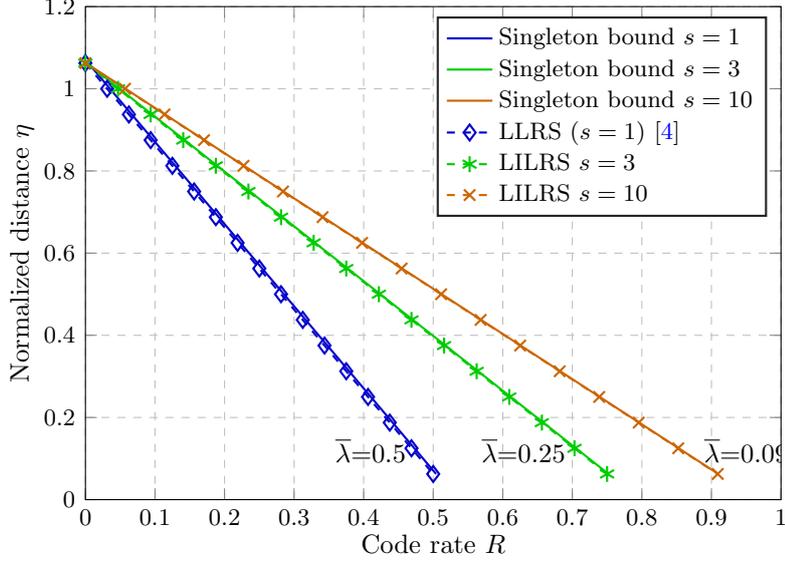


Fig. 2 Normalized distance η over the code rate R for an LILRS code $\text{LILRS}[\beta, \mathbf{a}, \ell = 2, s; \mathbf{n}_t = (8, 8), k = 4]$ over \mathbb{F}_{3^8} for interleaving orders $s \in \{1, 3, 10\}$ with the corresponding (average) normalized weight $\bar{\lambda}$. The case $s = 1$ corresponds to the LLRS codes from [4].

The concept of the Loidreau–Overbeck decoder was generalized to decoding ILRS codes in the sum-rank metric [31]. In the sum-rank-metric case an \mathbb{F}_q -linear transformation matrix is obtained for each block.

Based on the previous decoders for the rank and sum-rank metric, we derive a Loidreau–Overbeck-like decoder for LILRS codes. Similar to the original decoder and its sum-rank-metric analogue we set up a decoding matrix that allows to compute \mathbb{F}_q -linear transformation matrices for each shot $\mathcal{U}^{(i)}$. The obtained transformation matrices allow to compute particular bases for the received subspaces $\mathcal{U}^{(i)}$ that can be split into a basis for the *corrupted* part (corresponding to the error space $\mathcal{E}^{(i)}$) and a *noncorrupted* part (i.e. a basis for $\mathcal{V}^{(i)} \cap \mathcal{U}^{(i)}$) for each shot. The noncorrupted part is then used to reconstruct the message polynomials via Lagrange interpolation. The qualitative structure of the tuple $\hat{\mathbf{U}} = (\hat{\mathbf{U}}^{(1)}, \hat{\mathbf{U}}^{(2)}, \dots, \hat{\mathbf{U}}^{(\ell)})$ containing transformed basis matrices is illustrated in Figure 3.

$$\hat{\mathbf{U}} = \left(\begin{array}{c} \boxed{\phantom{\text{green}}} \\ \boxed{\phantom{\text{red}}} \end{array}, \begin{array}{c} \boxed{\phantom{\text{green}}} \\ \boxed{\phantom{\text{red}}} \end{array}, \dots, \begin{array}{c} \boxed{\phantom{\text{green}}} \\ \boxed{\phantom{\text{red}}} \end{array} \right)$$

Fig. 3 Qualitative illustration of the structure of the tuple $\hat{\mathbf{U}}$ containing transformed basis matrices. The green parts form a basis for the non-corrupted spaces whereas the red parts indicate a basis for the erroneous spaces. The green part is used to reconstruct the message polynomials.

The main motivation to derive the Loidreau–Overbeck-like decoder is to obtain an upper bound on the decoding failure probability that incorporates the distribution of the error spaces in \mathcal{E} . In Section 4.3 we will reduce the interpolation-based decoder for LILRS codes to the Loidreau–Overbeck-like decoder in order to obtain an upper bound on the decoding failure probability of the interpolation-based probabilistic unique decoder.

Up to our knowledge, this is the first Loidreau–Overbeck-like decoding scheme in the (sum-) subspace metric. It includes lifted interleaved Gabidulin (or interleaved Kötter–Kschischang) codes [30, 46] as special case for $\ell = 1$. Hence, the results give a strict upper bound⁴ on the decoding failure probability of the decoders in [30, 46].

Suppose we transmit the tuple of subspaces

$$\mathcal{V}(\mathbf{f}) = \left(\mathcal{V}^{(1)}(\mathbf{f}), \dots, \mathcal{V}^{(\ell)}(\mathbf{f}) \right) \in \text{LILRS}[\beta, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$$

over an ℓ -shot operator channel with overall γ insertions and δ deletions and receive the tuple of subspaces

$$\mathcal{U} = \left(\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(\ell)} \right) \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r),$$

where the received subspaces $\mathcal{U}^{(i)}$ are as defined in (7) and $\mathbf{n}_r = \mathbf{n}_t + \gamma - \delta$ (see (4)). Define the vectors

$$\boldsymbol{\xi} = \left(\boldsymbol{\xi}^{(1)} \mid \boldsymbol{\xi}^{(2)} \mid \dots \mid \boldsymbol{\xi}^{(\ell)} \right) \in \mathbb{F}_{q^m}^{\mathbf{n}_r} \quad \text{and} \quad \mathbf{u}_j = \left(\mathbf{u}_j^{(1)} \mid \mathbf{u}_j^{(2)} \mid \dots \mid \mathbf{u}_j^{(\ell)} \right) \in \mathbb{F}_{q^m}^{\mathbf{n}_r},$$

for all $j = 1, \dots, s$ and consider the matrix

$$\mathbf{L} := \begin{pmatrix} \Lambda_{n_t - \delta - 1}(\boldsymbol{\xi})_{\mathbf{a}} \\ \Lambda_{n_t - \delta - k}(\mathbf{u}_1)_{\mathbf{a}} \\ \vdots \\ \Lambda_{n_t - \delta - k}(\mathbf{u}_s)_{\mathbf{a}} \end{pmatrix} \in \mathbb{F}_{q^m}^{((s+1)(n_t - \delta) - sk - 1) \times \mathbf{n}_r}. \quad (12)$$

Lemma 2 (Transformed of Decoding Matrix). *Consider the transmission of a tuple of subspaces $\mathcal{V}(\mathbf{f}) \in \text{LILRS}[\beta, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ over an ℓ -shot operator channel with overall γ insertions and δ deletions and receive the tuple of subspaces \mathcal{U} . Let \mathbf{L} be as in (12). Then there exist invertible matrices $\mathbf{W}^{(i)} \in \mathbb{F}_q^{\mathbf{n}_r^{(i)} \times \mathbf{n}_r^{(i)}}$ such that for $\mathbf{W} = \text{diag}(\mathbf{W}^{(1)}, \mathbf{W}^{(2)}, \dots, \mathbf{W}^{(\ell)})$ we have that*

$$\bar{\mathbf{L}} = \left(\bar{\mathbf{L}}^{(1)}, \dots, \bar{\mathbf{L}}^{(\ell)} \right) = \mathbf{L} \cdot \mathbf{W} \in \mathbb{F}_{q^m}^{((s+1)(n_t - \delta) - sk - 1) \times \mathbf{n}_r} \quad (13)$$

⁴In [30, 46] a heuristic upper bound on the decoding failure probability, which does *not* incorporate the distribution of the error spaces, was derived.

consists of component matrices of the form

$$\bar{\mathbf{L}}^{(i)} = \begin{pmatrix} \mathbf{V}_{n_t-\delta-1}(\bar{\boldsymbol{\xi}}_1^{(i)})_{a_i} & \mathbf{V}_{n_t-\delta-1}(\bar{\boldsymbol{\xi}}_2^{(i)})_{a_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{n_t-\delta-k}(\tilde{\boldsymbol{e}}_1^{(i)})_{a_i} & \mathbf{V}_{n_t-\delta-k}(\hat{\boldsymbol{e}}_1^{(i)})_{a_i} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{V}_{n_t-\delta-k}(\tilde{\boldsymbol{e}}_s^{(i)})_{a_i} & \mathbf{V}_{n_t-\delta-k}(\hat{\boldsymbol{e}}_s^{(i)})_{a_i} \end{pmatrix} \in \mathbb{F}_{q^m}^{((s+1)(n_t-\delta)-sk-1) \times n_r^{(i)}}$$

where $\bar{\boldsymbol{\xi}}_1^{(i)} \in \mathbb{F}_{q^m}^{n_t^{(i)}-\delta^{(i)}}$, $\bar{\boldsymbol{\xi}}_2^{(i)}, \tilde{\boldsymbol{e}}_l^{(i)} \in \mathbb{F}_{q^m}^{t^{(i)}}$, $\hat{\boldsymbol{e}}_l^{(i)} \in \mathbb{F}_{q^m}^{\varkappa^{(i)}}$ and

$$\begin{aligned} \text{rk}_q(\bar{\boldsymbol{\xi}}_1^{(i)}) &= n_t^{(i)} - \delta^{(i)}, & \text{rk}_q(\bar{\boldsymbol{\xi}}_2^{(i)}) &= t^{(i)} \\ \text{rk}_q(\tilde{\boldsymbol{e}}_l^{(i)}) &= t^{(i)}, & \text{rk}_q(\hat{\boldsymbol{e}}_l^{(i)}) &= \varkappa^{(i)} \\ \text{rk}_q((\tilde{\boldsymbol{e}}_l^{(i)} \mid \hat{\boldsymbol{e}}_l^{(i)})) &= t^{(i)} + \varkappa^{(i)}, \end{aligned}$$

for all $i = 1, \dots, \ell$ such that $\text{rk}_{q^m}(\mathbf{L}) = \text{rk}_{q^m}(\bar{\mathbf{L}})$.

The proof of Lemma 2 is based on particular bases for the received spaces in \mathbf{U} and properties of the intersection and error spaces in $\mathbf{V} \cap \mathbf{U}$ and \mathbf{E} , respectively, and can be found in Appendix A.1.

Lemma 3 (Properties of Decoding Matrix). *Consider the notation and definitions as in Lemma 2 and define the vectors*

$$\bar{\boldsymbol{e}}_l := \left((\tilde{\boldsymbol{e}}_l^{(1)} \mid \hat{\boldsymbol{e}}_l^{(1)}) \mid \dots \mid (\tilde{\boldsymbol{e}}_l^{(\ell)} \mid \hat{\boldsymbol{e}}_l^{(\ell)}) \right) \in \mathbb{F}_{q^m}^\gamma, \quad \forall l = 1, \dots, s,$$

and the matrix

$$\bar{\mathbf{Z}} = \left(\bar{\mathbf{Z}}^{(1)} \mid \dots \mid \bar{\mathbf{Z}}^{(\ell)} \right) := \begin{pmatrix} \Lambda_{n_t-\delta-k}(\bar{\boldsymbol{e}}_1)_{\mathbf{a}} \\ \vdots \\ \Lambda_{n_t-\delta-k}(\bar{\boldsymbol{e}}_s)_{\mathbf{a}} \end{pmatrix} \in \mathbb{F}_{q^m}^{s(n_t-\delta-k) \times \gamma}. \quad (14)$$

Let $\mathbf{h} = (\mathbf{h}^{(1)} \mid \mathbf{h}^{(2)} \mid \dots \mid \mathbf{h}^{(\ell)}) \in \mathbb{F}_{q^m}^{n_r}$ with $\mathbf{h}^{(i)} \in \mathbb{F}_{q^m}^{n_r^{(i)}}$ for all $i = 1, \dots, \ell$ be a nonzero vector in the right kernel of the decoding matrix \mathbf{L} and suppose that $\bar{\mathbf{Z}}$ has \mathbb{F}_{q^m} -rank γ . Then:

1. We have $\text{rk}_{q^m}(\mathbf{L}) = n_r - 1$.
2. We have $\text{rk}_q(\mathbf{h}^{(i)}) = n_t^{(i)} - \delta^{(i)}$ for all $i = 1, \dots, \ell$, i.e., \mathbf{h} has sum-rank weight $\text{wt}_{\Sigma R}^{(n_r)}(\mathbf{h}) = n_t - \delta$.
3. There are invertible matrices $\mathbf{T}^{(i)} \in \mathbb{F}_{q^m}^{n_r^{(i)} \times n_r^{(i)}}$, for all $i = 1, \dots, \ell$, such that the last (rightmost) $\gamma^{(i)}$ positions of $\mathbf{h}^{(i)} \mathbf{T}^{(i)}$ are zero.
4. The first (upper) $n_t^{(i)} - \delta^{(i)}$ rows of $\hat{\mathbf{U}}^{(i)} = \left(\mathbf{T}^{(i)} \right)^{-1} \mathbf{U}^{(i)}$ form a basis for the non-corrupted received space $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$ for all $i = 1, \dots, \ell$.

5. The l -th message polynomial f_l can be uniquely reconstructed from the transformed basis $\hat{U}^{(i)}$ for the received space $\mathcal{U}^{(i)}$ by Lagrange interpolation on the first $n_t^{(i)} - \delta^{(i)}$ rows of $\hat{U}^{(i)}$ for all $l = 1, \dots, s$ and $i = 1, \dots, \ell$.

We now provide a sketch of the proof. The full proof of Lemma 3 can be found in Appendix A.2.

Sketch of the Proof.

- Ad 1): The matrix $\bar{\mathbf{L}}$ can be rearranged into an upper block-triangular matrix whose rank is determined by the two blocks on the diagonal, which have rank $n_t - \delta - 1$ and γ and thus imply that the \mathbb{F}_{q^m} -rank of the whole matrix equals $n_t - \delta - 1 + \gamma = n_r - 1$. The statement follows since by Lemma 2 we have that $\text{rk}_{q^m}(\mathbf{L}) = \text{rk}_{q^m}(\bar{\mathbf{L}})$.
- Ad 2): By assumption the \mathbb{F}_{q^m} -rank of $\bar{\mathbf{Z}}$ equals γ which implies that $\text{rk}_{q^m}(\bar{\mathbf{Z}}^{(i)}) = \gamma^{(i)}$ for all $i = 1, \dots, \ell$. Thus, for any $\bar{\mathbf{h}} \in \ker_r(\bar{\mathbf{L}}) \setminus \{\mathbf{0}\}$ the $\gamma^{(i)}$ rightmost entries of $\bar{\mathbf{h}}^{(i)}$ must be zero which implies that $\text{rk}_q(\bar{\mathbf{h}}^{(i)}) \leq n_t^{(i)} - \delta^{(i)}$ for all $i = 1, \dots, \ell$. On the other hand $\bar{\mathbf{h}}$ is contained in a code with minimum sum-rank distance $n_t - \delta$ which is the dual of the code spanned by the first $n_t - \delta - 1$ rows of $\bar{\mathbf{L}}$. The statement follows by combining these two facts.
- Ad 3): By 2) the \mathbb{F}_q -rank of $\mathbf{h}^{(i)} \in \mathbb{F}_{q^m}^{n_r^{(i)}}$ equals $n_t^{(i)} - \delta^{(i)}$ for all $i = 1, \dots, \ell$. Hence there exist matrices $\mathbf{T}^{(i)} \in \mathbb{F}_q^{n_r^{(i)} \times n_r^{(i)}}$ such that the $n_r^{(i)} - (n_t^{(i)} - \delta^{(i)}) = \gamma^{(i)}$ rightmost entries of $\mathbf{h}^{(i)}\mathbf{T}^{(i)}$ are equal to zero.
- Ad 4): Define the matrices $\mathbf{D}^{(i)} = \left(\mathbf{T}^{(i)-1}\right)^\top$ for all $i = 1, \dots, \ell$ and observe that $\mathbf{h}\mathbf{T} \in \ker_r(\mathbf{L} \cdot \text{diag}(\mathbf{D}^{(1)}, \dots, \mathbf{D}^{(\ell)}))$. By using the \mathbb{F}_{q^m} -rank condition on $\bar{\mathbf{Z}}$ one can show that the span of the $\gamma^{(i)}$ rightmost columns of the matrices $\bar{\mathbf{L}}^{(i)}$ and $\mathbf{L}^{(i)}\mathbf{D}^{(i)}$ coincides. These columns correspond to the insertions which in turn implies that the last $\gamma^{(i)}$ rows of $\hat{U}^{(i)} = (\mathbf{D}^{(i)})^\top \mathbf{U}^{(i)}$ form a basis for $\mathcal{E}^{(i)}$. The statement follows since by the definition of the operator channel we have that $\mathcal{V}^{(i)} \cap \mathcal{E}^{(i)} = \{\mathbf{0}\}$ for all $i = 1, \dots, \ell$.
- Ad 5): By 4) the first $n_t^{(i)} - \delta^{(i)}$ rows of the transformed basis $\hat{U}^{(i)}$ form a basis for the noncorrupted intersection space $\mathcal{V}^{(i)} \cap \mathcal{U}^{(i)}$ for all $i = 1, \dots, \ell$. Due to the \mathbb{F}_q -linearity of the generalized operator evaluation for a fixed evaluation parameter (i.e. per shot), the message polynomials can be reconstructed by constructing the corresponding Lagrange interpolation polynomials (see Figure 4).

□

The complete procedure for the Loidreau–Overbeck-like decoder for LILRS codes is given in Algorithm 1. The structure of the transformed basis matrices $\hat{U}^{(i)}$ for all $i = 1, \dots, \ell$ is illustrated in Figure 4.

An execution of the Loidreau–Overbeck-like decoder for LILRS codes is illustrated in Example 1.

Example 1 (Loidreau–Overbeck-like Decoder). *Consider the finite field \mathbb{F}_{32} with primitive element α defined by the primitive polynomial $x^2 + 2x + 2$. Consider the LILRS code $\text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ over \mathbb{F}_{32} with code locators $\boldsymbol{\beta} = ((1, \alpha) \mid (1, \alpha))$, evaluation parameters $\mathbf{a} = (1, \alpha)$, number of shots $\ell = 2$, interleaving order $s = 2$,*

Algorithm 1 LOIDREAU–OVERBECK-LIKE DECODER FOR LILRS CODES

Input: A tuple containing the basis matrices $\mathbf{U} = (\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(\ell)}) \in \prod_{i=1}^{\ell} \mathbb{F}_{q^n}^{n_r^{(i)} \times (s+1)}$ for the output $\mathbf{U} = (\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \dots, \mathcal{U}^{(\ell)}) \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r)$ of an ℓ -shot operator channel with overall γ insertions and δ deletions for input $\mathcal{V}(\mathbf{f}) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$

Output: Message polynomial vector $\mathbf{f} = (f_1, \dots, f_s) \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$ or “decoding failure”

```

1: Set up the matrix  $\mathbf{L}$  as in (12)
2: Compute right kernel  $\mathcal{H} = \ker_r(\mathbf{L})$ 
3: if  $\dim(\mathcal{H}) > 1$  then
4:   return “decoding failure”
5: else
6:   Compute an element  $\mathbf{h} = (\mathbf{h}^{(1)} \mid \dots \mid \mathbf{h}^{(\ell)}) \in \mathcal{H} \setminus \{\mathbf{0}\}$ 
7:   for  $i = 1, \dots, \ell$  do
8:     Compute  $n_t^{(i)} - \delta^{(i)} \leftarrow \text{rk}_q(\mathbf{h}^{(i)})$ 
9:     Compute full-rank matrix  $\mathbf{T}^{(i)} \in \mathbb{F}_q^{n_r^{(i)} \times n_r^{(i)}}$  s.t.
       the rightmost  $\gamma^{(i)} = n_r^{(i)} - (n_t^{(i)} - \delta^{(i)})$  entries of  $\mathbf{h}^{(i)}\mathbf{T}^{(i)}$  are zero.
10:     $\hat{\mathbf{U}}^{(i)} \leftarrow (\mathbf{T}^{(i)})^{-1} \mathbf{U}^{(i)} = (\hat{\boldsymbol{\xi}}^{(i)\top} \ \hat{\mathbf{u}}_1^{(i)\top} \ \dots \ \hat{\mathbf{u}}_s^{(i)\top}) \in \mathbb{F}_{q^m}^{n_r^{(i)} \times (s+1)}$ 
11:    end for
12:    for  $l = 1, \dots, s$  do
13:       $f_l \leftarrow \mathcal{I}_{\mathcal{B}_l}^{\text{op}}$  where  $\mathcal{B}_l = \{(\hat{\xi}_\mu^{(i)}, \hat{u}_{l,\mu}^{(i)}, a_i) : i = 1, \dots, \ell, \mu = 1, \dots, n_t^{(i)} - \delta^{(i)}\}$ 
14:    end for
15:    return  $\mathbf{f} = (f_1, \dots, f_s)$ 
16: end if

```

sum-dimension partition $\mathbf{n}_t = (2, 2)$ and dimension $k = 2$. Consider the transmission of a codeword $\mathcal{V}(\mathbf{f}) = (\mathcal{V}^{(1)}(\mathbf{f}), \mathcal{V}^{(2)}(\mathbf{f})) \in \text{ILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}, k]$ with

$$\mathcal{V}^{(1)}(\mathbf{f}) = \left\langle \begin{pmatrix} 1 & \alpha + 2 & 2 \\ \alpha & 2\alpha + 2 & 2 \end{pmatrix} \right\rangle_q \quad \text{and} \quad \mathcal{V}^{(2)}(\mathbf{f}) = \left\langle \begin{pmatrix} 1 & \alpha & 1 \\ \alpha & \alpha & \alpha + 1 \end{pmatrix} \right\rangle_q \quad (15)$$

that corresponds to the message polynomial vector $\mathbf{f} = (f_1, f_2)$ with

$$f_1 = \alpha x + 2 \quad \text{and} \quad f_2 = 2\alpha x + \alpha + 2$$

over a multishot operator channel (3) with an overall number of $\gamma = 2$ insertions and $\delta = 1$ deletion. Note, that a bounded minimum distance (BMD) decoder could only correct $\gamma = 1$ insertions and $\delta = 1$ deletion.

$$\hat{\mathbf{U}}^{(i)} = (\mathbf{T}^{(i)})^{-1} \mathbf{U}^{(i)} = \begin{pmatrix} \hat{\xi}_1^{(i)} & f_1(\hat{\xi}_1^{(i)})_{a_i} & \cdots & f_s(\hat{\xi}_1^{(i)})_{a_i} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\xi}_{n_t^{(i)} - \delta^{(i)}}^{(i)} & f_1(\hat{\xi}_{n_t^{(i)} - \delta^{(i)}}^{(i)})_{a_i} & \cdots & f_s(\hat{\xi}_{n_t^{(i)} - \delta^{(i)}}^{(i)})_{a_i} \\ \hat{\xi}_{n_t^{(i)} - \delta^{(i)} + 1}^{(i)} & \hat{u}_{1, n_t^{(i)} - \delta^{(i)} + 1}^{(i)} & \cdots & \hat{u}_{s, n_t^{(i)} - \delta^{(i)} + 1}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\xi}_{n_r^{(i)}}^{(i)} & \hat{u}_{1, n_r^{(i)}}^{(i)} & \cdots & \hat{u}_{s, n_r^{(i)}}^{(i)} \end{pmatrix} \begin{matrix} \uparrow \\ n_t^{(i)} - \delta^{(i)} \\ \uparrow \\ \gamma^{(i)} \end{matrix}$$

Fig. 4 Illustration of the structure of the transformed basis matrices $\hat{\mathbf{U}}^{(i)}$. The green part forms a basis for the non-corrupted space $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$ whereas the red part forms a basis for the error space $\mathcal{E}^{(i)}$.

Suppose we receive the tuple of subspaces $\mathcal{U} = (\mathcal{U}^{(1)}, \mathcal{U}^{(2)})$ with

$$\mathcal{U}^{(1)} = \left\langle \left(\begin{array}{ccc} \alpha + 2 & \alpha & 0 \\ 1 & 1 & 1 \\ 2\alpha + 2 & 2 & 2 \end{array} \right) \right\rangle_q \quad \text{and} \quad \mathcal{U}^{(2)} = \left\langle \left(\begin{array}{ccc} 2\alpha + 1 & 0 & 2 \\ 2\alpha + 2 & \alpha & \alpha + 2 \end{array} \right) \right\rangle_q \quad (16)$$

where $\mathbf{n}_r = (3, 2)$ and $n_r = 5$.

According to (12) the Loidreau–Overbeck-like decoding matrix is

$$\mathbf{L} = \left(\begin{array}{ccc|cc} \alpha + 2 & 1 & 2\alpha + 2 & 2\alpha + 1 & 2\alpha + 2 \\ 2\alpha & 1 & \alpha + 1 & \alpha + 1 & 2\alpha + 1 \\ \alpha & 1 & 2 & 0 & \alpha \\ 0 & 1 & 2 & 2 & \alpha + 2 \end{array} \right).$$

The decoding matrix \mathbf{L} has \mathbb{F}_{q^m} -rank $n_r - 1 = 4$ implying the right \mathbb{F}_{3^2} -kernel of \mathbf{L} has dimension one. We pick

$$\mathbf{h} = (\mathbf{h}^{(1)} \mid \mathbf{h}^{(2)}) = ((\alpha, 0, 2) \mid (\alpha + 1, \alpha + 1)) \in \ker_r(\mathbf{L})$$

as nonzero element from the right \mathbb{F}_{q^m} -kernel of \mathbf{L} and recover the partition of insertions γ as

$$\gamma = \left(n_r^{(1)} - \text{rk}_q(\mathbf{h}^{(1)}) \mid n_r^{(2)} - \text{rk}_q(\mathbf{h}^{(2)}) \right) = (1 \mid 1).$$

Next we compute the transformation matrices

$$\mathbf{T}^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{T}^{(2)} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

such that the $\gamma^{(1)} = \gamma^{(2)} = 1$ rightmost entry of $\mathbf{h}^{(1)}\mathbf{T}^{(1)}$ and $\mathbf{h}^{(2)}\mathbf{T}^{(2)}$ is zero, i.e. we have

$$\mathbf{h}^{(1)}\mathbf{T}^{(1)} = (1, \alpha + 2, 0) \quad \text{and} \quad \mathbf{h}^{(2)}\mathbf{T}^{(2)} = (2\alpha + 2, 0).$$

Now we can compute the transformed bases for the received spaces $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$ as

$$\begin{aligned} \hat{\mathbf{U}}^{(1)} &= \left(\mathbf{T}^{(1)}\right)^{-1} \mathbf{U}^{(1)} = \begin{pmatrix} 2\alpha & \alpha + 1 & 1 \\ \alpha + 2 & \alpha & 0 \\ \alpha + 2 & 2 & 2 \end{pmatrix}, \\ \hat{\mathbf{U}}^{(2)} &= \left(\mathbf{T}^{(2)}\right)^{-1} \mathbf{U}^{(2)} = \begin{pmatrix} 2\alpha & 2\alpha & 2\alpha + 2 \\ \alpha + 1 & 2\alpha & 2\alpha + 1 \end{pmatrix}, \end{aligned}$$

where $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ are bases for $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$, respectively. Then the upper $n_t^{(1)} - \delta^{(1)} = 2$ rows of $\hat{\mathbf{U}}^{(1)}$ form a basis for the noncorrupted space $\mathcal{V}^{(1)} \cap \mathcal{U}^{(1)}$ and the upper $n_t^{(2)} - \delta^{(2)} = 1$ rows of $\hat{\mathbf{U}}^{(2)}$ form a basis for the noncorrupted space $\mathcal{V}^{(2)} \cap \mathcal{U}^{(2)}$.

Applying Lagrange interpolation to the rows of $\hat{\mathbf{U}}^{(1)}$ and $\hat{\mathbf{U}}^{(2)}$ corresponding to the noncorrupted spaces we can recover the message polynomials $f_1 = \alpha x + 2$ and $f_2 = 2\alpha x + \alpha + 2$.

Lemma 4 (Decoding Failure Probability). *Suppose that a tuple of subspaces*

$$\mathcal{V}(\mathbf{f}) = \left(\mathcal{V}^{(1)}(\mathbf{f}), \dots, \mathcal{V}^{(\ell)}(\mathbf{f})\right) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$$

is transmitted over a random instance of the ℓ -shot operator channel (see Remark 2) with overall γ insertions and δ deletions, where γ and δ satisfy $\gamma \leq \gamma_{\max} := s(n_t - \delta - k)$. Let $\bar{\mathbf{Z}}$ be defined as in (14) (see Lemma 3). Then, we have

$$\Pr(\text{rk}_{q^m}(\Lambda_{n_t - \delta - k}(\bar{\mathbf{Z}})\mathbf{a}) < \gamma) \leq \kappa_q^{\ell+1} q^{-m(\gamma_{\max} - \gamma + 1)}.$$

Proof. Let $\boldsymbol{\gamma} = (\gamma^{(1)}, \dots, \gamma^{(\ell)}) \in \mathbb{Z}_{\geq 0}^{\ell}$ and $\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(\ell)}) \in \mathbb{Z}_{\geq 0}^{\ell}$ be the partition of the insertions and deletions of the ℓ -shot operator channel, respectively. By assumption, the tuple of error spaces $\mathcal{E} \in \mathcal{G}_q(\mathbf{N}, \boldsymbol{\gamma})$ is chosen uniformly at random from the set $\mathcal{E}^{(\boldsymbol{\gamma})}$ as defined in (5). Each instance of the ℓ -shot operator channel yields a decoding matrix $\bar{\mathbf{L}}$ of the form (13) for some $\mathbf{t} = (t_1, \dots, t_{\ell}) \in \mathbb{Z}_{\geq 0}^{\ell}$, $\boldsymbol{\varkappa} = (\varkappa_1, \dots, \varkappa_{\ell}) \in \mathbb{Z}_{\geq 0}^{\ell}$. Now let us fix $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, which yields a particular instance of \mathbf{t} and $\boldsymbol{\varkappa}$. By Lemma 3 the decoder succeeds if the \mathbb{F}_{q^m} -rank of the matrix

$$\bar{\mathbf{Z}} := \begin{pmatrix} \Lambda_{n_t - \delta - k}(\bar{\mathbf{e}}_1)\mathbf{a} \\ \vdots \\ \Lambda_{n_t - \delta - k}(\bar{\mathbf{e}}_s)\mathbf{a} \end{pmatrix} \in \mathbb{F}_{q^m}^{s(n_t - \delta - k) \times \gamma},$$

equals γ given that

$$\text{wt}_{\Sigma R} \left(\tilde{\mathbf{E}}^{(1)} \hat{\mathbf{E}}^{(1)} \mid \dots \mid \tilde{\mathbf{E}}^{(\ell)} \hat{\mathbf{E}}^{(\ell)} \right) = \mathbf{t} + \boldsymbol{\varkappa} = \boldsymbol{\gamma}. \quad (17)$$

Drawing \mathcal{E} uniformly at random with a fixed rank partition γ from $\mathcal{E}^{(\gamma)}$ corresponds to drawing $\tilde{\mathbf{E}}$ and $\hat{\mathbf{E}}$ uniformly at random from all matrices that satisfy (17). This implies that the probability

$$\Pr(\text{rk}_{q^m}(\Lambda_{n_t-\delta-1}(\overline{\mathbf{Z}})_{\mathbf{a}}) < \gamma)$$

depends on γ and *not* on \mathbf{t} and $\boldsymbol{\kappa}$. Hence, we can use [31, Lemma 7] with $t = \gamma$, $\mathbf{t} = \gamma$, $n = n_r$ and get

$$\Pr(\text{rk}_{q^m}(\Lambda_{n_t-\delta-1}(\overline{\mathbf{Z}})_{\mathbf{a}}) < \gamma) \leq \kappa_q^{\ell+1} q^{-m(\gamma_{\max}-\gamma+1)}.$$

Note that this expression is independent of the rank partition γ , so it is also an upper bound for $\Pr(\text{rk}_{q^m}(\Lambda_{n_t-\delta-k}(\overline{\mathbf{Z}})_{\mathbf{a}}) < \gamma)$ with \mathcal{E} drawn according to Remark 2. \square

Theorem 3 (Loidreau–Overbeck-like Decoder for LILRS Codes). *Suppose we transmit the tuple of subspaces*

$$\mathcal{V}(\mathbf{f}) = \left(\mathcal{V}^{(1)}(\mathbf{f}), \dots, \mathcal{V}^{(\ell)}(\mathbf{f}) \right) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$$

over a random instance of the ℓ -shot operator channel (see Remark 2)

$$\mathbf{U} = \mathcal{H}_{n_t-\delta}(\mathcal{V}) \oplus \mathcal{E},$$

with overall γ insertions and δ deletions, where

$$\gamma \leq \gamma_{\max} := s(n_t - \delta - k). \quad (18)$$

Then, Algorithm 1 with input $\mathcal{V}(\mathbf{f})$ returns the correct message polynomial vector \mathbf{f} with success probability at least

$$\Pr(\text{success}) \geq 1 - \kappa_q^{\ell+1} q^{-m(\gamma_{\max}-\gamma+1)}.$$

Furthermore, the algorithm has complexity $O(sn_r^\omega)$ operations in \mathbb{F}_{q^m} plus $O(mn_r^{\omega-1})$ operations in \mathbb{F}_q .

Proof. Due to Proposition 3, the algorithm returns the correct message polynomial vector \mathbf{f} if the \mathbb{F}_{q^m} -rank of $\Lambda_{n_t-\delta-k}(\overline{\mathbf{Z}})_{\mathbf{a}}$ is at least γ . Hence, the success probability is lower bounded by the probability that $\text{rk}_{q^m}(\Lambda_{n_t-\delta-k}(\overline{\mathbf{Z}})_{\mathbf{a}}) = \gamma$, which is given in Lemma 4.

The lines of the algorithm have the following complexities:

- Lines 3 and 6: This can be done by solving the linear system of equations $\mathbf{L}\mathbf{h}^\top = \mathbf{0}$. Since $\mathbf{L} \in \mathbb{F}_{q^m}^{((s+1)(n_t-\delta)-sk-1) \times n_r}$, it costs $O(sn_r^\omega)$ operations in \mathbb{F}_{q^m} .
- Line 8 can be implemented by transforming the matrix representation of $\mathbf{h}^{(i)}$, which is an $m \times n_r^{(i)}$ matrix over \mathbb{F}_q , into column echelon form. For each i ,

this costs $O\left(mn_r^{(i)\omega-1}\right)$ operations in \mathbb{F}_q . In total, all ℓ calls of this line cost $O\left(\ell m \sum_i n_r^{(i)\omega-1}\right) \subseteq O(mn_r^{\omega-1})$ operations in \mathbb{F}_q .

- Line 9 can be implemented by transforming the matrix representation of $\mathbf{h}^{(i)}$ into column echelon form, which was already accomplished in Line 8.
- Line 10 requires $O\left(sn_r^{(i)2}\right)$ multiplications over \mathbb{F}_{q^m} and thus $O\left(s \sum_i n_r^{(i)2}\right) \subseteq O(sn_r^2)$ operations in \mathbb{F}_{q^m} in total.
- Line 13 computes s interpolation polynomials of degree less than $k \leq n_t$ point tuples. This costs in total $\tilde{O}(s\mathcal{M}(n_t))$ operations in \mathbb{F}_{q^m} (c.f. [47, 48]).

This proves the complexity statement. \square

The decoding region of the Loidreau–Overbeck-like decoder for LILRS codes is illustrated in Figure 5.

4.3 An Interpolation-Based Decoding Approach

We now derive an interpolation-based decoding approach for LILRS codes. The decoding principle consists of an interpolation step and a root-finding step. In [4], (lifted) linearized Reed–Solomon codes are decoded using the isometry between the sum-rank and the skew metric. In this work we consider an interpolation-based decoding scheme in the generalized operator evaluation domain. The new decoder is a generalization of [25] (interleaved Gabidulin codes in the rank metric) and [30] (lifted interleaved Gabidulin codes in the subspace metric). Compared to the Loidreau–Overbeck-like decoder from Section 4.2, which requires $O(s^\omega n_r^2)$ operations in \mathbb{F}_{q^m} , the proposed interpolation based decoder has a reduced computational complexity in the order of $\tilde{O}(s^\omega \mathcal{M}(n_r))$ operations in \mathbb{F}_{q^m} .

4.3.1 Interpolation Step

Suppose we transmit the tuple of subspaces

$$\mathcal{V}(\mathbf{f}) = \left(\mathcal{V}^{(1)}(\mathbf{f}), \dots, \mathcal{V}^{(\ell)}(\mathbf{f})\right) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$$

over an ℓ -shot operator channel (3) with γ insertions and δ deletions and receive the tuple of subspaces

$$\mathbf{u} = \left(\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \dots, \mathcal{U}^{(\ell)}\right) \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r)$$

where the received subspaces $\mathcal{U}^{(i)}$ are represented as in (7). We describe \mathbf{u} by the tuple containing the basis matrices of the received subspaces as

$$\mathbf{U} := \left(\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(\ell)}\right) \in \prod_{i=1}^{\ell} \mathbb{F}_{q^m}^{n_r^{(i)} \times (s+1)}$$

where

$$\mathbf{U}^{(i)} := \left(\boldsymbol{\xi}^{(i)\top} \quad \mathbf{u}_1^{(i)\top} \quad \mathbf{u}_2^{(i)\top} \quad \dots \quad \mathbf{u}_s^{(i)\top}\right) \in \mathbb{F}_{q^m}^{n_r^{(i)} \times (s+1)}$$

has \mathbb{F}_q -rank $\text{rk}_q(\mathbf{U}^{(i)}) = n_r^{(i)}$ and satisfies $\mathcal{U}^{(i)} = \langle \mathbf{U}^{(i)} \rangle_q$ for all $i = 1, \dots, \ell$.

Remark 4. In contrast to [4, Section V.H] we do not need the assumption that the \mathbb{F}_q -rank of $\boldsymbol{\xi}^{(i)}$ equals $n_r^{(i)}$ for all $i = 1, \dots, \ell$, which is not the case in general (see also [41, Section 5.1.2]).

For a multivariate skew polynomial of the form

$$Q(x, y_1, \dots, y_s) = Q_0(x) + Q_1(x)y_1 + \dots + Q_s(x)y_s$$

where $Q_l(x) \in \mathbb{F}_{q^m}[x; \sigma]$ for all $l \in [0, s]$ define the n_r generalized operator evaluation maps

$$\begin{aligned} \mathbb{F}_{q^m}[x, y_1, \dots, y_s; \sigma] \times \mathbb{F}_{q^m}^{s+1} &\rightarrow \mathbb{F}_{q^m} \\ \left(Q, (\xi_j^{(i)}, u_{1,l}^{(i)}, \dots, u_{s,l}^{(i)}) \right) &\mapsto \mathcal{E}_j^{(i)}(Q) := Q_0(\xi_j^{(i)})_{a_i} + \sum_{l=1}^s Q_l(u_{j,l}^{(i)})_{a_i}. \end{aligned} \quad (19)$$

for all $j = 1, \dots, n_r^{(i)}$ and $i = 1, \dots, \ell$. Now consider the following interpolation problem in $\mathbb{F}_{q^m}[x; \sigma]$ (see also [32, Problem 1]).

Problem 1 (LILRS Interpolation Problem). *Given the integers $D, s, \ell \in \mathbb{Z}_{\geq 0}$, a set*

$$\mathcal{E} = \left\{ \mathcal{E}_j^{(i)} : i = 1, \dots, \ell, j = 1, \dots, n_i \right\}$$

containing the generalized operator evaluation maps defined in (19) and a vector $\mathbf{w} = (0, k-1, \dots, k-1) \in \mathbb{Z}_{\geq 0}^{s+1}$, find a nonzero polynomial of the form

$$Q(x, y_1, \dots, y_s) = Q_0(x) + Q_1(x)y_1 + \dots + Q_s(x)y_s$$

with $Q_l(x) \in \mathbb{F}_{q^m}[x; \sigma]$ for all $l \in [0, s]$ that satisfies:

1. $\mathcal{E}_j^{(i)}(Q) = 0, \quad \forall i = 1, \dots, \ell, j = 1, \dots, n_r^{(i)},$
2. $\deg_{\mathbf{w}}(Q(x, y_1, \dots, y_s)) < D.$

Define the skew polynomials

$$Q_0(x) = \sum_{i=0}^{D-1} q_{0,i} x^i \quad \text{and} \quad Q_j(x) = \sum_{i=0}^{D-k} q_{j,i} x^i,$$

and the vectors

$$\boldsymbol{\xi} = \left(\boldsymbol{\xi}^{(1)} \mid \boldsymbol{\xi}^{(2)} \mid \dots \mid \boldsymbol{\xi}^{(\ell)} \right) \in \mathbb{F}_{q^m}^{n_r} \quad \text{and} \quad \mathbf{u}_j = \left(\mathbf{u}_j^{(1)} \mid \mathbf{u}_j^{(2)} \mid \dots \mid \mathbf{u}_j^{(\ell)} \right) \in \mathbb{F}_{q^m}^{n_r}$$

for all $j = 1, \dots, s$. Then a solution of Problem 1 can be found by solving the \mathbb{F}_{q^m} -linear system

$$\mathbf{R}_I \mathbf{q} = \mathbf{0} \quad (20)$$

for

$$\mathbf{q} = (q_{0,0}, q_{0,1}, \dots, q_{0,D-1} \mid q_{1,0}, q_{1,1}, \dots, q_{1,D-k} \mid \dots \mid q_{s,0}, q_{s,1}, \dots, q_{s,D-k}) \quad (21)$$

where the interpolation matrix $\mathbf{R}_I \in \mathbb{F}_{q^m}^{n_r \times D(s+1) - s(k-1)}$ is given by

$$\mathbf{R}_I = (\Lambda_D(\boldsymbol{\xi})_{\mathbf{a}}^\top \ \Lambda_{D-k+1}(\mathbf{u}_1)_{\mathbf{a}}^\top \ \dots \ \Lambda_{D-k+1}(\mathbf{u}_s)_{\mathbf{a}}^\top). \quad (22)$$

Problem 1 can be solved by the skew Kötter interpolation [49] with the generalized operator evaluation maps $\mathcal{E}_j^{(i)}$ as defined in (19) requiring $O(s^2 n_r^2)$ operations in \mathbb{F}_{q^m} . A solution of Problem 1 can be found efficiently requiring only $\tilde{O}(s^\omega \mathcal{M}(n_r))$ operations in \mathbb{F}_{q^m} using a variant of the minimal approximant bases approach from [31]. Another approach yielding the same computational complexity of $\tilde{O}(s^\omega \mathcal{M}(n_r))$ operations in \mathbb{F}_{q^m} is given by the fast divide-and-conquer Kötter interpolation from [50].

Lemma 5 (Existence of Solution [32, Lemma 2]). *A nonzero solution of Problem 1 exists if $D = \lceil \frac{n_r + s(k-1) + 1}{s+1} \rceil$.*

Proof. Problem 1 corresponds to a system of n_r \mathbb{F}_{q^m} -linear equations in $D(s+1) - s(k-1)$ unknowns (see (20)) which has a nonzero solution if the number of equations is less than the number of unknowns, i.e. if

$$n_r < D(s+1) - s(k-1) \iff D \geq \frac{n_r + s(k-1) + 1}{s+1}. \quad (23)$$

□

The \mathbb{F}_{q^m} -linear solution space \mathcal{Q} of Problem 1 is defined as

$$\mathcal{Q} := \{Q \in \mathbb{F}_{q^m}[x, y_1, \dots, y_s; \sigma] : \mathbf{q}(Q) \in \ker_r(\mathbf{R}_I)\}$$

where $\mathbf{q}(Q) \in \mathbb{F}_{q^m}^{D(s+1) - s(k-1)}$ denotes the coefficient vector of Q as defined in (21). The dimension of the \mathbb{F}_{q^m} -linear solution space \mathcal{Q} of Problem 1 (i.e. the dimension of the right kernel of \mathbf{R}_I in (22)) is denoted by

$$d_I := \dim(\mathcal{Q}) = \dim(\ker_r(\mathbf{R}_I)).$$

4.3.2 Root-Finding Step

The goal of the root-finding step is to recover the message polynomials $f_1, \dots, f_s \in \mathbb{F}_{q^m}[x; \sigma]_{<k}$ from the multivariate polynomial constructed in the interpolation step. We now derive a condition for the recovery of the message polynomials (see also [32, Lemma 3, Theorem 1]).

Lemma 6 (Roots of Polynomial). *Let*

$$P(x) := Q_0(x) + Q_1(x)f_1(x) + \dots + Q_s(x)f_s(x).$$

Then there exist elements $\zeta_1^{(i)}, \dots, \zeta_{n_t^{(i)} - \delta^{(i)}}^{(i)}$ in \mathbb{F}_{q^m} that are \mathbb{F}_q -linearly independent for each $i = 1, \dots, \ell$ such that

$$P(\zeta_j^{(i)})_{a_i} = 0$$

for all $i = 1, \dots, \ell$ and $j = 1, \dots, n_t^{(i)} - \delta^{(i)}$.

Proof. In each shot the non-corrupted intersection space has dimension $\dim(\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}) = n_t^{(i)} - \delta^{(i)}$ for all $i = 1, \dots, \ell$. A basis for each intersection space $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$ can be represented as

$$\left\{ \left(\zeta_j^{(i)}, f_1(\zeta_j^{(i)})_{a_i}, \dots, f_s(\zeta_j^{(i)})_{a_i} \right) : j \in [1, n_t^{(i)} - \delta^{(i)}] \right\}$$

where $\zeta_1^{(i)}, \dots, \zeta_{n_t^{(i)} - \delta^{(i)}}^{(i)}$ are $(n_t^{(i)} - \delta^{(i)})$ \mathbb{F}_q -linearly independent elements from \mathbb{F}_{q^m} for all $i = 1, \dots, \ell$. Since each intersection space $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$ is a subspace of the received space $\mathcal{U}^{(i)}$ we have that

$$P(\zeta_j^{(i)})_{a_i} := Q_0(\zeta_j^{(i)})_{a_i} + \sum_{l=1}^s Q_l(f_l(\zeta_j^{(i)})_{a_i})_{a_i} = 0$$

for all $i = 1, \dots, \ell, j = 1, \dots, n_t^{(i)} - \delta^{(i)}$. \square

Theorem 4 (Decoding Region). *Let $\mathbf{U} \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r)$ be the tuple containing the received subspaces and let $Q(x, y_1, \dots, y_s) \neq 0$ fulfill the constraints in Problem 1. Then for all codewords $\mathbf{V}(\mathbf{f}) \in \text{LILRS}[\beta, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ that are (γ, δ) -reachable from \mathbf{U} , where γ and δ satisfy*

$$\gamma + s\delta < s(n_t - k + 1), \quad (24)$$

we have that

$$P(x) = Q_0(x) + Q_1(x)f_1(x) + \dots + Q_s(x)f_s(x) = 0. \quad (25)$$

Proof. By Lemma 6 there exist elements $\zeta_1^{(i)}, \dots, \zeta_{n_t^{(i)} - \delta^{(i)}}^{(i)}$ in \mathbb{F}_{q^m} that are \mathbb{F}_q -linearly independent for each $i = 1, \dots, \ell$ such that

$$P(\zeta_j^{(i)})_{a_i} = 0$$

for all $i = 1, \dots, \ell$ and $j = 1, \dots, n_t^{(i)} - \delta^{(i)}$. By choosing

$$D \leq n_t - \delta \quad (26)$$

the degree of $P(x)$ exceeds the degree bound from [40]) which is possible only if $P(x) = 0$. Combining (23) and (26) we get

$$\begin{aligned} n_r + s(k-1) &< D(s+1) \leq (s+1)(n_t - \delta) \\ \iff \gamma + s\delta &< s(n_t - k + 1). \end{aligned}$$

\square

4.3.3 List Decoding

In general, the root-finding matrix \mathbf{Q}_R in (30) can be rank deficient. In this case we obtain a *list* of potential message polynomials f_1, \dots, f_s . By [31, Proposition 4] the root-finding system in (25) has at most $q^{m(k(s-1))}$ solutions $f, \dots, f_s \in \mathbb{F}_{q^m}[x; \sigma]_{<k}$. In general, we have that $k \leq n_t$, where $n_t \leq \ell m$. Hence, for $m \approx n_t/\ell$ we get a worst-case list size of $q^{\frac{n_t}{\ell}(k(s-1))}$.

Algorithm 2 LIST DECODING OF LILRS CODES

Input: A tuple containing the basis matrices $\mathbf{U} = (\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(\ell)}) \in \prod_{i=1}^{\ell} \mathbb{F}_{q^m}^{n_r^{(i)} \times (s+1)}$ for the output $\mathbf{U} = (\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(\ell)}) \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r)$ of an ℓ -shot operator channel with overall γ insertions and δ deletions for input $\mathbf{V}(\mathbf{f}) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$

Output: A list \mathcal{L} containing message polynomial vectors $\mathbf{f} = (f_1, \dots, f_s) \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$ that satisfy (25)

- 1: Find left $\mathbb{F}_{q^m}[x; \sigma]$ -linearly independent $Q^{(1)}, \dots, Q^{(s')} \in \mathcal{Q} \setminus \{0\}$ whose left $\mathbb{F}_{q^m}[x; \sigma]$ -span contains the \mathbb{F}_{q^m} -linear solution space \mathcal{Q} of Problem 1
 - 2: Using $Q^{(1)}, \dots, Q^{(s')}$, find the list $\mathcal{L} \subseteq \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$ of all $\mathbf{f} = (f_1, \dots, f_s) \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$ that satisfy (25)
 - 3: **return** \mathcal{L}
-

Theorem 5 (List Decoding of LILRS Codes). *Let $\mathbf{U} \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r)$ be a tuple of received subspaces of a transmission of a codeword $\mathbf{V} \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ over an ℓ -shot operator channel with overall γ insertions and δ deletions. If the number of overall insertions γ and deletions δ satisfy*

$$\gamma + s\delta < s(n_t - k + 1),$$

then a list \mathcal{L} of size

$$|\mathcal{L}| \leq q^{m(k(s-1))}$$

containing all message polynomial vectors $\mathbf{f} \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$ corresponding to codewords $\mathbf{V}(\mathbf{f}) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ that are (γ, δ) -reachable from \mathbf{U} can be found requiring at most $\tilde{O}(s^\omega \mathcal{M}(n_r))$ operations in \mathbb{F}_{q^m} .

Proof. The proof follows directly from Lemma 6, Theorem 4 and the discussion above. \square

4.3.4 Probabilistic Unique Decoding

We now consider the interpolation-based decoder from Section 4.3 as a probabilistic unique decoder which either returns a unique solution (if the list size is equal to one) or a decoding failure. The main idea is to use a basis for the d_I -dimensional \mathbb{F}_{q^m} -linear

solution space \mathcal{Q} of the interpolation system (20) to set up the root-finding matrix (27) which in turn facilitates that the root-finding matrix \mathbf{Q}_R can have full rank.

Using similar arguments as in [30, 31, 46] we can lower bound the dimension d_I of the \mathbb{F}_{q^m} -linear solution space \mathcal{Q} of Problem 1.

Lemma 7 (Dimension of Solution Space). *Let γ and δ satisfy (24). Then the dimension $d_I = \dim(\mathcal{Q})$ of the \mathbb{F}_{q^m} -linear solution space \mathcal{Q} of Problem 1 satisfies*

$$d_I \geq s(D+1) - sk - \gamma.$$

Proof. Let

$$\left\{ \left(\zeta_j^{(i)}, f_1(\zeta_j^{(i)})_{a_i}, \dots, f_s(\zeta_j^{(i)})_{a_i} \right) : j \in [1, n_t^{(i)} - \delta^{(i)}] \right\}$$

be a basis for each non-corrupted intersection space $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$ where $\zeta_1^{(i)}, \dots, \zeta_{n_t^{(i)} - \delta^{(i)}}^{(i)}$ are $(n_t^{(i)} - \delta^{(i)})$ \mathbb{F}_q -linearly independent elements from \mathbb{F}_{q^m} for all $i = 1, \dots, \ell$. Define the vector $\boldsymbol{\zeta} := (\boldsymbol{\zeta}^{(1)} \mid \boldsymbol{\zeta}^{(2)} \mid \dots \mid \boldsymbol{\zeta}^{(\ell)}) \in \mathbb{F}_{q^m}^{n_t - \delta}$ where $\boldsymbol{\zeta}^{(i)} = (\zeta_1^{(i)}, \dots, \zeta_{n_t^{(i)} - \delta^{(i)}}^{(i)})$ for all $i = 1, \dots, \ell$. Let

$$\left\{ \left(\epsilon_j^{(i)}, \tilde{e}_{j,1}^{(i)}, \dots, \tilde{e}_{j,s}^{(i)} \right) : j \in [1, \gamma^{(i)}] \right\}$$

be a basis for the error space $\mathcal{E}^{(i)}$ for all $i = 1, \dots, \ell$ and define

$$\begin{aligned} \boldsymbol{\epsilon} &= (\boldsymbol{\epsilon}^{(1)} \mid \boldsymbol{\epsilon}^{(2)} \mid \dots \mid \boldsymbol{\epsilon}^{(\ell)}) \in \mathbb{F}_{q^m}^{\gamma} \quad \text{with} \quad \boldsymbol{\epsilon}^{(i)} = (\epsilon_1^{(i)}, \epsilon_2^{(i)}, \dots, \epsilon_{\gamma^{(i)}}^{(i)}), \\ \tilde{\boldsymbol{e}}_j &= (\tilde{e}_j^{(1)} \mid \tilde{e}_j^{(2)} \mid \dots \mid \tilde{e}_j^{(\ell)}) \in \mathbb{F}_{q^m}^{\gamma^{(i)}} \quad \text{with} \quad \tilde{\boldsymbol{e}}_j^{(i)} = (\tilde{e}_{1,j}^{(i)}, \tilde{e}_{2,j}^{(i)}, \dots, \tilde{e}_{\gamma^{(i)},j}^{(i)}). \end{aligned}$$

Then the matrix

$$\tilde{\mathbf{R}}_I = \begin{pmatrix} \Lambda_D(\boldsymbol{\zeta})_{\mathbf{a}}^\top & \mathbf{0} & \dots & \mathbf{0} \\ \Lambda_D(\boldsymbol{\epsilon})_{\mathbf{a}}^\top & \Lambda_{D-k+1}(\tilde{\boldsymbol{e}}_1)_{\mathbf{a}}^\top & \dots & \Lambda_{D-k+1}(\tilde{\boldsymbol{e}}_s)_{\mathbf{a}}^\top \end{pmatrix} \in \mathbb{F}_{q^m}^{n_r \times D(s+1) - s(k-1)}$$

has the same column space as the matrix \mathbf{R}_I in (22). Since $\text{wt}_{\Sigma R}(\boldsymbol{\zeta}) = n_t - \delta$ and $D \leq n_t - \delta$ (see (26)) we have that the matrix $\Lambda_D(\boldsymbol{\zeta})_{\mathbf{a}}^\top$ has \mathbb{F}_{q^m} -rank D . The last γ rows of $\tilde{\mathbf{R}}_I$ can increase the \mathbb{F}_{q^m} -rank of $\tilde{\mathbf{R}}_I$ by at most γ . Thus we have that $\text{rk}_{q^m}(\mathbf{R}_I) = \text{rk}_{q^m}(\tilde{\mathbf{R}}_I) \leq D + \gamma$. Hence, the dimension d_I of the \mathbb{F}_{q^m} -linear solution space \mathcal{Q} of Problem 1 satisfies

$$\begin{aligned} d_I &= \dim(\ker_r(\mathbf{R}_I)) = D(s+1) - s(k-1) - \text{rk}_{q^m}(\mathbf{R}_I) \\ &\geq s(D+1) - sk - \gamma. \end{aligned}$$

□

The rank of the root-finding matrix \mathbf{Q}_R can be full if and only if the dimension of the solution space of the interpolation problem d_I is at least s , i.e. if

$$\begin{aligned} d_I \geq s &\iff \gamma \leq sD - sk \\ &\iff \gamma + s\delta \leq s(n_t - k). \end{aligned} \tag{31}$$

The probabilistic unique decoding region in (31) is only slightly smaller than the list decoding region in (24). The improved decoding region for LILRS codes is illustrated in Figure 5. Recall from Remark 4 that unlike the proposed decoder, the decoder in [4] has the restriction that $n_r = n_t$, which corresponds to the case that $\gamma = \delta$.

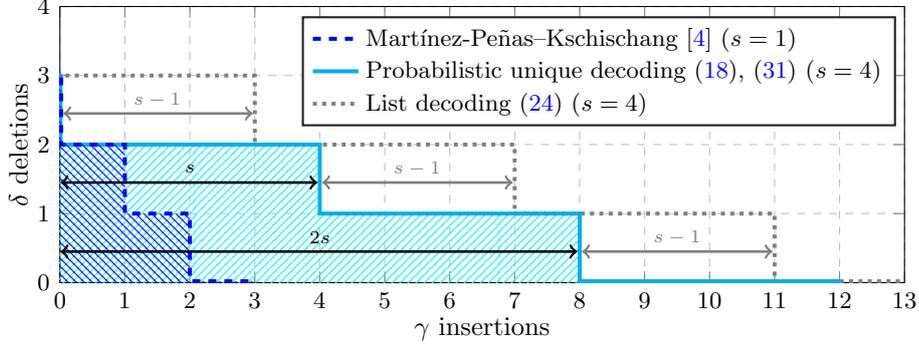


Fig. 5 Decoding region for Martínez-Peñas-Kschischang [4] codes ($s = 1$) and for decoding of lifted ($s = 4$)-interleaved linearized Reed-Solomon codes. The decoding region for insertions increases with the interleaving order s .

Combining (26) and (31) we get the degree constraint for the probabilistic unique decoder (see [46])

$$D_u = \left\lceil \frac{n_r + sk}{s + 1} \right\rceil.$$

In order to get an estimate of the probability of successful decoding, we use similar assumptions as in [25, 46] to derive a heuristic upper bound on the decoding failure probability P_f .

The root-finding matrix \mathbf{Q}_R in (27) contains a lower block-diagonal matrix with $\mathbf{Q}_0, \sigma^{-1}(\mathbf{Q}_0), \dots, \sigma^{-(k-1)}(\mathbf{Q}_0)$ on the diagonal. Since $\text{rk}_{q^m}(\sigma^i(\mathbf{Q}_0)) = \text{rk}_{q^m}(\mathbf{Q}_0)$ for all i , this implies that $\text{rk}_{q^m}(\mathbf{Q}_R) = sk$ if $\text{rk}_{q^m}(\mathbf{Q}_0) = s$.

Under the assumption that the coefficients $q_{i,j}^{(r)}$ are uniformly distributed over \mathbb{F}_{q^m} (see [25, 46, Lemma 9]), we can upper bound the decoding failure probability $P_f := \Pr(\text{rk}_{q^m}(\mathbf{Q}_R) < sk)$ by the probability that the $(d_I \times s)$ matrix \mathbf{Q}_0 with uniformly distributed elements from \mathbb{F}_{q^m} has \mathbb{F}_{q^m} -rank less than s and get

$$P_f \leq \kappa_q q^{-m(d_I - s + 1)} \leq \kappa_q q^{-m(s(\lceil \frac{n_r + sk}{s+1} \rceil - k) - \gamma + 1)}. \quad (32)$$

Note, that for $D = n_t - \delta$ (see (26)) we get

$$P_f \leq \kappa_q q^{-m(\gamma_{\max} - \gamma + 1)}.$$

Note, that the assumption that the coefficients $q_{i,j}^{(r)}$ are uniformly distributed over \mathbb{F}_{q^m} does not reflect the distribution of the error space tuple \mathcal{E} . Although there is evidence that this assumption is reasonable (see e.g. [51] for folded LRS codes), it does not reflect the actual error model of the multishot operator channel.

Similar as in [25, Lemma 8] for interleaved Gabidulin codes and [31, Theorem 4] for ILRS codes, the conditions of successful decoding of the interpolation-based decoder can be reduced to the conditions of the Loidreau–Overbeck-like decoder from Section 4.2. This reduction allows to obtain an upper bound on the decoding failure probability since the distribution of the error space tuple \mathcal{E} is considered in the derivation. The results of the interpolation-based probabilistic unique decoder are summarized in Algorithm 3 and Theorem 6.

Algorithm 3 PROBABILISTIC UNIQUE DECODING OF LILRS CODES

Input: A tuple containing the basis matrices $\mathbf{U} = (\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(\ell)}) \in \prod_{i=1}^{\ell} \mathbb{F}_{q^m}^{n_r^{(i)} \times (s+1)}$ for the output $\mathbf{U} = (\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \dots, \mathcal{U}^{(\ell)}) \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r)$ of an ℓ -shot operator channel with overall γ insertions and δ deletions for input $\mathcal{V}(\mathbf{f}) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$

Output: Message polynomial vector $\mathbf{f} = (f_1, \dots, f_s) \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$ or “decoding failure”

- 1: Define the generalized operator evaluation maps as in (19)
 - 2: Find left $\mathbb{F}_{q^m}[x; \sigma]$ -linearly independent $Q^{(1)}, \dots, Q^{(s')} \in \mathcal{Q} \setminus \{0\}$ whose left $\mathbb{F}_{q^m}[x; \sigma]$ -span contains the \mathbb{F}_{q^m} -linear solution space \mathcal{Q} of Problem 1
 - 3: **if** $s' = s$ **then**
 - 4: Use $Q^{(1)}, \dots, Q^{(s)}$ to find the unique vector $\mathbf{f} = (f_1, \dots, f_s) \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$ that satisfies (25)
 - 5: **else**
 - 6: **return** “decoding failure”
 - 7: **end if**
 - 8: **return** Message polynomial vector $\mathbf{f} = (f_1, \dots, f_s) \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$
-

Theorem 6 (Probabilistic Unique Decoding of LILRS Codes). *Let $\mathbf{U} \in \mathcal{G}_q(\mathbf{N}, \mathbf{n}_r)$ be a tuple of received subspaces of a transmission of a codeword $\mathcal{V} \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ over random instance of the ℓ -shot operator channel (see Remark 2) with overall γ insertions and δ deletions. If the number of overall insertions γ and deletions δ satisfy*

$$\gamma + s\delta \leq s(n_t - k),$$

then a the unique message polynomial vector $\mathbf{f} \in \mathbb{F}_{q^m}[x; \sigma]_{<k}^s$ corresponding to the codeword $\mathcal{V}(\mathbf{f}) \in \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ satisfying $d_{\Sigma S}(\mathcal{V}(\mathbf{f}), \mathbf{U}) = \gamma + \delta$ can be found with probability at least

$$\Pr(\text{success}) \geq 1 - \kappa_q^{\ell+1} q^{-m(\gamma_{\max} - \gamma + 1)} \quad (33)$$

requiring at most $\tilde{O}(s^\omega \mathcal{M}(n_r))$ operations in \mathbb{F}_{q^m} .

Proof. For the purpose of the proof (but not algorithmically), we consider the root-finding problem set up with an \mathbb{F}_{q^m} -basis $Q^{(1)}, \dots, Q^{(d_t)}$ of \mathcal{Q} . The unique decoder

fails if there are at least two distinct roots \mathbf{f} and \mathbf{f}' . In this case, the \mathbb{F}_{q^m} -linear system $\mathbf{Q}_R \cdot \mathbf{f}_R = -\mathbf{q}_0$ in (30) set up with the \mathbb{F}_{q^m} -basis $\tilde{Q}^{(r)} \in \mathcal{Q}$ for $r = 1, \dots, d_I$ has at least two solutions. This means that $\mathbf{Q}_R \in \mathbb{F}_q^{D d_I \times sk}$ must have rank $< sk$.

The matrix \mathbf{Q}_R contains a lower block triangular matrix with matrices $\mathbf{Q}_0, \sigma^{-1}(\mathbf{Q}_0), \dots, \sigma^{-(k-1)}(\mathbf{Q}_0)$ on the upper diagonal, which have all \mathbb{F}_{q^m} -rank $\text{rk}_{q^m}(\mathbf{Q}_0)$ (see [31]). Thus, if $\text{rk}_{q^m}(\mathbf{Q}_0) = s$ the matrix \mathbf{Q}_R has full \mathbb{F}_{q^m} -rank sk . Therefore, $\text{rk}_{q^m}(\mathbf{Q}_R) < sk$ implies that \mathbf{Q}_0 has rank $< s$.

Since the root-finding system (30) has at least one solution \mathbf{f}_R , there is a vector $\mathbf{f}_0 \in \mathbb{F}_{q^m}^s$ such that

$$\mathbf{Q}_0 \mathbf{f}_0 = -\mathbf{q}_{0,0}^\top.$$

Thus, the matrix

$$\overline{\mathbf{Q}}_0 := (\mathbf{Q}_0 \ \mathbf{q}_{0,0}^\top) \in \mathbb{F}_{q^m}^{d_I \times (s+1)}$$

has rank $\text{rk}_{q^m}(\overline{\mathbf{Q}}_0) = \text{rk}_{q^m}(\mathbf{Q}_0) < s$. Hence, there are at least $d_I - s + 1$ \mathbb{F}_{q^m} -linearly independent polynomials $\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(d_I - s + 1)} \in \mathcal{Q}$ such that their zeroth coefficients $\tilde{q}_{l,0}^{(1)}, \dots, \tilde{q}_{l,0}^{(d_I - s + 1)}$ are zero for all $l = 0, \dots, s$ (obtained by suitable) \mathbb{F}_{q^m} -linear combinations of the original basis polynomials $Q^{(1)}, \dots, Q^{(d_I)}$, such that the corresponding \mathbb{F}_{q^m} -linear row operations on $\overline{\mathbf{Q}}_0$ give a $(d_I - s + 1) \times (s + 1)$ zero matrix (recall that $\overline{\mathbf{Q}}_0$ has d_I rows, but rank at most $s - 1$).

The $d_I - s + 1$ \mathbb{F}_{q^m} -linearly independent coefficient vectors of $\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(d_I - s + 1)}$ of the form (21) are in the left kernel of the matrix

$$\mathbf{R}_I^\top = \begin{bmatrix} \Lambda_D(\boldsymbol{\xi})_{\mathbf{a}} \\ \Lambda_{D-k+1}(\mathbf{u}_1)_{\mathbf{a}} \\ \vdots \\ \Lambda_{D-k+1}(\mathbf{u}_s)_{\mathbf{a}} \end{bmatrix} \in \mathbb{F}_{q^m}^{D(s+1) - s(k-1) \times n_r}.$$

Since the zeroth components $\tilde{q}_{l,0}^{(r)}$ of all $\tilde{Q}^{(r)}$ are zero for all $l = 0, \dots, s$ and $r = 1, \dots, d_I - s + 1$, this means that the left kernel of the matrix

$$\tilde{\mathbf{R}}_I^\top = \begin{bmatrix} \mathcal{D}_{\mathbf{a}}(\Lambda_{D-1}(\boldsymbol{\xi})_{\mathbf{a}}) \\ \mathcal{D}_{\mathbf{a}}(\Lambda_{D-k}(\mathbf{u}_1)_{\mathbf{a}}) \\ \vdots \\ \mathcal{D}_{\mathbf{a}}(\Lambda_{D-k}(\mathbf{u}_s)_{\mathbf{a}}) \end{bmatrix} \in \mathbb{F}_{q^m}^{D(s+1) - sk - 1 \times n}$$

has dimension at least $d_I - s + 1$. The maximum decoding region corresponds to the degree constraint $D = n_r - \gamma_{\max} = n_t - \delta$ (see (26)) and thus

$$\dim(\ker_l(\tilde{\mathbf{R}}_I^\top)) \geq s(n_t - \delta + 1) - sk - \gamma_{\max} - s + 1 \geq 1.$$

Therefore, we have that

$$\begin{aligned} \text{rk}_{q^m}(\tilde{\mathbf{R}}_I^\top) &\leq D(s+1) - sk - 1 - \dim(\ker_l(\tilde{\mathbf{R}}_I^\top)) \\ &< (n_t - \delta)(s+1) - sk - 1 \end{aligned}$$

$$= n_r - 1.$$

Observe, that for $D = n_t - \delta$ we have that

$$\tilde{\mathbf{R}}_I^\top = \mathcal{D}_a(\mathbf{L})$$

where \mathbf{L} is the Loidreau–Overbeck decoding matrix from (12). By [31, Lemma 3] the \mathbb{F}_{q^m} -rank of \mathbf{L} and $\mathcal{D}_a(\mathbf{L})$ is the same and thus we have that

$$\text{rk}_{q^m}(\mathbf{L}) = \text{rk}_{q^m}(\tilde{\mathbf{R}}_I) < n_r - 1$$

which shows that in this case the Loidreau–Overbeck-like decoder fails as well. Therefore, we conclude that

$$\Pr(\text{rk}_{q^m}(\mathbf{Q}_R) < sk) \leq \Pr(\text{rk}_{q^m}(\mathbf{Q}_0) < s) \leq \Pr(\text{rk}_{q^m}(\mathbf{L}) < n_r - 1)$$

and thus the lower bound on the probability of successful decoding follows from Theorem 3. The complexity statement follows from [31, Corollary 1] and [31, Corollary 2]. \square

The lower bound on the probability of successful decoding in (33) yields an upper bound on the decoding failure probability P_f , i.e. we have that

$$P_f \leq \kappa_q^{\ell+1} q^{-m(\gamma_{\max} - \gamma + 1)}. \quad (34)$$

The simulation results in Section 4.5 show that the upper bound on the decoding failure probability in (34) gives a good estimate of the performance of the probabilistic unique decoder.

The interpolation-based probabilistic unique decoding scheme for LILRS codes is illustrated in Example 2.

Example 2 (Interpolation-Based Decoding). *Consider again the code LILRS $[\beta, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$, the codeword $\mathbf{V}(\mathbf{f})$ from (15) and received spaces in \mathcal{U} from (16) considered in Example 1.*

First, we compute $\mathbb{F}_{q^m}[x; \sigma]$ -linearly independent polynomials of minimal $\mathbf{w} = (0, 1, 1)$ -weighted degree that span the solution space of Problem 1 as

$$Q^{(1)} = (\alpha + 2)x^2 + (2\alpha + 1)x + 1 + (x + 1)y_1$$

$$Q^{(2)} = (\alpha + 1)x^2 + (\alpha + 2)x + \alpha + 2 + ((2\alpha + 2)x + 2\alpha)y_1 + (x + 2\alpha + 1)y_2$$

using e.g. the skew Kötter interpolation from [49].

Using the coefficients of $Q^{(1)}$ and $Q^{(2)}$ we set up the root-finding matrix \mathbf{Q}_R as (see (27))

$$\mathbf{Q}_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2\alpha & 2\alpha + 1 & 0 & 0 \\ \alpha + 1 & 1 & \alpha + 2 & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2\alpha + 2 & 1 \end{pmatrix}$$

and the vector \mathbf{q}_0 as (see (29))

$$\mathbf{q}_0 = (1, \alpha + 2, \alpha, 2\alpha, \alpha + 2, \alpha + 1)^\top.$$

The unique solution of the \mathbb{F}_{q^m} -linear root-finding system $\mathbf{Q}_R \cdot \mathbf{f}_R = -\mathbf{q}_0$ in (30) is

$$\begin{aligned} \mathbf{f}_R &:= (f_0^{(1)}, f_0^{(2)}, \sigma^{-1}(f_1^{(1)}), \sigma^{-1}(f_1^{(2)})) \\ &= (2, \alpha + 2, 2\alpha + 1, \alpha + 2)^\top. \end{aligned}$$

Considering the structure of \mathbf{f}_R (cf. (28)) we can recover the message polynomials

$$\begin{aligned} f_1(x) &= \alpha x + 2, \\ f_2(x) &= 2\alpha x + \alpha + 2 \end{aligned}$$

which correspond to the transmitted codeword $\mathbf{C}(\mathbf{f})$.

4.4 Insertion/Deletion-Correction with the Complementary Code

In [30, Section 4.4] it was shown that the complementary of an interleaved (single-shot) subspace code is capable of correcting more deletions than insertions. We will now briefly describe how to extend the concept from [30] to the multishot scenario. In particular, we show that the complementary code of a LILRS code is more resilient against *deletions* than *insertions*. By using the arguments from [30, Lemma 14] and [30, Theorem 4] on each of the components of \mathbf{U}^\perp (and \mathbf{U}) we obtain the following result. **Proposition 7.** *Consider a LILRS code $\mathcal{C} = \text{LILRS}[\boldsymbol{\beta}, \mathbf{a}, \ell, s; \mathbf{n}_t, k]$ and the corresponding complementary code \mathcal{C}^\perp . Suppose we transmit a tuple $\mathbf{V}^\perp \in \mathcal{C}^\perp$ over a multishot operator channel (3) with overall γ insertions and δ deletions and receive*

$$\mathbf{U}^\perp = \mathcal{H}_{N-n_t-\delta}(\mathbf{V}^\perp) \oplus \mathcal{E}_\gamma$$

where $\dim_\Sigma(\mathcal{E}_\gamma) = \gamma$. Then we have that

$$\mathbf{U} = \mathcal{H}_{n_t-\gamma}(\mathbf{V}) \oplus \mathcal{E}_\delta$$

where $\dim_\Sigma(\mathcal{E}_\delta) = \delta$.

The proof of Proposition 7 can be found in Appendix B.1.

Proposition 7 shows, that the dual of the received tuple \mathbf{u}^\perp , which is \mathbf{u} , is a codeword $\mathbf{v} \in \mathcal{C}$ that is corrupted by δ insertions and γ deletions. Therefore we can use the decoder from Section 4.3.3 on \mathbf{u} to perform list decoding of γ insertions and δ deletions up to

$$\delta + s\gamma < s(n_t - k + 1)$$

or the decoders from Sections 4.2 & 4.3.4 to perform probabilistic unique decoding up to

$$\delta + s\gamma \leq s(n_t - k).$$

The decoding steps can be summarized as follows:

1. Transmit a tuple $\mathbf{v}^\perp \in \mathcal{C}^\perp$ over a multishot operator channel with overall γ insertions and δ deletions.
2. Compute the dual of the received tuple $(\mathbf{u}^\perp)^\perp = \mathbf{u}$.
3. Use a decoder from Sections 4.3.3, 4.2 & 4.3.4 to recover the tuple $\mathbf{v} \in \mathcal{C}$.
4. Compute the dual tuple of \mathbf{v} to obtain \mathbf{v}^\perp .

4.5 Simulation Results

In order to verify the upper bound on the decoding failure probability in (33) we performed a Monte Carlo simulation (100 errors) of a code LILRS $[\beta, \mathbf{a}, \ell = 2, s = 3; \mathbf{n}_t = (3, 3), k = 3]$ over \mathbb{F}_{3^3} over a multishot operator channel with overall $\delta = 1$ deletion and $\gamma \in \{4, 5, 6\}$ insertions.

The channel realization is chosen uniformly at random from all possible realizations of the multishot operator channel with exactly this number of deletions and insertions (see Remark 2). The drawing procedure was implemented using the adapted dynamic-programming routine in Appendix C.

The results in Figure 6 show, that the upper bound in (33) gives a good estimate of the decoding failure probability. Although the heuristic upper bound from (32) looks tighter for the considered parameters, it is *not* a strict upper bound for the considered multishot operator channel. For the same parameters a (non-interleaved) lifted linearized Reed–Solomon code [4] (i.e. $s = 1$) can only correct γ insertions and δ deletions up to $\gamma + \delta < 4$.

5 Conclusion

5.1 Summary

We considered lifted s -interleaved linearized Reed–Solomon (LILRS) codes for error-control in noncoherent multishot network coding and showed that the relative overhead due to lifting can be reduced significantly compared to the construction by Martínez–Peñas–Kschischang. We proposed two decoding schemes for the multishot operator channel that are capable of correcting insertions and deletions beyond the unique decoding region in the sum-subspace metric.

We proposed an efficient interpolation-based decoding scheme for LILRS codes, which can be used as a list decoder or as a probabilistic unique decoder and can

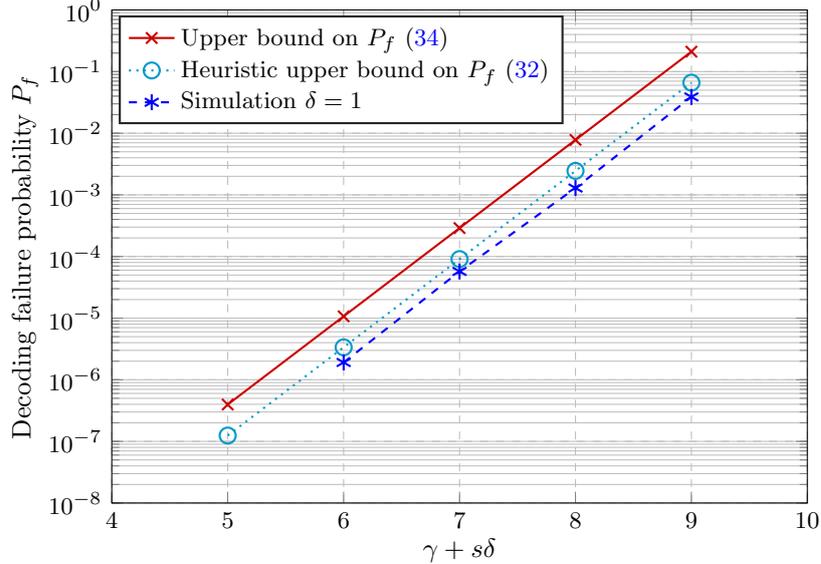


Fig. 6 Result of a Monte Carlo simulation of the code LILRS $[\beta, \mathbf{a}, \ell = 2, s = 3; \mathbf{n}_t = (3, 3), k = 3]$ over \mathbb{F}_{33} transmitted over a multishot operator channel with overall $\delta = 1$ deletions and $\gamma = 2, 3, 4, 5, 6$ insertions.

correct a total number of γ insertions and δ deletions up to $\gamma + s\delta < s(n_t - k + 1)$ and $\gamma + s\delta \leq s(n_t - k)$, respectively, where s is the interleaving order, n_t the sum of the dimensions of the transmitted spaces and k the dimension of the code. We derived a Loidreau–Overbeck-like decoder for LILRS codes, which provides arguments to upper bound on the decoding failure probability for the interpolation-based probabilistic unique decoder.

We showed how to construct and decode lifted *s-interleaved* linearized Reed–Solomon codes for error control in random linear multishot network coding. Compared to the construction by Martínez-Peñas–Kschischang, interleaving allows to increase the decoding region significantly (especially w.r.t. the number of insertions) and decreases the overhead due to the lifting (i.e., increases the code rate), at the cost of an increased packet size.

Up to our knowledge, the proposed decoding schemes are the first being able to correct errors beyond the unique decoding region in the sum-subspace metric efficiently. The tightness of the upper bounds on the decoding failure probability of the proposed decoding schemes for LILRS codes were validated via Monte Carlo simulations.

5.2 Remarks on Generality

In this paper, we considered codes constructed by skew polynomials with zero derivations, i.e. polynomials from $\mathbb{F}_{q^m}[x; \sigma]$, only. The main reason for this is that for operations in $\mathbb{F}_{q^m}[x; \sigma]$ we can give complexity bounds, which are of interest in the implementation point of view. However, the complexity analysis has to be performed w.r.t. this setup (computational complexity may be larger).

We considered decoding of *homogeneous* LILRS codes, respectively, i.e. interleaved codes where the component codes have the same code dimension. We consider these simpler code classes in order to not further complicate the quite involved notation. The decoding schemes proposed in this paper can be generalized to *heterogeneous* interleaved codes, where each component code may have a different dimension, in a straight-forward manner like e.g. in [45, 46]. Denote by k_1, \dots, k_s the dimensions of the component codes and define $\bar{k} := \frac{1}{s} \sum_{l=1}^s k_l$. The resulting decoding regions are then $\gamma + s\delta < s(n_t - \bar{k} + 1)$ for list decoding and $\gamma + s\delta \leq s(n_t - \bar{k})$ for probabilistic unique decoding.

5.3 Outlook & Future Work

For future work it would be interesting [to see the considered concepts](#) applied to interleaved LILRS codes that are based on the construction of linearized Reed–Solomon (LRS) codes over smaller fields. [Further, the investigation of lifted interleaved variants of other maximum sum-rank distance \(MSRD\) codes such as e.g. one-weight codes in the sum-rank metric and the application of subspace designs to sum-subspace-metric codes, is subject to future work.](#)

So far, no results on the list-decodability of random sum-subspace-metric codes, like e.g. for single-shot subspace codes [52], are available. Once such results are available it would be interesting to compare the list-decodability of random sum-subspace-metric codes with constructive results proposed in this paper.

6 Funding and Competing Interests

H. Bartz acknowledges the financial support by the Federal Ministry of Education and Research of Germany in the programme of “Souverän. Digital. Vernetzt.” Joint project 6G-RIC, project identification number: 16KISK022. The authors have no competing interests to declare that are relevant to the content of this article.

References

- [1] Ahlswede, R., Cai, N., Li, S.-Y., Yeung, R.W.: Network Information Flow. *IEEE Transactions on Information Theory* **46**(4), 1204–1216 (2000)
- [2] Koetter, R., Kschischang, F.R.: Coding for Errors and Erasures in Random Network Coding. *IEEE Transactions on Information Theory* **54**(8), 3579–3591 (2008)
- [3] Nóbrega, R.W., Uchôa-Filho, B.F.: Multishot Codes for Network Coding: Bounds and a Multilevel Construction. In: *IEEE International Symposium on Information Theory (ISIT)*, pp. 428–432 (2009)
- [4] Martínez-Peñas, U., Kschischang, F.R.: Reliable and Secure Multishot Network Coding using Linearized Reed-Solomon Codes. *IEEE Transactions on Information Theory* **65**(8), 4785–4803 (2019)

- [5] Krachkovsky, V.Y., Lee, Y.X.: Decoding for Iterative Reed–Solomon Coding Schemes. *IEEE Transactions on Magnetics* **33**(5), 2740–2742 (1997)
- [6] Bleichenbacher, D., Kiayias, A., Yung, M.: Decoding of Interleaved Reed Solomon Codes Over Noisy Data. In: *International Colloquium on Automata, Languages, and Programming*, pp. 97–108 (2003)
- [7] Coppersmith, D., Sudan, M.: Reconstructing Curves in Three (and Higher) Dimensional Space from Noisy Data. In: *ACM Symposium on the Theory of Computing* (2003)
- [8] Parvaresh, F., Vardy, A.: Multivariate Interpolation Decoding Beyond the Guruswami–Sudan Radius. In: *Allerton Conference on Communication, Control and Computing* (2004)
- [9] Brown, A., Minder, L., Shokrollahi, A.: Probabilistic Decoding of Interleaved RS-Codes on the q -Ary Symmetric Channel. In: *IEEE International Symposium on Information Theory (ISIT)*, pp. 326–326 (2004)
- [10] Parvaresh, F.: Algebraic List-Decoding of Error-Correcting Codes. PhD thesis, University of California, San Diego (2007)
- [11] Schmidt, G., Sidorenko, V., Bossert, M.: Enhancing the Correcting Radius of Interleaved Reed–Solomon Decoding Using Syndrome Extension Techniques. In: *IEEE International Symposium on Information Theory (ISIT)*, pp. 1341–1345 (2007)
- [12] Schmidt, G., Sidorenko, V.R., Bossert, M.: Collaborative Decoding of Interleaved Reed–Solomon Codes and Concatenated Code Designs. *IEEE Transactions on Information Theory* **55**(7), 2991–3012 (2009)
- [13] Cohn, H., Heninger, N.: Approximate Common Divisors via Lattices. *The Open Book Series* **1**(1), 271–293 (2013)
- [14] Nielsen, J.S.: Generalised Multi-Sequence Shift-Register Synthesis Using Module Minimisation. In: *IEEE International Symposium on Information Theory (ISIT)*, pp. 882–886 (2013)
- [15] Wachter-Zeh, A., Zeh, A., Bossert, M.: Decoding Interleaved Reed–Solomon Codes Beyond Their Joint Error-Correcting Capability. *Designs, Codes and Cryptography* **71**(2), 261–281 (2014)
- [16] Puchinger, S., Rosenkilde né Nielsen, J.: Decoding of Interleaved Reed–Solomon Codes Using Improved Power Decoding. In: *IEEE International Symposium on Information Theory (ISIT)* (2017)
- [17] Yu, J.-H., Loeliger, H.-A.: Simultaneous Partial Inverses and Decoding Interleaved

- Reed–Solomon Codes. *IEEE Transactions on Information Theory* **64**(12), 7511–7528 (2018)
- [18] Brown, A., Minder, L., Shokrollahi, A.: Improved Decoding of Interleaved AG Codes. In: *IMA International Conference on Cryptography and Coding*, pp. 37–46 (2005)
- [19] Kampf, S.: Bounds on Collaborative Decoding of Interleaved Hermitian Codes and Virtual Extension. *Designs, Codes and Cryptography* **70**(1-2), 9–25 (2014)
- [20] Puchinger, S., Rosenkilde, J., Bouw, I.: Improved Power Decoding of Interleaved One-Point Hermitian Codes. *Designs, Codes and Cryptography* **87**(2-3), 589–607 (2019)
- [21] Loidreau, P., Overbeck, R.: Decoding Rank Errors Beyond the Error-Correction Capability. *International Workshop on Algebraic and Combinatorial Coding Theory (ACCT)* (2006)
- [22] Sidorenko, V., Bossert, M.: Decoding Interleaved Gabidulin Codes and Multisequence Linearized Shift-Register Synthesis. In: *IEEE International Symposium on Information Theory (ISIT)*, pp. 1148–1152 (2010)
- [23] Sidorenko, V., Jiang, L., Bossert, M.: Skew-Feedback Shift-Register Synthesis and Decoding Interleaved Gabidulin Codes. *IEEE Transactions on Information Theory* **57**(2), 621–632 (2011)
- [24] Sidorenko, V., Bossert, M.: Fast Skew-Feedback Shift-Register Synthesis. *Designs, Codes and Cryptography* **70**(1-2), 55–67 (2014)
- [25] Wachter-Zeh, A., Zeh, A.: List and Unique Error-Erasure Decoding of Interleaved Gabidulin Codes with Interpolation Techniques. *Designs, Codes and Cryptography* **73**(2), 547–570 (2014)
- [26] Puchinger, S., Rosenkilde né Nielsen, J., Li, W., Sidorenko, V.: Row Reduction Applied to Decoding of Rank-Metric and Subspace Codes. *Designs, Codes and Cryptography* **82**(1-2), 389–409 (2017)
- [27] Puchinger, S., Muelich, S., Mödinger, D., Rosenkilde né Nielsen, J., Bossert, M.: Decoding Interleaved Gabidulin Codes Using Alekhovich’s Algorithm. *Electronic Notes in Discrete Mathematics* **57**, 175–180 (2017)
- [28] Bartz, H., Jerkovits, T., Puchinger, S., Rosenkilde, J.: Fast Decoding of Codes in the Rank, Subspace, and Sum-Rank Metric. *IEEE Transactions on Information Theory*, 1–1 (2021) <https://doi.org/10.1109/TIT.2021.3067318>
- [29] Silva, D., Kschischang, F.R., Koetter, R.: A Rank-Metric Approach to Error Control in Random Network Coding. *IEEE Transactions on Information Theory*

54(9), 3951–3967 (2008)

- [30] Bartz, H., Wachter-Zeh, A.: Efficient Decoding of Interleaved Subspace and Gabidulin Codes Beyond their Unique Decoding Radius using Gröbner Bases. *Advances in Mathematics of Communications* **12**(4), 773 (2018)
- [31] Bartz, H., Puchinger, S.: Fast Decoding of Interleaved Linearized Reed-Solomon Codes and Variants. arXiv preprint arXiv:2201.01339 (2022)
- [32] Bartz, H., Puchinger, S.: Decoding of Interleaved Linearized Reed-Solomon Codes with Applications to Network Coding. In: *IEEE International Symposium on Information Theory (ISIT)* (2021)
- [33] Nóbrega, R.W., Uchôa-Filho, B.F.: Multishot Codes for Network Coding using Rank-Metric Codes. In: *2010 Third IEEE International Workshop on Wireless Network Coding*, pp. 1–6 (2010)
- [34] Ore, O.: Theory of Non-Commutative Polynomials. *Annals of Mathematics*, 480–508 (1933)
- [35] Martínez-Peñas, U.: Skew and Linearized Reed–Solomon Codes and Maximum Sum Rank Distance Codes over any Division Ring. *Journal of Algebra* **504**, 587–612 (2018)
- [36] Lam, T.-Y., Leroy, A.: Vandermonde and Wronskian Matrices over Division Rings. *Journal of Algebra* **119**(2), 308–336 (1988)
- [37] Leroy, A., *et al.*: Pseudolinear Transformations and Evaluation in Ore Extensions. *Bulletin of the Belgian Mathematical Society-Simon Stevin* **2**(3), 321–347 (1995)
- [38] Martínez-Peñas, U.: Private Information Retrieval from Locally Repairable Databases with Colluding Servers. In: *IEEE International Symposium on Information Theory (ISIT)*, pp. 1057–1061 (2019)
- [39] Lam, T.-Y.: *A General Theory of Vandermonde Matrices*. Center for Pure and Applied Mathematics, University of California, Berkeley, California (1985)
- [40] Caruso, X.: Residues of Skew Rational Functions and Linearized Goppa Codes. arXiv preprint arXiv:1908.08430 (2019)
- [41] Silva, D.: Error Control for Network Coding. PhD thesis, University of Toronto, Toronto, Canada (2009)
- [42] Puchinger, S., Renner, J., Rosenkilde, J.: Generic Decoding in the Sum-Rank Metric. arXiv preprint arXiv:2001.04812 (2020)
- [43] Overbeck, R.: Public Key Cryptography based on Coding Theory. PhD thesis, Technische Universität (2007)

- [44] Overbeck, R.: Structural Attacks for Public Key Cryptosystems based on Gabidulin Codes. *Journal of Cryptology* **21**(2), 280–301 (2008)
- [45] Wachter-Zeh, A.: Decoding of Block and Convolutional Codes in Rank Metric. PhD thesis, Ulm University and Université Rennes 1 (2013)
- [46] Bartz, H.: Algebraic Decoding of Subspace and Rank-Metric Codes. PhD thesis, Technische Universität München (2017)
- [47] Puchinger, S., Wachter-Zeh, A.: Sub-Quadratic Decoding of Gabidulin Codes. In: *IEEE International Symposium on Information Theory (ISIT)*, pp. 2554–2558 (2016)
- [48] Caruso, X., Le Borgne, J.: Fast Multiplication for Skew Polynomials. In: *International Symposium on Symbolic and Algebraic Computation (ISSAC)* (2017)
- [49] Liu, S., Manganiello, F., Kschischang, F.R.: Kötter Interpolation in Skew Polynomial Rings. *Designs, Codes and Cryptography* **72**(3), 593–608 (2014)
- [50] Bartz, H., Jerkovits, T., Rosenkilde, J.: Fast Kötter–Nielsen–Høholdt Interpolation over Skew Polynomial Rings and its Application in Coding Theory (2021)
- [51] Hörmann, F., Bartz, H.: Interpolation-Based Decoding of Folded Variants of Linearized and Skew Reed–Solomon Codes. *Designs, Codes and Cryptography* (2023) <https://doi.org/10.1007/s10623-023-01214-8>
- [52] Ding, Y.: On List-Decodability of Random Rank Metric Codes and Subspace Codes. *IEEE Transactions on Information Theory* **61**(1), 51–59 (2014)

Appendix A Proofs from Section 4.2

A.1 Proof of Lemma 2

Proof. First observe, that every received space $\mathcal{U}^{(i)}$ can be represented by a matrix $\tilde{\mathcal{U}}^{(i)}$ of the form

$$\tilde{\mathcal{U}}^{(i)} = \left(\begin{array}{c|ccc} \boldsymbol{\xi}_1^{(i)\top} & \mathbf{x}_1^{(i)\top} & \dots & \mathbf{x}_s^{(i)\top} \\ \boldsymbol{\xi}_2^{(i)\top} & \tilde{\mathbf{u}}_1^{(i)\top} & \dots & \tilde{\mathbf{u}}_s^{(i)\top} \\ \mathbf{0} & \hat{\mathbf{e}}_1^{(i)\top} & \dots & \hat{\mathbf{e}}_s^{(i)\top} \end{array} \right) \in \mathbb{F}_{q^m}^{n_t^{(i)} \times (s+1)}$$

such that first $n_t^{(i)} - \delta^{(i)}$ rows of $\tilde{\mathcal{U}}^{(i)}$ form a basis for the non-corrupted received space $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$ whereas the last $t^{(i)} + \boldsymbol{\varkappa}^{(i)} = \gamma^{(i)}$ rows correspond to the insertions for all

$i = 1, \dots, \ell$, i.e. we have that

$$\begin{aligned} \left\langle \left(\xi_1^{(i)\top} \mid \mathbf{x}_1^{(i)\top} \dots \mathbf{x}_s^{(i)\top} \right) \right\rangle_q &= \mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}, \\ \left\langle \left(\begin{array}{c|ccc} \xi_2^{(i)\top} & \tilde{\mathbf{u}}_1^{(i)\top} & \dots & \tilde{\mathbf{u}}_s^{(i)\top} \\ \mathbf{0} & \hat{\mathbf{e}}_1^{(i)\top} & \dots & \hat{\mathbf{e}}_s^{(i)\top} \end{array} \right) \right\rangle_q &= \mathcal{E}^{(i)} \end{aligned} \quad (\text{A1})$$

for all $i = 1, \dots, \ell$ where $\xi_1^{(i)}, \mathbf{x}_j^{(i)} \in \mathbb{F}_{q^m}^{n_t^{(i)} - \delta^{(i)}}$, $\xi_2^{(i)}, \tilde{\mathbf{u}}_j^{(i)} \in \mathbb{F}_{q^m}^{t^{(i)}}$ and $\hat{\mathbf{E}}^{(i)} = (\hat{\mathbf{e}}_1^{(i)\top} \dots \hat{\mathbf{e}}_s^{(i)\top}) \in \mathbb{F}_{q^m}^{\varkappa^{(i)} \times s}$.

Since the rows of the matrices $\mathbf{U}^{(i)}$ also form a basis of $\mathcal{U}^{(i)}$, there exist invertible matrices $\mathbf{W}^{(i)} \in \mathbb{F}_q^{n_r^{(i)}}$ such that

$$\left(\mathbf{W}^{(i)} \right)^\top \mathbf{U}^{(i)} = \tilde{\mathbf{U}}^{(i)}, \quad \forall i = 1, \dots, \ell.$$

Since $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$ is an $n_t^{(i)} - \delta^{(i)}$ -dimensional subspace $\mathcal{V}^{(i)}$, we have by the definition of LILRS codes that $\text{rk}_q(\xi_1^{(i)}) = n_t^{(i)} - \delta^{(i)}$ for all $i = 1, \dots, \ell$. We also have that $\text{rk}_q(\hat{\mathbf{E}}^{(i)}) = \varkappa^{(i)}$ for all $i = 1, \dots, \ell$ since otherwise the matrix $\tilde{\mathbf{U}}^{(i)}$ would not have full \mathbb{F}_q -rank $n_r^{(i)}$, which contradicts that it is a basis for $\mathcal{U}^{(i)}$ that is obtained from $\mathbf{U}^{(i)}$ via \mathbb{F}_q -elementary row operations. Since $\mathcal{V}^{(i)} \cap \mathcal{E}^{(i)} = \{\mathbf{0}\}$, we have that $\tilde{\mathbf{u}}_l^{(i)} \neq f_1(\xi_2^{(i)})_{a_i}$ for all $i = 1, \dots, \ell$ and $l = 1, \dots, s$, since otherwise there would exist a vector such that, for some j , we have

$$\left\langle \left(\xi_{2,j}^{(i)} \mid f_1(\xi_{2,j}^{(i)})_{a_i} \dots f_1(\xi_{2,j}^{(i)})_{a_i} \right) \right\rangle_q \in \mathcal{V}^{(i)}$$

which contradicts that $\mathcal{V}^{(i)} \cap \mathcal{E}^{(i)} = \{\mathbf{0}\}$. We now show that

$$\text{rk}_q(\xi_2^{(i)}) = t^{(i)}, \quad \text{and} \quad \text{rk}_q((\xi_1^{(i)}, \xi_2^{(i)})) = n_t^{(i)} - \delta^{(i)} + t^{(i)}, \quad \forall i = 1, \dots, \ell.$$

Suppose that $\text{rk}_q(\xi_2^{(i)}) < t^{(i)}$. Then there exists an \mathbb{F}_q -linear combination of the rows of the matrix in (A1) such that the first entry of the vector becomes zero. In this case it would belong to the lower part of the matrix having a leading zero. If $\text{rk}_q((\xi_1^{(i)}, \xi_2^{(i)})) < n_t^{(i)} - \delta^{(i)} + t^{(i)}$, then there exist an element $\xi_{2,j}$ such that $\langle \xi_{2,j} \rangle_q \subseteq \langle \xi_1^\top \rangle_q$. In this case we can subtract the corresponding \mathbb{F}_q -linear combination from the row having $\xi_{2,j}$ as first entry, which again yields a row with leading zero element.

Using the matrices $\tilde{\mathbf{U}}^{(i)}$ for all $i = 1, \dots, \ell$ and performing \mathbb{F}_{q^m} -linear row operations, we can set up a matrix of the form

$$\bar{\mathbf{L}} = \left(\mathbf{L}^{(1)}, \dots, \mathbf{L}^{(\ell)} \right) := \begin{pmatrix} \Lambda_{n_t - \delta - 1}(\bar{\boldsymbol{\xi}})_{\mathbf{a}} \\ \Lambda_{n_t - \delta - k}(\bar{\mathbf{u}}_1)_{\mathbf{a}} \\ \vdots \\ \Lambda_{n_t - \delta - k}(\bar{\mathbf{u}}_s)_{\mathbf{a}} \end{pmatrix} \in \mathbb{F}_{q^m}^{((s+1)(n_t - \delta) - sk - 1) \times n_r}$$

where

$$\begin{aligned} \bar{\boldsymbol{\xi}} &= \left((\boldsymbol{\xi}_1^{(1)} \mid \boldsymbol{\xi}_2^{(1)} \mid \mathbf{0}) \mid \dots \mid (\boldsymbol{\xi}_1^{(\ell)} \mid \boldsymbol{\xi}_2^{(\ell)} \mid \mathbf{0}) \right), \\ \bar{\mathbf{u}}_l &= \left((\mathbf{0} \mid \tilde{\mathbf{e}}_l^{(1)} \mid \hat{\mathbf{e}}_l^{(1)}) \mid \dots \mid (\mathbf{0} \mid \tilde{\mathbf{e}}_l^{(\ell)} \mid \hat{\mathbf{e}}_l^{(\ell)}) \right), \quad \forall l = 1, \dots, s. \end{aligned}$$

The component matrices are then of the form

$$\bar{\mathbf{L}}^{(i)} = \begin{pmatrix} \mathbf{V}_{n_t - \delta - 1}(\bar{\boldsymbol{\xi}}_1^{(i)})_{a_i} & \mathbf{V}_{n_t - \delta - 1}(\bar{\boldsymbol{\xi}}_2^{(i)})_{a_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{n_t - \delta - k}(\tilde{\mathbf{e}}_1^{(i)})_{a_i} & \mathbf{V}_{n_t - \delta - k}(\hat{\mathbf{e}}_1^{(i)})_{a_i} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{V}_{n_t - \delta - k}(\tilde{\mathbf{e}}_s^{(i)})_{a_i} & \mathbf{V}_{n_t - \delta - k}(\hat{\mathbf{e}}_s^{(i)})_{a_i} \end{pmatrix} \in \mathbb{F}_{q^m}^{((s+1)(n_t - \delta) - sk - 1) \times n_r^{(i)}}$$

where $\mathbf{x}_l^{(i)} - f_l(\boldsymbol{\xi}_1^{(i)})_{a_i} = \mathbf{0}$ (yielding the zero matrices in the leftmost block) and $\tilde{\mathbf{e}}_l^{(i)} = \tilde{\mathbf{u}}_l^{(i)} - f_l(\boldsymbol{\xi}_2^{(i)})_{a_i}$ for all $l = 1, \dots, s$ and $i = 1, \dots, \ell$. Define the matrices

$$\tilde{\mathbf{E}}^{(i)} = \begin{pmatrix} \tilde{\mathbf{e}}_1^{(i)} \\ \vdots \\ \tilde{\mathbf{e}}_s^{(i)} \end{pmatrix} \in \mathbb{F}_{q^m}^{s \times t^{(i)}}, \quad \hat{\mathbf{E}}^{(i)} = \begin{pmatrix} \hat{\mathbf{e}}_1^{(i)} \\ \vdots \\ \hat{\mathbf{e}}_s^{(i)} \end{pmatrix} \in \mathbb{F}_{q^m}^{s \times \varkappa^{(i)}}$$

for all $i = 1, \dots, \ell$. Suppose that the \mathbb{F}_q -rank of $\tilde{\mathbf{E}}^{(i)}$ is less than $t^{(i)}$. Then there exist a nontrivial \mathbb{F}_q -linear combination of the columns of the matrix

$$\begin{pmatrix} \mathbf{V}_{n_t - \delta - 1}(\bar{\boldsymbol{\xi}}_2^{(i)})_{a_i} \\ \mathbf{V}_{n_t - \delta - k}(\tilde{\mathbf{e}}_1^{(i)})_{a_i} \\ \vdots \\ \mathbf{V}_{n_t - \delta - k}(\tilde{\mathbf{e}}_s^{(i)})_{a_i} \end{pmatrix} \in \mathbb{F}_{q^m}^{((s+1)(n_t - \delta) - sk - 1) \times t^{(i)}}$$

such that the first (upper) $(n_t - \delta - 1)$ rows are nonzero and the last (lower) $s(n_t - \delta - k)$ rows are all zero. This contradicts that $\mathcal{V}^{(i)} \cap \mathcal{E}^{(i)} = \{\mathbf{0}\}$ since each $\bar{\mathbf{L}}^{(i)}$ is obtained by subtracting the evaluations of each f_l at the corresponding values in $\boldsymbol{\xi}_1^{(i)}$ and $\boldsymbol{\xi}_2^{(i)}$ (and the corresponding row-operator powers thereof). Now assume that $\text{rk}_q(\tilde{\mathbf{E}}^{(i)} \mid \hat{\mathbf{E}}^{(i)}) < \gamma^{(i)}$ for some $i \in \{1, \dots, \ell\}$. Then there exist nontrivial \mathbb{F}_q -linear combinations of the

$t^{(i)} + \varkappa^{(i)} = \gamma^{(i)}$ rightmost columns of $\bar{\mathbf{L}}^{(i)}$ such that the $s(n_t - \delta - k)$ rows are all zero and the upper $n_t - \delta - 1$ rows are nonzero. This contradicts that $\mathcal{V}^{(i)} \cap \mathcal{E}^{(i)} = \{\mathbf{0}\}$ by the same argument as above.

Since $\bar{\mathbf{L}}$ is obtained from \mathbf{L} via \mathbb{F}_{q^m} -elementary row operations and \mathbb{F}_q -elementary column operations, we have that

$$\text{rk}_{q^m}(\mathbf{L}) = \text{rk}_{q^m}(\bar{\mathbf{L}}).$$

□

A.2 Proof of Lemma 3

Proof. – Ad 1): By column permutations, i.e. by rearranging the columns of $\bar{\mathbf{L}}$ according to the vectors

$$\begin{aligned} \boldsymbol{\xi}_1 &:= \left(\boldsymbol{\xi}_1^{(1)} \mid \boldsymbol{\xi}_1^{(2)} \mid \cdots \mid \boldsymbol{\xi}_1^{(\ell)} \right) \in \mathbb{F}_{q^m}^{n_t - \delta}, & \boldsymbol{\xi}_2 &:= \left(\boldsymbol{\xi}_2^{(1)} \mid \boldsymbol{\xi}_2^{(2)} \mid \cdots \mid \boldsymbol{\xi}_2^{(\ell)} \right) \in \mathbb{F}_{q^m}^t, \\ \tilde{\mathbf{e}}_l &:= \left(\tilde{\mathbf{e}}_l^{(1)} \mid \tilde{\mathbf{e}}_l^{(2)} \mid \cdots \mid \tilde{\mathbf{e}}_l^{(\ell)} \right) \in \mathbb{F}_{q^m}^t, & \hat{\mathbf{e}}_l &:= \left(\hat{\mathbf{e}}_l^{(1)} \mid \hat{\mathbf{e}}_l^{(2)} \mid \cdots \mid \hat{\mathbf{e}}_l^{(\ell)} \right) \in \mathbb{F}_{q^m}^\varkappa, \end{aligned}$$

we obtain another (equivalent) matrix

$$\tilde{\mathbf{L}} := \left(\begin{array}{c|c|c} \Lambda_{n_t - \delta - 1}(\boldsymbol{\xi}_1)_{\mathbf{a}} & \Lambda_{n_t - \delta - 1}(\boldsymbol{\xi}_2)_{\mathbf{a}} & \mathbf{0} \\ \mathbf{0} & \Lambda_{n_t - \delta - k}(\tilde{\mathbf{E}})_{\mathbf{a}} & \Lambda_{n_t - \delta - k}(\hat{\mathbf{E}})_{\mathbf{a}} \end{array} \right)$$

where

$$\tilde{\mathbf{E}} = \left(\tilde{\mathbf{E}}^{(1)} \mid \cdots \mid \tilde{\mathbf{E}}^{(\ell)} \right) \in \mathbb{F}_{q^m}^{s \times t} \quad \text{and} \quad \hat{\mathbf{E}} = \left(\hat{\mathbf{E}}^{(1)} \mid \cdots \mid \hat{\mathbf{E}}^{(\ell)} \right) \in \mathbb{F}_{q^m}^{s \times \varkappa}$$

such that $\text{rk}_{q^m}(\mathbf{L}) = \text{rk}_{q^m}(\bar{\mathbf{L}}) = \text{rk}_{q^m}(\tilde{\mathbf{L}})$. By Lemma 2 we have that

$$\text{wt}_{\Sigma R}(\tilde{\mathbf{E}}) = t, \quad \text{wt}_{\Sigma R}(\hat{\mathbf{E}}) = \varkappa$$

and

$$\text{wt}_{\Sigma R} \left(\left(\tilde{\mathbf{E}}^{(1)} \mid \hat{\mathbf{E}}^{(1)} \right) \mid \cdots \mid \left(\tilde{\mathbf{E}}^{(\ell)} \mid \hat{\mathbf{E}}^{(\ell)} \right) \right) = t + \varkappa = \gamma.$$

Defining the matrix

$$\mathbf{Z} := \left(\Lambda_{n_t - \delta - k}(\tilde{\mathbf{E}})_{\mathbf{a}} \mid \Lambda_{n_t - \delta - k}(\hat{\mathbf{E}})_{\mathbf{a}} \right) \in \mathbb{F}_{q^m}^{s(n_t - \delta - k) \times \gamma}$$

we can write $\tilde{\mathbf{L}}$ as a block matrix of the form

$$\tilde{\mathbf{L}} = \left(\begin{array}{c|c} \Lambda_{n_t - \delta - 1}(\boldsymbol{\xi}_1)_{\mathbf{a}} & \star \\ \mathbf{0} & \mathbf{Z} \end{array} \right),$$

where \star denotes an arbitrary matrix, and thus we have that $\text{rk}_{q^m}(\tilde{\mathbf{L}}) = \text{rk}_{q^m}(\Lambda_{n_t-\delta-1}(\boldsymbol{\xi}_1)_{\mathbf{a}}) + \text{rk}_{q^m}(\mathbf{Z})$. By Lemma 2 the vector $\boldsymbol{\xi}_1 \in \mathbb{F}_{q^m}^{n_t-\delta}$ has sum-rank $n_t - \delta$. Recall that we have

$$\text{rk}_{q^m}(\Lambda_{n_t-\delta-1}(\boldsymbol{\xi}_1)_{\mathbf{a}}) = \min\{n_t - \delta - 1, n_t - \delta\} = n_t - \delta - 1.$$

The matrix \mathbf{Z} can be obtained from $\bar{\mathbf{Z}}$ via column permutations, which implies that $\text{rk}_{q^m}(\mathbf{Z}) = \text{rk}_{q^m}(\bar{\mathbf{Z}})$. Since the \mathbb{F}_{q^m} -rank of $\bar{\mathbf{Z}}$ (and thus also of \mathbf{Z}) equals γ , we conclude by the upper block triangular structure of $\tilde{\mathbf{L}}$ that

$$\text{rk}_{q^m}(\tilde{\mathbf{L}}) = \text{rk}_{q^m}(\mathbf{L}) = n_t - \delta - 1 + \gamma = n_r - 1.$$

– Ad 2): Let $\bar{\mathbf{h}} = \left(\mathbf{h}^{(1)} \mid \mathbf{h}^{(2)} \mid \dots \mid \mathbf{h}^{(\ell)} \right) \in \ker_r(\bar{\mathbf{L}}) \setminus \{\mathbf{0}\}$ be a nonzero element in the right kernel of $\bar{\mathbf{L}}$. If the \mathbb{F}_{q^m} -rank of $\bar{\mathbf{Z}}$ equals γ , we have that

$$\text{rk}_{q^m}(\bar{\mathbf{Z}}^{(i)}) = \gamma^{(i)}$$

for all $i = 1, \dots, \ell$. Thus, for any $\bar{\mathbf{h}} \in \ker_r(\bar{\mathbf{L}}) \setminus \{\mathbf{0}\}$ we must have that the last (rightmost) $\gamma^{(i)}$ entries of each $\bar{\mathbf{h}}^{(i)}$ must be zero, which implies that

$$\text{rk}_q(\bar{\mathbf{h}}^{(i)}) \leq n_t^{(i)} - \delta^{(i)}, \quad \forall i = 1, \dots, \ell. \quad (\text{A2})$$

Let \mathcal{C} be the $[n_t - \delta + t, n_t - \delta - 1]$ code generated by $(\Lambda_{n_t-\delta-1}(\boldsymbol{\xi}_1)_{\mathbf{a}} \mid \Lambda_{n_t-\delta-1}(\boldsymbol{\xi}_2)_{\mathbf{a}})$ and let \mathcal{C}' be the $[n_r, n_t - \delta - 1]$ code generated by $(\Lambda_{n_t-\delta-1}(\boldsymbol{\xi}_1)_{\mathbf{a}} \mid \Lambda_{n_t-\delta-1}(\boldsymbol{\xi}_2)_{\mathbf{a}} \mid \mathbf{0})$. Since $(\Lambda_{n_t-\delta-1}(\boldsymbol{\xi}_1)_{\mathbf{a}} \mid \Lambda_{n_t-\delta-1}(\boldsymbol{\xi}_2)_{\mathbf{a}})$ is a generator matrix of an $[n_t - \delta + t, n_t - \delta - 1]$ LRS code (up to column permutations), we have that the dual code \mathcal{C}^\perp is equivalent to an LRS code with minimum sum-rank distance $n_t - \delta$. The dual code of \mathcal{C}' is an $[n_r, \gamma + 1]$ code \mathcal{C}'^\perp that can be decomposed into

$$\begin{aligned} \mathcal{C}'_1{}^\perp &= \{(\mathbf{c} \mid \mathbf{0}) : \mathbf{c} \in \mathcal{C}^\perp\} \\ \mathcal{C}'_2{}^\perp &= \{(\mathbf{0} \mid \tilde{\mathbf{c}}) : \tilde{\mathbf{c}} \in \mathbb{F}_{q^m}^\gamma\} \end{aligned}$$

such that $\mathcal{C}'^\perp = \mathcal{C}'_1{}^\perp + \mathcal{C}'_2{}^\perp$ and $\mathcal{C}'_1{}^\perp \cap \mathcal{C}'_2{}^\perp = \{\mathbf{0}\}$. Therefore, we have that the minimum distance of the dual code \mathcal{C}'^\perp is $n_t - \delta$.

The first $n_t - \delta - 1$ rows of $\bar{\mathbf{L}}$ span code $\bar{\mathcal{C}}$ which is equivalent to \mathcal{C}' and thus we have that the minimum sum-rank distance of the dual code $\bar{\mathcal{C}}^\perp$ equals $n_t - \delta$. Since any nonzero element $\bar{\mathbf{h}}$ in the right kernel of $\bar{\mathbf{L}}$ is contained in $\bar{\mathcal{C}}^\perp$, we have that

$$\text{wt}_{\Sigma R}(\bar{\mathbf{h}}) \geq n_t - \delta. \quad (\text{A3})$$

Therefore, (A2) and (A3) are satisfied only if

$$\text{rk}_q(\bar{\mathbf{h}}^{(i)}) = n_t^{(i)} - \delta^{(i)}, \quad \forall i = 1, \dots, \ell.$$

This proves the statement, since by Lemma 2 we have that $\ker_r(\mathbf{L}) = \ker_r(\overline{\mathbf{L}}\mathbf{W}^{-1})$ and thus

$$\overline{\mathbf{h}} \in \ker_r(\overline{\mathbf{L}}) \iff \overline{\mathbf{h}}(\mathbf{W}^{-1})^\top \in \ker_r(\mathbf{L}).$$

- Ad 3): Expand $\mathbf{h}^{(i)}$ into an $m \times n_r^{(i)}$ matrix over \mathbb{F}_q , which has \mathbb{F}_q -rank $n_t^{(i)} - \delta^{(i)}$ by 2). Then, we can perform elementary column operations on this matrix to bring it into reduced column echelon form, where the $n_t^{(i)} - \delta^{(i)}$ nonzero columns are the leftmost ones. The matrix $\mathbf{T}^{(i)} \in \mathbb{F}_q^{n_r^{(i)} \times n_r^{(i)}}$ is then chosen to be the matrix that, by multiplication from the right, performs the used sequence of elementary column operations. Note that the $n_t^{(i)} - \delta^{(i)}$ nonzero entries of $\mathbf{h}^{(i)}\mathbf{T}^{(i)}$ are linearly independent over \mathbb{F}_q .
- Ad 4): The matrices $\mathbf{T}^{(i)} \in \mathbb{F}_q^{n_r^{(i)} \times n_r^{(i)}}$ are chosen such that the rightmost $\gamma^{(i)}$ positions of $\mathbf{h}^{(i)}\mathbf{T}^{(i)}$ are zero. Recall from 2), that if $\text{rk}_{q^m}(\overline{\mathbf{Z}}) = \gamma$, we have that for any nonzero element $\overline{\mathbf{h}} = (\overline{\mathbf{h}}^{(1)} \mid \dots \mid \overline{\mathbf{h}}^{(\ell)})$ in the right kernel of $\overline{\mathbf{L}}$ the $\gamma^{(i)}$ rightmost positions of each $\overline{\mathbf{h}}^{(i)}$ are zero. Define the matrices $\mathbf{D}^{(i)} = \left(\mathbf{T}^{(i)-1}\right)^\top$ for all $i = 1, \dots, \ell$. Then we have that

$$\mathbf{h}\mathbf{T} \in \ker_r(\mathbf{L} \cdot \text{diag}(\mathbf{D}^{(1)}, \dots, \mathbf{D}^{(\ell)})).$$

We will now show that the column span of the matrices

$$\left[\overline{\mathbf{L}}^{(i)}\right]_{\{n_t^{(i)} - \delta^{(i)} + 1, \dots, n_r^{(i)}\}} \quad \text{and} \quad \left[\mathbf{L}^{(i)}\mathbf{D}^{(i)}\right]_{\{n_t^{(i)} - \delta^{(i)} + 1, \dots, n_r^{(i)}\}}$$

must be the same. Since $\text{rk}_{q^m}(\overline{\mathbf{Z}}) = \gamma$ implies $\text{rk}_{q^m}(\overline{\mathbf{Z}}^{(i)}) = \gamma^{(i)}$ for all $i = 1, \dots, \ell$, the $s(n_t - \delta - k)$ last (lower) rows of $\mathbf{L}^{(i)}\mathbf{D}^{(i)}$ have \mathbb{F}_{q^m} -rank $\gamma^{(i)}$ for all $i = 1, \dots, \ell$. Now assume that the column span of the matrices above are not the same for some $i \in \{1, \dots, \ell\}$. Then there exist \mathbb{F}_{q^m} -linearly independent columns for some index $j \in \{1, \dots, n_t^{(i)} - \delta^{(i)}\}$, which contradicts that $\mathbf{h}\mathbf{T}$ (which has the $\gamma^{(i)}$ rightmost entries per block equal to zero) is in the right kernel of $\mathbf{L} \cdot \text{diag}(\mathbf{D}^{(1)}, \dots, \mathbf{D}^{(\ell)})$. Therefore, we must have that the column space is the same.

Since the $\gamma^{(i)}$ rightmost columns of each $\overline{\mathbf{L}}^{(i)}$ correspond to the insertions, we have that the rightmost $\gamma^{(i)}$ columns of $\mathbf{L}^{(i)}\mathbf{D}^{(i)}$ also correspond to the insertions. Since \mathbf{L} is set up using the matrices $\mathbf{U}^{(i)}$ for all $i = 1, \dots, \ell$, we have that the last (lower) $\gamma^{(i)}$ rows of $\left(\mathbf{T}^{(i)}\right)^{-1}\mathbf{U}^{(i)}$ span the error space $\mathcal{E}^{(i)}$. This implies that the first $n_t^{(i)} - \delta^{(i)}$ rows of $\left(\mathbf{T}^{(i)}\right)^{-1}\mathbf{U}^{(i)}$ form a basis of the noncorrupted space $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$, since by the definition of the operator channel we have that $\mathcal{V}^{(i)} \cap \mathcal{E}^{(i)} = \{\mathbf{0}\}$ for all $i = 1, \dots, \ell$.

- Ad 5): By 4) the first $n_t^{(i)} - \delta^{(i)}$ rows of

$$\hat{\mathbf{U}}^{(i)} = \left(\mathbf{T}^{(i)}\right)^{-1}\mathbf{U}^{(i)} = \left(\hat{\boldsymbol{\xi}}^{(i)\top} \hat{\mathbf{u}}_1^{(i)\top} \dots \hat{\mathbf{u}}_s^{(i)\top}\right) \in \mathbb{F}_{q^m}^{n_r^{(i)} \times (s+1)}.$$

form a basis for the noncorrupted subspace $\mathcal{U}^{(i)} \cap \mathcal{V}^{(i)}$, i.e. we have that

$$\hat{u}_{l,\mu}^{(i)} = f_l(\hat{\xi}^{(i)})_{a_i}, \quad \forall \mu = 1, \dots, n_t^{(i)} - \delta^{(i)}, \forall l = 1, \dots, s, \forall i = 1, \dots, \ell.$$

Therefore, the message polynomials $f_1, \dots, f_s \in \mathbb{F}_{q^m}[x; \sigma]_{<k}$ can be reconstructed via Lagrange interpolation (cf. Lemma 1). \square

Appendix B Proofs from Section 4.4

B.1 Proof of Proposition 7

Proof. The proof proceed similar as the proof of [30, Theorem 4]. At the output of the operator channel we have

$$\mathbf{u}^\perp = \mathcal{H}_{N-n_t-\delta}(\mathbf{v}^\perp) \oplus \mathcal{E}_\gamma$$

where $\dim_\Sigma(\mathcal{E}_\gamma) = \gamma$. Let γ and δ denote the partition of the insertions and deletions, respectively.

For all $i = 1, \dots, \ell$ we have that

$$\begin{aligned} \mathcal{U}^{(i)} &= \left((\mathcal{U}^{(i)})^\perp \right)^\perp = \left(\mathcal{H}_{N_i-n_t^{(i)}-\delta^{(i)}} \left((\mathcal{V}^{(i)})^\perp \right) \oplus \mathcal{E}_{\gamma^{(i)}}^{(i)} \right)^\perp \\ &= \left(\mathcal{H}_{N_i-n_t^{(i)}-\delta^{(i)}} \left((\mathcal{V}^{(i)})^\perp \right) \right)^\perp \cap (\mathcal{E}_{\gamma^{(i)}}^{(i)})^\perp. \end{aligned}$$

By [30, Lemma 14] we have that

$$\left(\mathcal{H}_{N_i-n_t^{(i)}-\delta^{(i)}} \left((\mathcal{V}^{(i)})^\perp \right) \right)^\perp = \mathcal{V}^{(i)} \oplus \mathcal{E}_{\delta^{(i)}}^{(i)} \quad (\text{B4})$$

where $\dim(\mathcal{E}_{\delta^{(i)}}^{(i)}) = \delta^{(i)}$ and therefore

$$\begin{aligned} \mathcal{U}^{(i)} &= \left(\mathcal{V}^{(i)} \oplus \mathcal{E}_{\delta^{(i)}}^{(i)} \right) \cap (\mathcal{E}_{\gamma^{(i)}}^{(i)})^\perp \\ &= \left(\mathcal{V}^{(i)} \cap (\mathcal{E}_{\gamma^{(i)}}^{(i)})^\perp \right) \oplus \mathcal{E}_{\delta^{(i)}}^{(i)} \end{aligned} \quad (\text{B5})$$

for all $i = 1, \dots, \ell$ since $\mathcal{E}_{\delta^{(i)}}^{(i)} \cap (\mathcal{E}_{\gamma^{(i)}}^{(i)})^\perp = \mathcal{E}_{\delta^{(i)}}^{(i)}$. From the operator channel we have that

$$(\mathcal{V}^{(i)})^\perp \cap \mathcal{E}_{\gamma^{(i)}}^{(i)} = 0 \quad \implies \quad (\mathcal{V}^{(i)})^\perp \subseteq (\mathcal{E}_{\gamma^{(i)}}^{(i)})^\perp$$

since $(\mathcal{V}^{(i)})^\perp \oplus \mathcal{V}^{(i)} = \mathbb{F}_q^{N_i}$ and $\dim((\mathcal{V}^{(i)})^\perp) \leq \dim((\mathcal{E}_{\gamma^{(i)}}^{(i)})^\perp)$. From (B4) we get $\mathcal{V}^{(i)} \cap \mathcal{E}_{\delta^{(i)}}^{(i)} = 0$ and therefore

$$\mathcal{E}_{\delta^{(i)}}^{(i)} \subseteq (\mathcal{V}^{(i)})^\perp \subseteq (\mathcal{E}_{\gamma^{(i)}}^{(i)})^\perp$$

for all $i = 1, \dots, \ell$. We have that $\dim((\mathcal{E}_{\gamma^{(i)}}^{(i)})^\perp) = N_i - \gamma^{(i)}$ which therefore deletes exactly $\gamma^{(i)}$ dimensions from $\mathcal{V}^{(i)}$ since $\dim(\mathcal{U}^{(i)}) = N_i - \dim((\mathcal{U}^{(i)})^\perp) = \dim(\mathcal{V}^{(i)}) - \gamma^{(i)} + \delta^{(i)}$ for all $i = 1, \dots, \ell$. Hence, we can write (B5) as

$$\mathcal{U}^{(i)} = \mathcal{H}_{n_t^{(i)} - \gamma^{(i)}}(\mathcal{V}^{(i)}) \oplus \mathcal{E}_{\delta^{(i)}}^{(i)}, \quad \forall i = 1, \dots, \ell$$

which proves the statement. \square

Appendix C Efficient Implementation of the Multishot Operator Channel

For implementing the multishot operator channel we need to draw tuples that contain subspaces and have a particular constant sum-dimension uniformly at random from the set of all such tuples efficiently. **In the following we assume that $\ell \mid N$.** In order to accomplish this subspace-tuple drawing algorithm efficiently we adapted the enumerative coding approach from [42] for the case where the dimension of the ambient space as well as the dimensions of the transmitted spaces are equal in each shot, i.e. for $\bar{N} = (\bar{N}, \bar{N}, \dots, \bar{N})$ with $\bar{N} := \frac{N}{\ell}$ and $\bar{\mathbf{n}}_t = (\bar{n}_t, \bar{n}_t, \dots, \bar{n}_t)$.

Suppose that we want to draw tuples with fixed sum-dimension γ uniformly at random from the set

$$\{\mathcal{E} \in \mathcal{P}_q(\bar{N}) : \dim_\Sigma(\mathcal{E}) = \gamma\}.$$

Define the set $\mathcal{T}_{\bar{N}, \gamma, \ell} := \left\{ \gamma \in \{0, \dots, \min\{\bar{N}, \gamma\}\}^\ell : \sum_{i=1}^\ell \gamma^{(i)} = \gamma \right\}$.

Then the number $N_{\Sigma S}(\bar{N}, t, \ell)$ of tuples in $\mathcal{P}_q(\bar{N})$ with sum-dimension γ is given by

$$N_{\Sigma S}(\bar{N}, \gamma, \ell) = \sum_{\gamma \in \mathcal{T}_{\bar{N}, \gamma, \ell}} \prod_{i=1}^\ell \left[\begin{matrix} \bar{N} \\ \gamma^{(i)} \end{matrix} \right]_q.$$

Since the cardinality of the set $\mathcal{T}_{\bar{N}, \gamma, \ell}$ and thus the number of terms in the sum above is large for most parameters (see [42, Section III]) we may compute $N_{\Sigma S}(\bar{N}, \gamma, \ell)$ recursively as

$$N_{\Sigma S}(\bar{N}, \gamma, \ell) = \begin{cases} \left[\begin{matrix} \bar{N} \\ \gamma \end{matrix} \right]_q, & \text{if } \ell = 1, \\ \sum_{\gamma' = \max\{0, \gamma - (\ell-1)\bar{N}\}}^{\min\{\bar{N}, \gamma\}} \left[\begin{matrix} \bar{N} \\ \gamma' \end{matrix} \right]_q N_{\Sigma S}(\bar{N}, \gamma - \gamma', \ell - 1) & \text{if } \ell > 1, \end{cases}$$

with $N_{\Sigma S}(\bar{N}, \gamma, \ell) = 0$ for $\gamma > \bar{N}\ell$.

For $\ell = 1$ the number coincides with the number of γ -dimensional **vector spaces** of $\mathbb{F}_q^{\bar{N}}$ which is $\left[\begin{matrix} \bar{N} \\ \gamma \end{matrix} \right]_q$. The recursion for $\ell > 1$ follows by summing over the product of all possible γ' -dimensional subspaces in the first block and the number of remaining $(\ell - 1)$ -tuples of sum-dimension $\gamma - \gamma'$.

As in [42] this recursion provides an efficient method to draw dimension partitions of ℓ -tuples with fixed sum-dimension that are chosen uniformly at random from $\mathcal{P}_q(\overline{\mathbf{N}})$ which is given in Algorithm 4.

Algorithm 4 DRAWDIMENSIONPARTITION

Input: Parameters $q, \overline{\mathbf{N}}, \gamma, \ell$

Output: Dimension partition $\gamma \in \mathcal{T}_{\overline{\mathbf{N}}, \gamma, \ell}$ of an ℓ -tuple \mathcal{E} with sum-dimension γ that is chosen uniformly at random from $\mathcal{G}_q(\overline{\mathbf{N}}, \overline{\mathbf{n}}_t)$

```

1:  $D^{(1)} \stackrel{\$}{\leftarrow} \{1, \dots, \mathbf{N}_{\Sigma S}(\overline{\mathbf{N}}, \gamma, \ell)\}$  ▷ Draw index uniformly at random
2:  $\gamma_1 \leftarrow \gamma$ 
3: for  $i = 1, \dots, \ell$  do
4:    $\gamma^{(i)} \leftarrow \max \left\{ \gamma'' \in \{0, \dots, \gamma_i\} : \sum_{\gamma' = \gamma_i - (\ell - i)\overline{\mathbf{N}}}^{\gamma'' - 1} \left[ \frac{\overline{\mathbf{N}}}{\gamma'} \right]_q \mathbf{N}_{\Sigma S}(\overline{\mathbf{N}}, \gamma_i - \gamma', \ell - i) < D^{(i)} \right\}$ 
5:    $D^{(i+1)} \leftarrow D^{(i)} - \sum_{\gamma' = \gamma_i - (\ell - i)\overline{\mathbf{N}}}^{\gamma^{(i)} - 1} \left[ \frac{\overline{\mathbf{N}}}{\gamma'} \right]_q \mathbf{N}_{\Sigma S}(\overline{\mathbf{N}}, \gamma_i - \gamma', \ell - i)$ 
6:    $\gamma_{i+1} \leftarrow \gamma_i - \gamma^{(i)}$ 
7: end for
8: return  $\gamma = (\gamma^{(1)}, \dots, \gamma^{(\ell)})$ 

```

Equipped with the efficient method to draw dimension partitions for sampling ℓ -tuples with constant sum-dimension from $\mathcal{P}_q(\overline{\mathbf{N}})$ uniformly at random (see Algorithm 4) we can implement the multishot operator channel (see (3))

$$\mathbf{U} = \mathcal{H}_{n_t - \delta}(\mathcal{V}) \oplus \mathcal{E}$$

with input alphabet $\mathcal{G}_q(\overline{\mathbf{N}}, \overline{\mathbf{n}}_t)$ and an overall number of γ insertions and δ deletions as follows.

First, we require the partition δ of the δ deletions. For each shot i there are $\left[\frac{\overline{n}_t}{\delta^{(i)}} \right]_q$ ways to chose an $\delta^{(i)}$ -dimensional subspace of the transmitted space $\mathcal{V}^{(i)} \in \mathcal{G}_q(\overline{\mathbf{N}}, \overline{\mathbf{n}}_t)$. Hence, we can use the routine DRAWDIMENSIONPARTITION($q, \overline{\mathbf{n}}_t, \delta, \ell$) in Algorithm 4 to draw the dimension partition δ for the deletions operator $\mathcal{H}_{n_t - \delta}(\mathcal{V})$.

Recall that for the error space \mathcal{E} we have the restriction that $\mathcal{V} \cap \mathcal{E} = \mathbf{0}$. Hence, we have to sample \mathcal{E} uniformly at random from $\mathcal{G}_q(\overline{\mathbf{N}} - \overline{\mathbf{n}}_t, \gamma)$ (rather than from $\mathcal{G}_q(\overline{\mathbf{N}}, \gamma)$) and thus have to call DRAWDIMENSIONPARTITION($q, \overline{\mathbf{N}} - \overline{\mathbf{n}}_t, \gamma, \ell$) in order to get the corresponding insertion partition γ .

Once the partitions of insertions γ and deletions δ are obtained we can apply the single-shot operator channel from [2] to each shot with the corresponding number of $\gamma^{(i)}$ insertions and $\delta^{(i)}$ deletions. The whole procedure is provided in Algorithm 5.

Algorithm 5 RANDOM INSTANCE OF MULTISHOT OPERATOR CHANNEL

Input: $\mathcal{V} \in \mathcal{G}_q(\overline{N}, \overline{n}_t)$, number of insertions γ , number of deletions δ , ℓ

Output: Received space $\mathcal{U} \in \mathcal{P}_q(\overline{N})$

- 1: $\gamma = (\gamma^{(1)}, \dots, \gamma^{(\ell)}) \leftarrow \text{DRAWDIMENSIONPARTITION}(q, \overline{N} - \overline{n}_t, \gamma, \ell)$
 - 2: $\delta = (\delta^{(1)}, \dots, \delta^{(\ell)}) \leftarrow \text{DRAWDIMENSIONPARTITION}(q, \overline{n}_t, \delta, \ell)$
 - 3: **for** $i=1, \dots, \ell$ **do**
 - 4: $\mathcal{E}^{(i)} \stackrel{\$}{\leftarrow} \mathcal{G}_q(\overline{N}, \gamma^{(i)}) \setminus \mathcal{V}^{(i)}$ \triangleright Choose random $\mathcal{E}^{(i)}$ s.t. $\mathcal{V}^{(i)} \cap \mathcal{E}^{(i)} = \{\mathbf{0}\}$
 - 5: $\mathcal{U}^{(i)} \leftarrow \mathcal{H}_{\overline{n}_t - \delta^{(i)}}(\mathcal{V}^{(i)}) \oplus \mathcal{E}^{(i)}$
 - 6: **end for**
 - 7: **return** $\mathcal{U} = (\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(\ell)})$
-