PRACTICAL EVALUATION OF THE OPTIMAL EMBEDDED ISING PROBLEM

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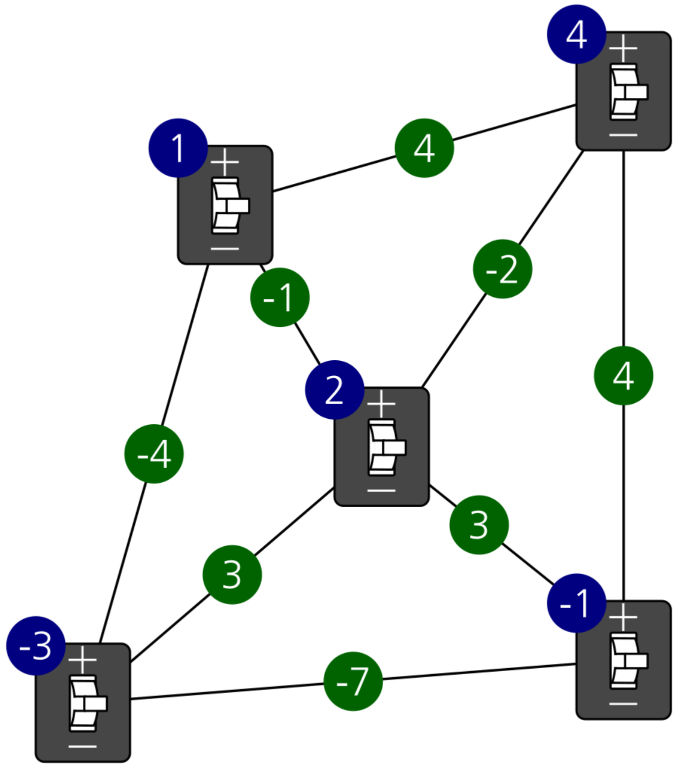
DLR

Ising Problem



Definition

An **Ising model** over graph G with **weights** $\boldsymbol{W} \in \mathbb{R}^{V(G)}$ and **strengths** $\boldsymbol{S} \in \mathbb{R}^{E(G)}_{\neq 0}$ is a function $\boldsymbol{I_{W,S}} : \{-1,1\}^{V(G)} \to \mathbb{R}$ with

$$I_{W,S}(s) := \sum_{v \in V(G)} W_v s_v + \sum_{vw \in E(G)} S_{vw} s_v s_w.$$

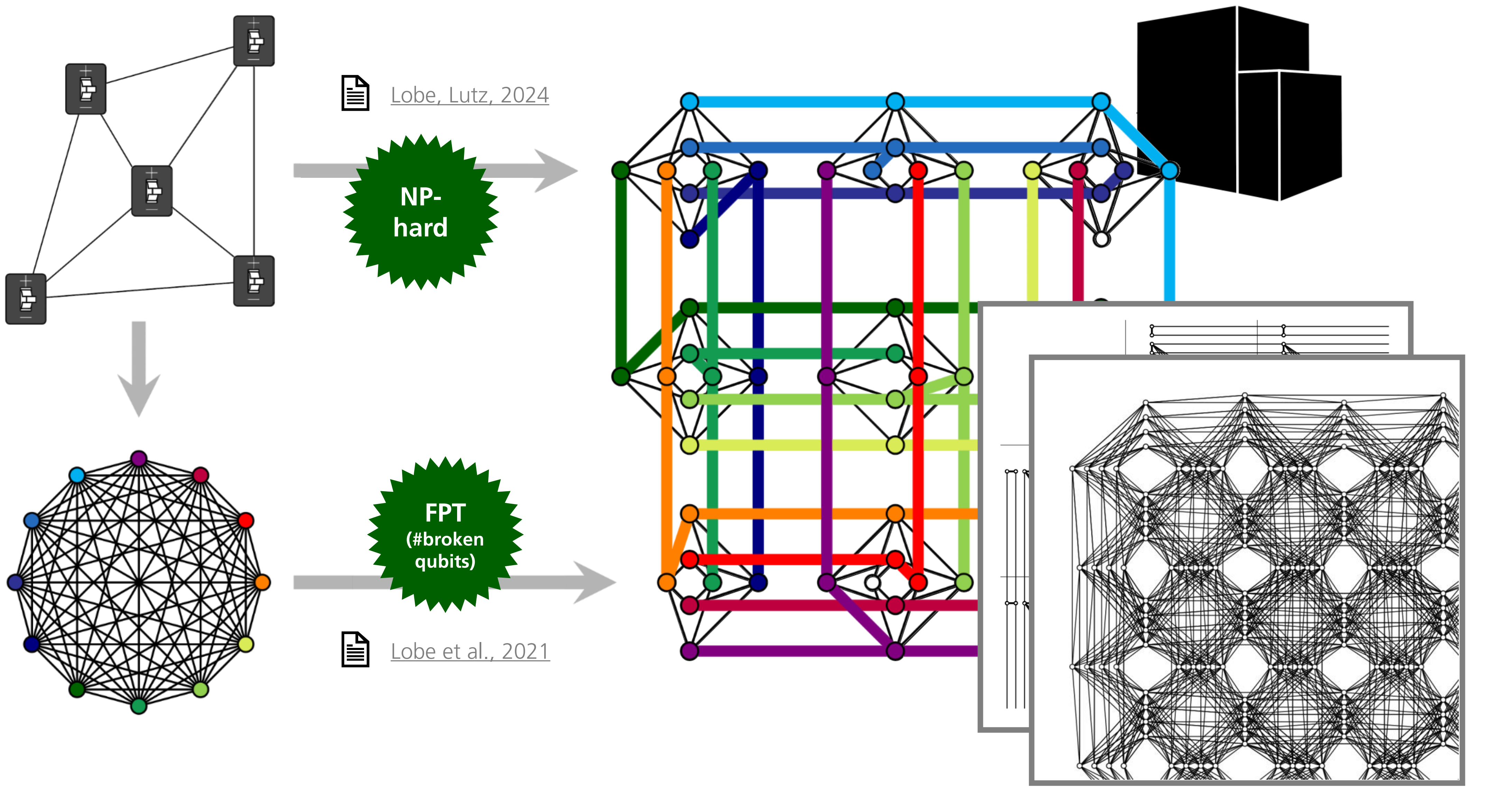
We call G the **interaction graph** of the Ising model.

Ising Problem

Given a graph G, $W \in \mathbb{R}^{V(G)}$ and $S \in \mathbb{R}^{E(G)}$, find s that solves

$$\min_{s \in \{-1,1\}^{V(G)}} I_{W,S}(s).$$

Embedding Problem for Restricted Hardware Graphs



Embedded Ising Problem

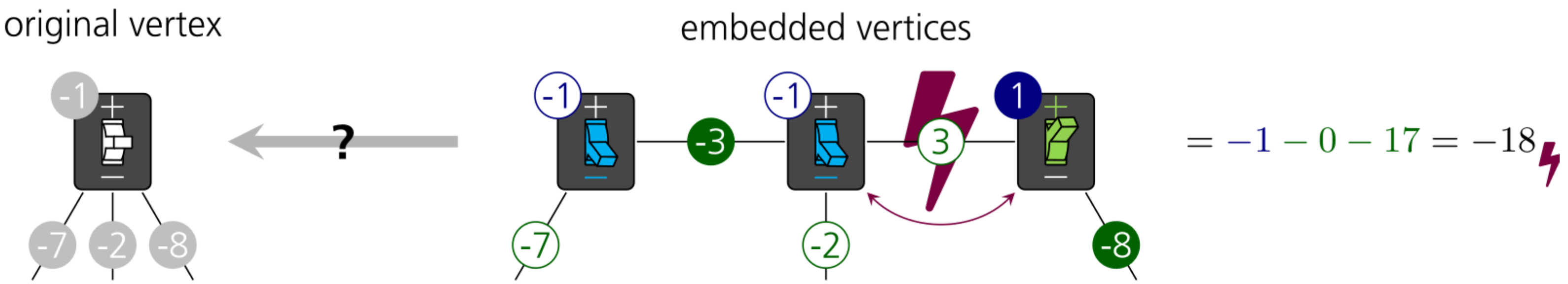
Lobe, Kaibel, 2023

$$\bar{I}_{\overline{W},\bar{S}}(s) := \sum_{q \in V(H_\varphi)} \overline{W}_q s_q + \sum_{pq \in E_\varphi \cup E_\delta} \bar{S}_{pq} s_p s_q$$

$$= \sum_{v \in V(G)} \left(\sum_{q \in \varphi_v} \overline{W}_q s_q + \sum_{pq \in E(H[\varphi_v])} \bar{S}_{pq} s_p s_q \right) + \sum_{vw \in E(G)} \left(\sum_{pq \in \delta_{vw}} \bar{S}_{pq} s_p s_q \right)$$

$? \; W_v t_v$ $\qquad$ $? \; S_{vw} t_v t_w$

- with **proven equivalence** to original problem
- such that (at least optimal) solutions correspond to each other
- based on **synchronization** of the embedded variables



How large does the coupling strength need to be, but how small can it be?

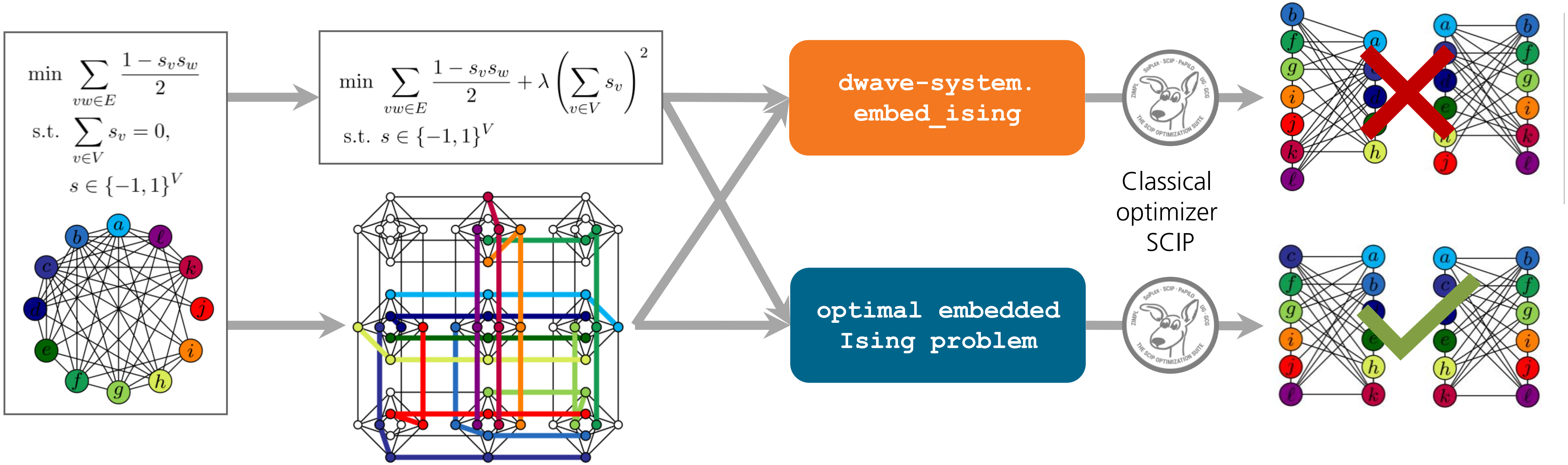
Get Embedded Ising via "Uniform Torque Compensation"

- Current implementation in D-Wave `dwave-ocean-sdk` with method `embed_ising`
  - calls `dwave-system.dwave.embedding.chain_strength.uniform_torque_compensation`

https://github.com/dwavesystems

- Found **counterexample**
  - where the method does **not** provide an equivalent embedded Ising problem
  - based on the Graph Partitioning Problem with 12 nodes
  - embedded by D-Wave using `minorminer.find_embedding`

Pelofske et al., 2022



Get the Optimal Embedded Ising Problem

Lobe, Kaibel, 2023

Requires to solve for each individual original vertex:

Gapped Weight Distribution Problem

Given graph $G = (V, E)$, $\sigma \in \mathbb{R}^V_{\geq 0}$, $W \in \mathbb{R}_{\geq 0}$ with $W < \sigma(V)$ and $\gamma \in \mathbb{R}_{>0}$:

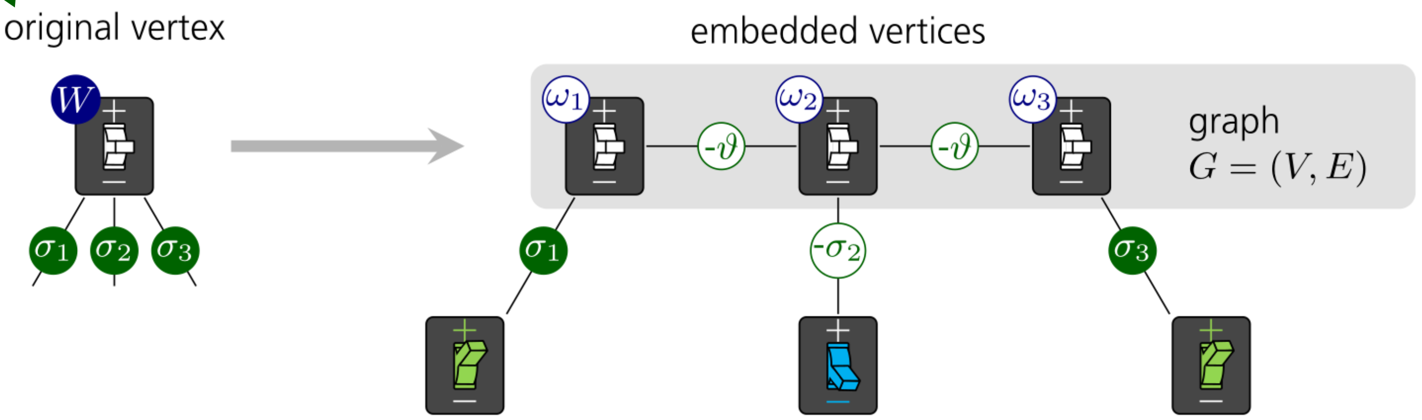
$$\min \vartheta$$

$$\text{s.t. } \vartheta \in \mathbb{R},\ \omega \in \mathbb{R}^V$$

$$\vartheta \geq \frac{\min\{\sigma(S) + \omega(S), \sigma(V \setminus S) - \omega(V \setminus S)\} + \gamma}{|\delta(S)|} \quad \forall \emptyset \neq S \subsetneq V,$$

$$\omega(V) = W$$

polynomial for trees



- cut constraints are redundant for S or $V \setminus S$ not being connected
- **efficiently solvable** in practical embedding setup

How large does the coupling strength ϑ need to be, but how small can it be?

Get the Optimal Embedded Ising Problem

Lobe, Kaibel, 2023

Lobe, PhD Thesis, 2022

Requires to solve for each individual original vertex:

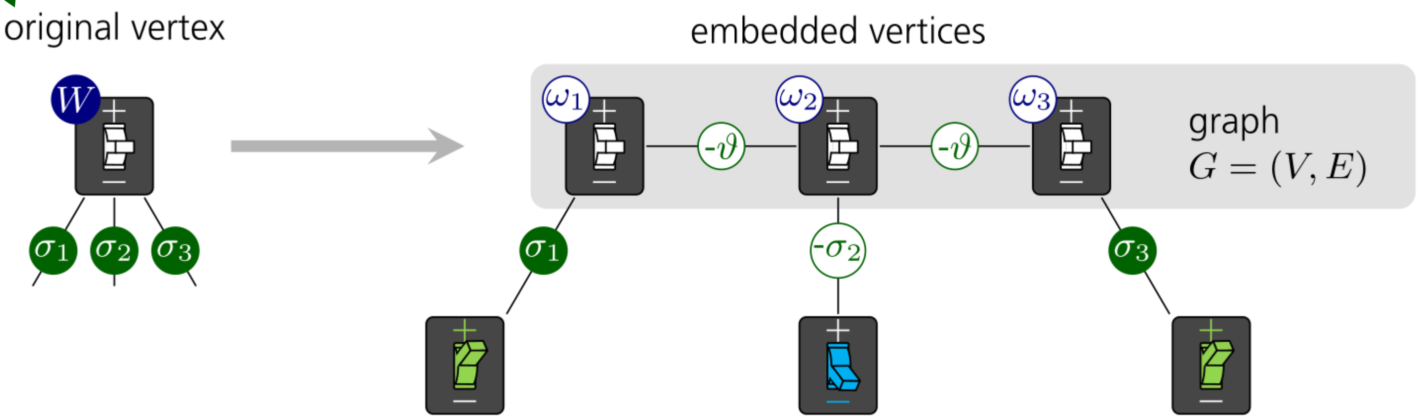
GAPPED INTEGER WEIGHT DISTRIBUTION PROBLEM

Given graph $G = (V, E)$, $\sigma \in \mathbb{N}^V$, $W \in \mathbb{N}$ with $W < \sigma(V)$ and $\gamma \in \mathbb{N}_+$:

$$\begin{aligned} &\min \vartheta \\ &\text{s.t. } \vartheta \in \mathbb{Z},\ \omega \in \mathbb{Z}^V, \\ &\vartheta \geq \frac{\min\{\sigma(S) + \omega(S), \sigma(V \setminus S) - \omega(V \setminus S)\} + \gamma}{|\delta(S)|} \quad \forall \emptyset \neq S \subsetneq V, \\ &\omega(V) = W \end{aligned}$$

polynomial for trees

- cut constraints are redundant for S or $V \setminus S$ not being connected
- **efficiently solvable** in practical embedding setup
- even in integer case

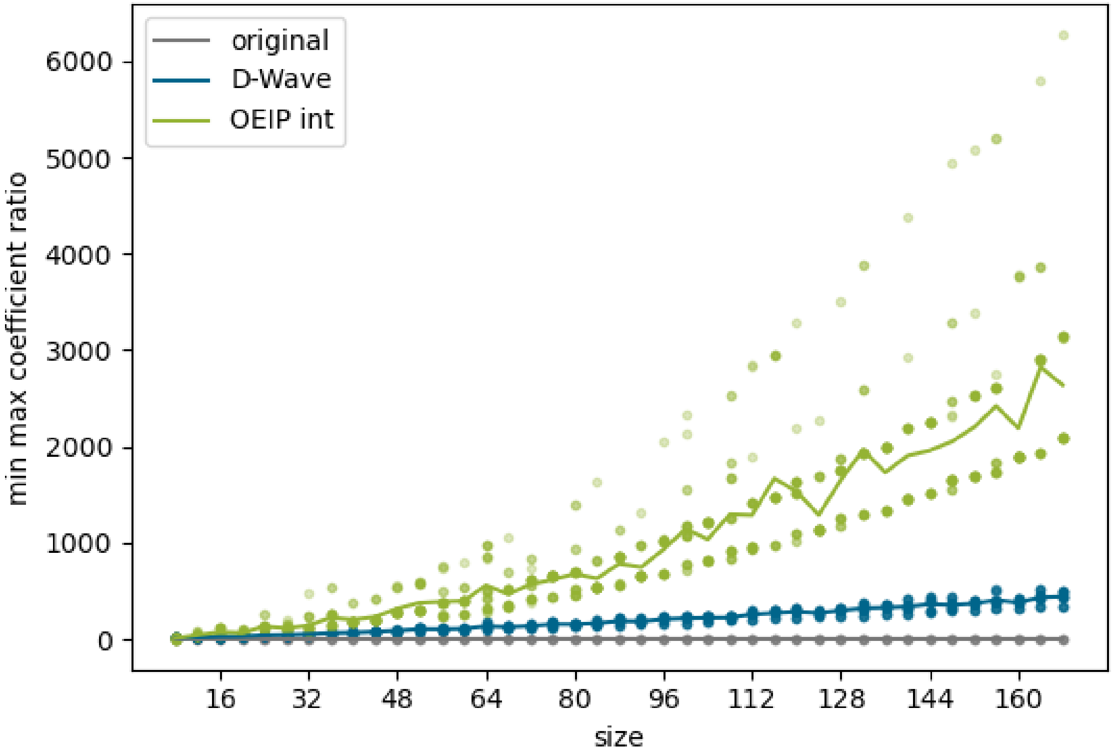


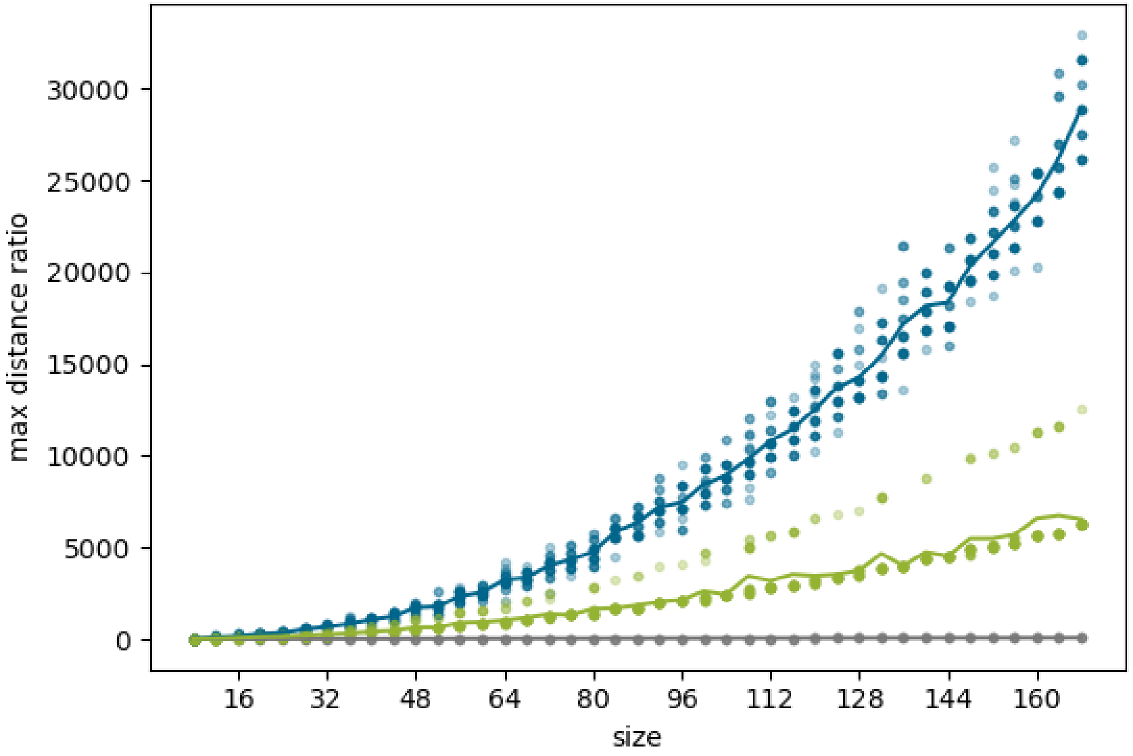
optimal embedded Ising problem

DLR-SC Quantum Computing Software
gitlab.com/quantum-computing-software

Preliminary Results of Experiments

- Graph Partitioning instances
  - with fixed density of 0.8 for increasing size
  - averaged over 5 random instances and 5 precalculated embeddings for each size





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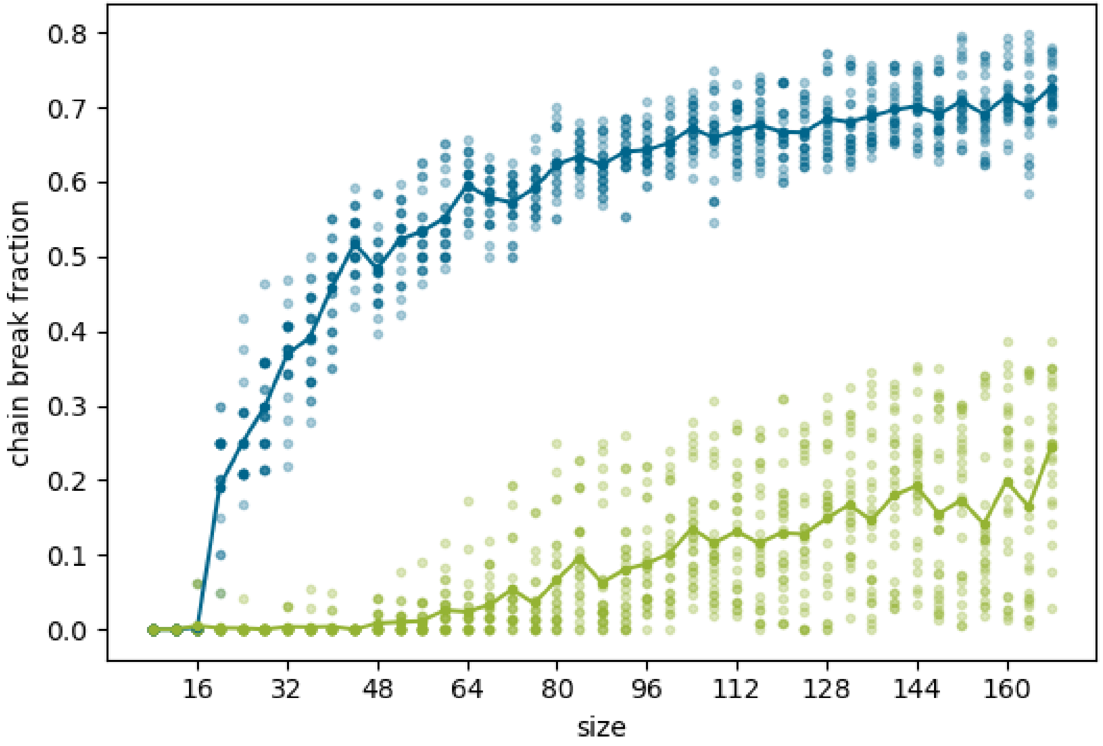
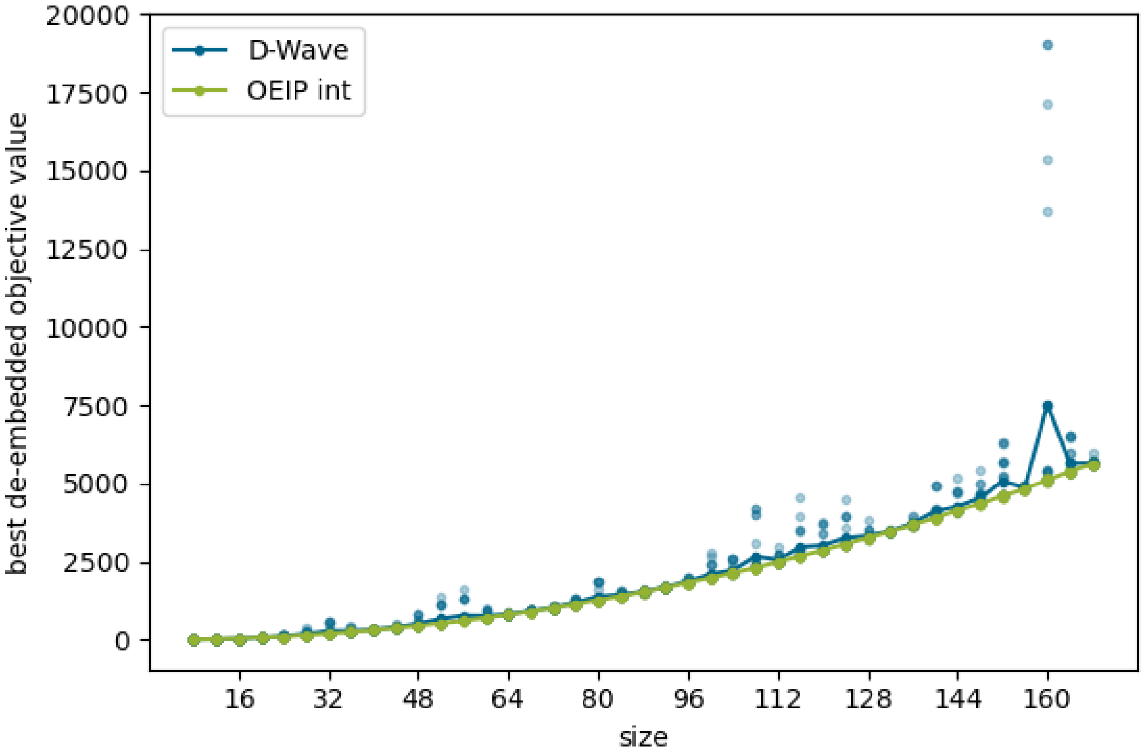
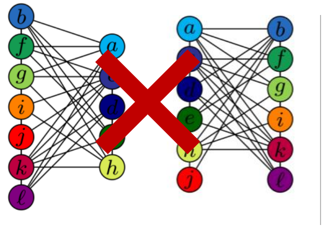


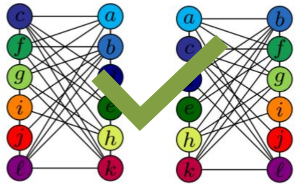


Conclusion & Outlook

- The Uniform Torque Compensation method does in general not provide provable equivalent embedded Ising problems
  - Spreads doubt in particular for larger problems, where the optimal solutions cannot be checked against anymore
  - How is the annealer supposed to solve the problem, if it was ill-defined?

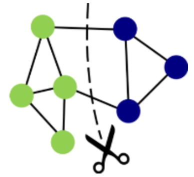


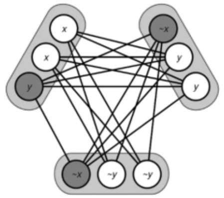
- The optimal embedded Ising problem formulation does
  - Improves the chain break fraction significantly and the success probability to some extent
  - Independent of hardware graph, only embedding structure required

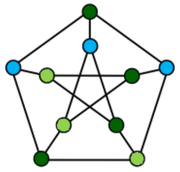


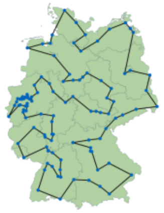
➔ Next Steps

- Actually **study the performance of the annealer** on well-defined problems
- Try more problems with different properties
- Investigate different de-embedding methods
- Explore the average rather than the worst case estimation









Imprint

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