Zeta Expansion for Long-Range Interactions under Periodic Boundary Conditions with Applications to Micromagnetics

Jonathan K. Busse

Joint work with Andreas A. Buchheit

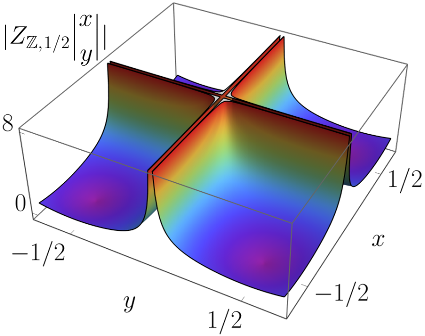
Preprint: `arXiv:2509.26274`
Python: `pip install epsteinlib`
GitHub: `github.com/epsteinlib`

October 2 2025

Epstein Zeta Function

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ and $\Lambda = A\mathbb{Z}^d$ be a lattice. The Epstein zeta function is the meromorphic continuation of the following lattice sum to $\nu \in \mathbb{C}$

$$Z_{\Lambda,\nu}\begin{vmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{vmatrix} = \sum_{\boldsymbol{z}\in\Lambda}{}' \frac{e^{-2\pi i \boldsymbol{y}\cdot\boldsymbol{z}}}{|\boldsymbol{z}-\boldsymbol{x}|^{\nu}}, \qquad \mathrm{Re}(\nu) > d.$$



- Private notes: Hurwitz[a]
- Foundational work: Epstein[b]
- 2D Bessel representation: Chowla–Selberg[c]
- ND Bessel representation: Elizalde[d]
- Incomplete Gamma representation: Crandall[e]

[a] Oswald et al. (2016), *From Arithmetic to Zeta-Functions: ...*

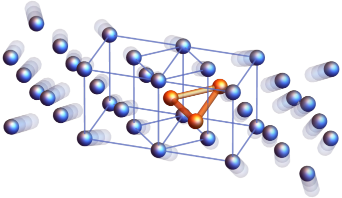
[b] Epstein (1903), *Math. Ann.* Epstein (1906), *Math. Ann.*

[c] Chowla et al. (1949), *Proceedings of the National Academy of Sciences*

[d] Elizalde (1998), *Communications in mathematical physics*

[e] Crandall (2012)

Broad Applicability in Long-Range Interacting Lattices

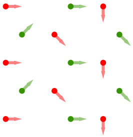


Theoretical Chemistry:

Cohesive energy for Lennard-Jones

Robles-Navarro, Busse, et al. (2025), *The Journal of Chemical Physics*

$$E_{\mathrm{LJ}}(\boldsymbol{r}) \propto \frac{1}{n}|\boldsymbol{r}|^{-n} - \frac{1}{m}|\boldsymbol{r}|^{-m}$$

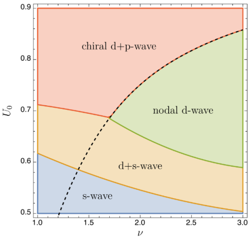


High Energy Physics:

Casimir Energy

Buchheit, Busse, et al. (2024) arXiv:2412.16317

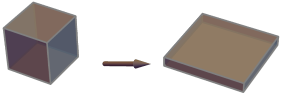
$$E_{\mathrm{cas}} = \frac{1}{2}\sum_{\boldsymbol{k}} \lambda_{\boldsymbol{k}}^{-\nu}\Big|_{\nu=-1}$$



Spin Systems:

$$H = \sum_{ij\in\Lambda} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

Buchheit, Busse, et al. (2024) arXiv:2412.16317



Superconductivity:

Long range gap equation

Buchheit et al. (2023), *Phys. Rev. Res.*

$$\Delta(\boldsymbol{k}) = \int_{\mathrm{BZ}} V_{\mathrm{LR}}(\boldsymbol{k}-\boldsymbol{q})\frac{\Delta(\boldsymbol{q})}{2E_\Delta(\boldsymbol{q})}\,\mathrm{d}\boldsymbol{q}$$

EpsteinLib

Evaluation of Epstein zeta function for

- every exponent ν,
- every lattice Λ,
- every shift vector $\boldsymbol{x}$,
- every wavevector $\boldsymbol{y}$.

In use by community:[a][b][c][d]

[a] Koziol et al. (2025), *Physical Review B*

[b] Kim et al. (2025), *Physical Review Letters*

[c] Dall'Agata et al. (2025) arXiv:2507.02339

[d] Yadav et al. (2025) arXiv:2405.12405

```c
double complex epsteinZeta(double nu, unsigned int dim, const double *A, const double *x, const double *y);
```
C

arXiv:2509.26274

pip install epsteinlib

Precision and Runtime

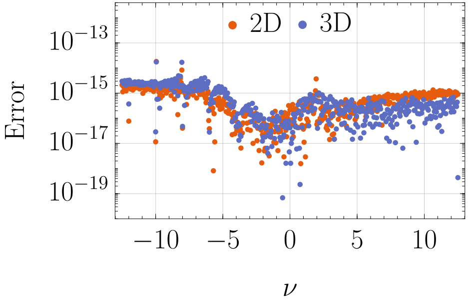
```shell
pip install epsteinlib
```

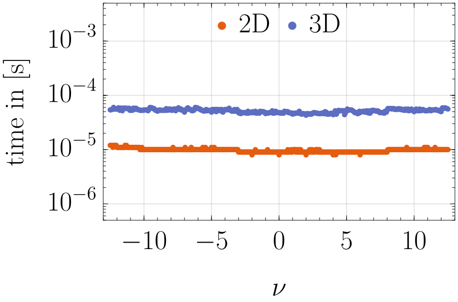
Shell

```python
import numpy as np
from epsteinlib import epstein_zeta

madelung = epstein_zeta(1.0, np.identity(3), np.zeros(3), np.full(3, 0.5))
```

Python





arXiv:2509.26274

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Crandall Representation of the Epstein Zeta Function

Theorem (Crandall representation)

Let $\Lambda = A\mathbb{Z}^d$, $A \in \mathbb{R}^{d\times d}$ *regular,* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, *and* $\nu \in \mathbb{C}$ $(\nu \neq d$ *if* $\boldsymbol{y} \in \Lambda^* = A^{-T}\mathbb{Z}^d)$

$$Z_{\Lambda,\nu}\begin{vmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{vmatrix} = \frac{\pi^{\nu/2}}{\Gamma(\nu/2)}\left[\sum_{\boldsymbol{z}\in\Lambda} G_\nu(\boldsymbol{z}-\boldsymbol{x})e^{-2\pi i\boldsymbol{y}\cdot\boldsymbol{z}} + \frac{1}{V_\Lambda}\sum_{\boldsymbol{k}\in\Lambda^*} G_{d-\nu}(\boldsymbol{k}+\boldsymbol{y})e^{-2\pi i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{y})}\right]$$

with ***upper Crandall*** *function*

$$G_\nu(0) = -\frac{2}{\nu} \qquad G_\nu(\boldsymbol{z}) = \frac{\Gamma(\nu/2, \pi\boldsymbol{z}^2)}{(\pi\boldsymbol{z}^2)^{\nu/2}} \propto \frac{\exp(-\pi\boldsymbol{z}^2)}{(\pi\boldsymbol{z}^2)^{\nu/2}} \ .$$

(incomplete Gamma function $\Gamma(\cdot,\cdot)$)

Epstein Zeta as a New Standard Function

Epstein zeta efficiently computable ✓

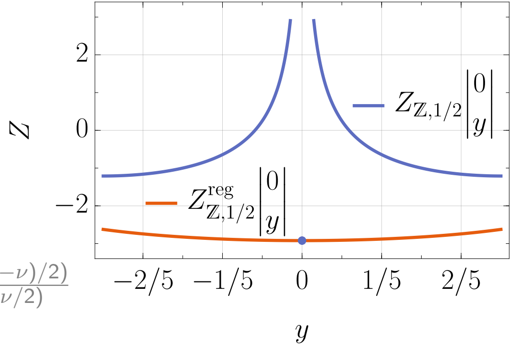
Analytical properties derived in arxiv:2412.16317 ✓

- Symmetries.
- Functional Equation.
- Joint holomorphy all arguments.

Singularity structure ✓

$$\underbrace{Z_{\Lambda,\nu}\begin{vmatrix}0\\ \boldsymbol{y}\end{vmatrix}}_{\substack{\text{singularity}\\ \text{in } \boldsymbol{y}=0}} = \underbrace{Z^{\mathrm{reg}}_{\Lambda,\nu}\begin{vmatrix}0\\ \boldsymbol{y}\end{vmatrix}}_{\substack{\text{smooth}\\ \text{in } \boldsymbol{y}\in\mathrm{BZ}}} + c_{\Lambda,\nu}|\boldsymbol{y}|^{\nu-d}$$

$$c_{\Lambda,\nu} = \frac{\pi^{\nu-d/2}}{|\det(A)|}\frac{\Gamma((d-\nu)/2)}{\Gamma(\nu/2)}$$



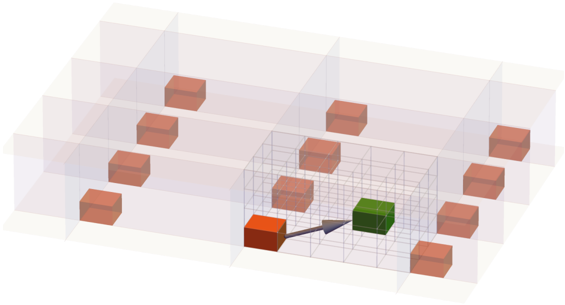
arXiv:2509.26274

`pip install epsteinlib`

Micromagnetics of a 2D Film in 3D

Consider Cuboids Ω at positions $\boldsymbol{z} \in L = A\mathbb{Z}^2 \times \{0\}$, $A \in \mathbb{R}^{2\times 2}$ regular and $\boldsymbol{r} \in \mathbb{R}^d$, where

$$\boldsymbol{z} + \Omega \xrightarrow{\text{influence}} \boldsymbol{r} + \Omega$$



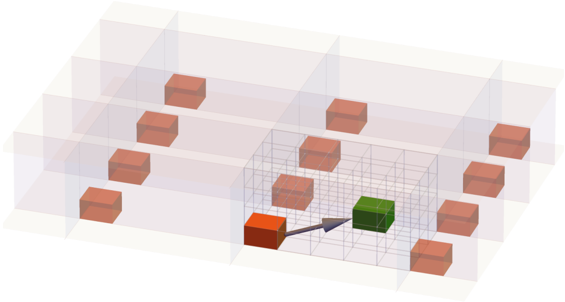
Buchheit, Busse, et al. (2025) arXiv:2509.26274

Generalized potential (multi-index notation $\nabla_{\boldsymbol{r}}^{\boldsymbol{m}} = \partial_{r_1}^{m_1} \dots \partial_{r_d}^{m_d}$)

$$U^{(\boldsymbol{m})}(\boldsymbol{r}) = \sum_{\boldsymbol{z}\in L} \int_{\Omega+\boldsymbol{z}} \nabla_{\boldsymbol{r}}^{\boldsymbol{m}} \int_{\Omega+\boldsymbol{r}} |\boldsymbol{r}' - \boldsymbol{r}''|^{-\nu} \,\mathrm{d}\boldsymbol{r}' \,\mathrm{d}\boldsymbol{r}''$$

with interaction $\nu > d$ and derivative order $\boldsymbol{m} \in \mathbb{N}^d$.

Micromagnetics of a 2D Film in 3D



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$$\boldsymbol{z} + \Omega \xrightarrow{\text{influence}} \boldsymbol{r} + \Omega$$

Generalized potential zeta decomposition for $L_{\mathrm{near}} \subset L$

$$U^{(\boldsymbol{m})}(\boldsymbol{r}) \approx \sum_{\boldsymbol{z} \in L_{\mathrm{near}}} S^{(\boldsymbol{m})}(\boldsymbol{r} + \boldsymbol{z})$$

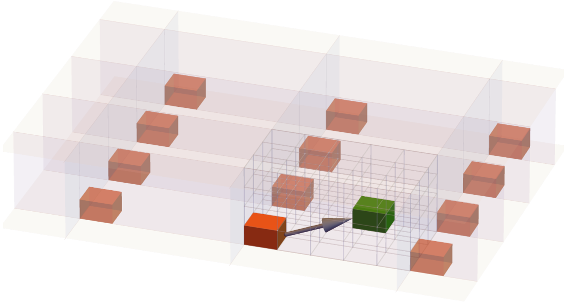
with interaction $\nu > d$ and derivative order $\boldsymbol{m} \in \mathbb{N}^d$

$$S^{(\boldsymbol{m})}(\boldsymbol{r}) = \int_{\Omega} \nabla_{\boldsymbol{r}}^{\boldsymbol{m}} \int_{\Omega + \boldsymbol{r}} |\boldsymbol{r}' - \boldsymbol{r}''|^{-\nu} \, \mathrm{d}\boldsymbol{r}' \, \mathrm{d}\boldsymbol{r}'' \, .$$

Micromagnetics of a 2D Film in 3D

Consider Cuboids Ω at positions
$\boldsymbol{z} \in L = A\mathbb{Z}^2 \times \{0\}$, $A \in \mathbb{R}^{2\times 2}$ regular
and $\boldsymbol{r} \in \mathbb{R}^d$, where

$$\boldsymbol{z} + \Omega \xrightarrow{\text{influence}} \boldsymbol{r} + \Omega$$



Buchheit, Busse, et al. (2025) arXiv:2509.26274

Generalized potential zeta decomposition for $L_{\mathrm{far}} = L \setminus \{L_{\mathrm{near}}\}$

$$U^{(\boldsymbol{m})}(\boldsymbol{r}) = \sum_{\boldsymbol{z} \in L_{\mathrm{near}}} S^{(\boldsymbol{m})}(\boldsymbol{r} + \boldsymbol{z}) + \sum_{\boldsymbol{\alpha} \geq 0} c_{\boldsymbol{\alpha}} Z_{L_{\mathrm{far}},\nu}^{(\boldsymbol{m}+2\boldsymbol{\alpha})}(\boldsymbol{r})$$

with interaction $\nu > d$ and derivative order $\boldsymbol{m} \in \mathbb{N}^d$

$$S^{(\boldsymbol{m})}(\boldsymbol{r}) = \int_\Omega \nabla_{\boldsymbol{r}}^{\boldsymbol{m}} \int_{\Omega + \boldsymbol{r}} |\boldsymbol{r}' - \boldsymbol{r}''|^{-\nu} \,\mathrm{d}\boldsymbol{r}' \,\mathrm{d}\boldsymbol{r}'' \qquad Z_{L_{\mathrm{far}},\nu}(\boldsymbol{r}) = \sum_{\boldsymbol{z} \in L_{\mathrm{far}}}{}' \frac{1}{|\boldsymbol{z} - \boldsymbol{r}|^\nu}.$$

Epstein Zeta Function and Set Zeta Derivatives

Epstein zeta function

$$Z_{\Lambda,\nu}\begin{vmatrix}\boldsymbol{r}\\ 0\end{vmatrix} = \sum_{\boldsymbol{z}\in\Lambda}{}' \, |\boldsymbol{z}-\boldsymbol{r}|^{-\nu}, \qquad \Lambda = A\mathbb{Z}^n$$

- Λ: n-dimensional lattice in n-dimensional system ✓
- no derivatives ✓

Epstein Zeta Function and Set Zeta Derivatives

Epstein zeta function

$$Z_{\Lambda,\nu}\begin{vmatrix} \boldsymbol{r} \\ 0 \end{vmatrix} = \sum_{\boldsymbol{z}\in\Lambda}{}' |\boldsymbol{z}-\boldsymbol{r}|^{-\nu}, \qquad \Lambda = A\mathbb{Z}^n$$

- Λ: n-dimensional lattice in n-dimensional system ✓
- no derivatives ✓

Set zeta derivatives

$$Z^{(\boldsymbol{\alpha})}_{L_{\mathrm{far}},\nu}(\boldsymbol{r}) = \sum_{\boldsymbol{z}\in L_{\mathrm{far}}}{}' \nabla^{\boldsymbol{\alpha}}_{\boldsymbol{r}} |\boldsymbol{z}-\boldsymbol{r}|^{-\nu}, \qquad L_{\mathrm{far}} = (\Lambda \times \{0\}^{d-n}) \setminus L_{\mathrm{near}}$$

- L_{far}: n-dimensional lattice in d-dimensional space
- $\nabla^{\boldsymbol{\alpha}}_{\boldsymbol{r}}$: Partial derivatives of positions

Crandall Representation for 2D Film in 3D

Theorem (Crandall representation)

Let $\Lambda = A\mathbb{Z}^2$, $A \in \mathbb{R}^{2\times 2}$ *regular,* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, $\nu \in \mathbb{C}$ *and* $L_{\mathrm{far}} = (\Lambda \times \{0\}) \setminus L_{\mathrm{near}}$

($\nu \neq 3$ if $\boldsymbol{y} \in \Lambda^* = A^{-T}\mathbb{Z}^3$)

$$Z^{(\boldsymbol{\alpha})}_{L_{\mathrm{far}},\nu}(\boldsymbol{r}) = \frac{\pi^{\nu/2}}{\Gamma(\nu/2)}\Bigg[\sum_{\boldsymbol{z}\in L_{\mathrm{far}}} G^{(\boldsymbol{\alpha})}_\nu(\boldsymbol{z}-\boldsymbol{r}) - \sum_{\boldsymbol{z}\in L_{\mathrm{near}}} g^{(\boldsymbol{\alpha})}_\nu(\boldsymbol{z}-\boldsymbol{r})$$
$$+ \frac{1}{|\det(A)|}\sum_{\boldsymbol{k}\in\Lambda^*}(2\pi i\boldsymbol{k})^{(\alpha_1,\alpha_2)} e^{2\pi i\boldsymbol{k}\cdot(r_1,r_2)} G^{(\alpha_3)}_{1-\nu}(\boldsymbol{k}, r_3)\Bigg].$$

- *Upper Crandall function* $\qquad G_\nu(\boldsymbol{z}) = \Gamma(\nu/2, \pi\boldsymbol{z}^2)/(\pi\boldsymbol{z}^2)^{\nu/2}$
- *Lower Crandall function* $\qquad g_\nu(\boldsymbol{z}) = \gamma(\nu/2, \pi\boldsymbol{z}^2)/(\pi\boldsymbol{z}^2)^{\nu/2}$
- ***Incomplete Bessel** function* $\qquad G_\nu(\boldsymbol{k}, \boldsymbol{r}) = 2\int_0^1 t^{-\nu-1} e^{-\pi\boldsymbol{k}^2/t^2} e^{-\pi\boldsymbol{r}^2 t^2}\,\mathrm{d}t.$

(Here, $\Gamma(\cdot,\cdot)$, $\gamma(\cdot,\cdot)$ upper/lower incomplete Gamma function and $|G_\nu(\boldsymbol{k},\boldsymbol{r})| \leq |G_\nu(\boldsymbol{r})|$)

Crandall Representation for 2D Film in 3D

Theorem (Crandall representation)

Let $\Lambda = A\mathbb{Z}^2$, $A \in \mathbb{R}^{2\times 2}$ *regular,* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, $\nu \in \mathbb{C}$ *and* $L_{\mathrm{far}} = (\Lambda \times \{0\}) \setminus L_{\mathrm{near}}$

($\nu \neq 3$ *if* $\boldsymbol{y} \in \Lambda^* = A^{-T}\mathbb{Z}^3$)

$$Z^{(\boldsymbol{\alpha})}_{L_{\mathrm{far}},\nu}(\boldsymbol{r}) = \frac{\pi^{\nu/2}}{\Gamma(\nu/2)}\Bigg[\sum_{\boldsymbol{z}\in L_{\mathrm{far}}} G^{(\boldsymbol{\alpha})}_{\nu}(\boldsymbol{z}-\boldsymbol{r}) - \sum_{\boldsymbol{z}\in L_{\mathrm{near}}} g^{(\boldsymbol{\alpha})}_{\nu}(\boldsymbol{z}-\boldsymbol{r})$$
$$+ \frac{1}{|\det(A)|}\sum_{\boldsymbol{k}\in\Lambda^*}(2\pi i\boldsymbol{k})^{(\alpha_1,\alpha_2)} e^{2\pi i \boldsymbol{k}\cdot(r_1,r_2)} G^{(\alpha_3)}_{1-\nu}(\boldsymbol{k}, r_3)\Bigg].$$

- *Upper Crandall derivatives* $G^{(\boldsymbol{\alpha})}_{\nu}(\boldsymbol{z}) = \sum_{|\boldsymbol{\beta}|\le|\boldsymbol{\alpha}|/2} p_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{r}) G_{\nu+2|\boldsymbol{\alpha}|-2|\boldsymbol{\beta}|}(\boldsymbol{r})$,
- *Lower Crandall derivatives* $g^{(\boldsymbol{\alpha})}_{\nu}(\boldsymbol{z}) = \sum_{|\boldsymbol{\beta}|\le|\boldsymbol{\alpha}|/2} p_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{r}) g_{\nu+2|\boldsymbol{\alpha}|-2|\boldsymbol{\beta}|}(\boldsymbol{r})$,
- ***Incomplete Bessel*** *derivatives* $G^{(\boldsymbol{\alpha})}_{\nu}(\boldsymbol{k}, \boldsymbol{r}) = \sum_{|\boldsymbol{\beta}|\le|\boldsymbol{\alpha}|/2} p_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{r}) G_{\nu-2|\boldsymbol{\alpha}|+2|\boldsymbol{\beta}|}(\boldsymbol{k}, \boldsymbol{r})$.

(*Here,* $p_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{r}) = (-\pi)^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\frac{(\boldsymbol{\alpha}-\boldsymbol{\beta})!}{(\boldsymbol{\alpha}-2\boldsymbol{\beta})!}(2\boldsymbol{r})^{\boldsymbol{\alpha}-2\boldsymbol{\beta}}$)

Benchmarks

Geometry:

- $\Omega = \prod_{i=1}^{3}[-\frac{c_i}{2}, \frac{c_i}{2}]$
- $L = \mathbb{Z}^2 \times \{0\}$

Maximum error over a grid of values:

- $\boldsymbol{r} = (1/4, 1/4, r),\ r = 0.1, 0.2, \ldots, 0.5$
- $\nu = 10, 10.1, \ldots, 14$

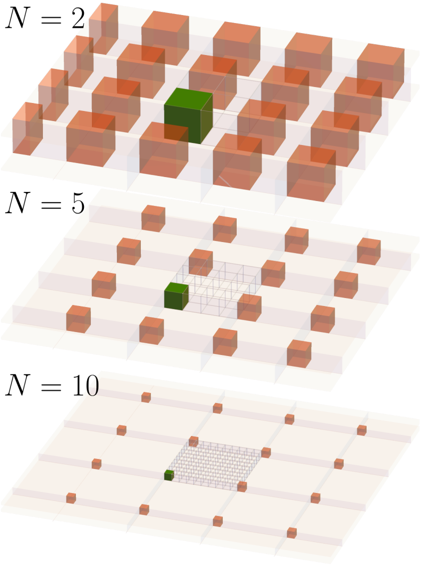
derivatives $\boldsymbol{m}$	cuboid geometry $\boldsymbol{c}$	$\max_{\boldsymbol{r},\nu} E_{\mathrm{rel}}$
$(1,1,0)$	$(1,1,1)/50$	$2.16 \cdot 10^{-16}$
	$(1,1,1)/100$	$2.12 \cdot 10^{-16}$
	$(1,2,3)/100$	$2.20 \cdot 10^{-16}$
$(1,0,1)$	$(1,1,1)/50$	$2.18 \cdot 10^{-16}$
	$(1,1,1)/100$	$2.11 \cdot 10^{-16}$
	$(1,2,3)/100$	$2.16 \cdot 10^{-16}$
$(0,0,2)$	$(1,1,1)/50$	$3.87 \cdot 10^{-16}$
	$(1,1,1)/100$	$2.14 \cdot 10^{-16}$
	$(1,2,3)/100$	$2.19 \cdot 10^{-16}$

$$U^{(\boldsymbol{m})}(\boldsymbol{r}) = \sum_{\boldsymbol{z}\in L_{\mathrm{near}}} S^{(\boldsymbol{m})}(\boldsymbol{r}+\boldsymbol{z}) + \sum_{\alpha\geq 0} c_\alpha Z_{L_{\mathrm{far}},\nu}^{(\boldsymbol{m}+2\alpha)}(\boldsymbol{r})$$

Demagnetization factor for a 2D Film in 3D

- Cuboids $\Omega = (\frac{1}{2}[-\frac{1}{N}, \frac{1}{N}])^3$
- Nb. of Cuboids per dim. $N \in \mathbb{N}$
- Embedded lattice $L = \mathbb{Z}^2 \times \{0\}$

$$\underset{(z \in L)}{z + \Omega} \xrightarrow{\text{influence}} \Omega$$

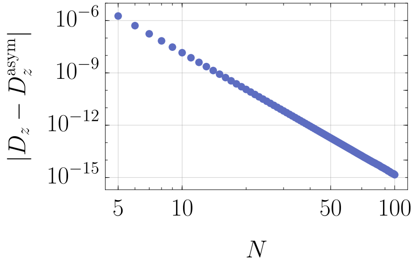


Demagnetization factor in z-direction

$$D_z = 1 + \frac{N^3}{4\pi}\left(U^{(2,0,0)}(0) + U^{(0,2,0)}(0)\right)$$

$$D_z^{\mathrm{asym}} = \frac{1}{3} + \frac{Z_{\mathbb{Z}^2,3}(0)}{4\pi}\,\frac{1}{N^3}$$

Demagnetization factor for a 2D Film in 3D

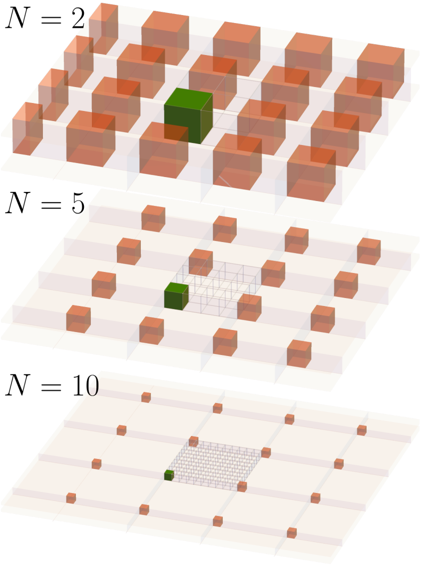


Demagnetization factor in z-direction

$$D_z = 1 + \frac{N^3}{4\pi}\left(U^{(2,0,0)}(0) + U^{(0,2,0)}(0)\right)$$

$$D_z^{\mathrm{asym}} = \frac{1}{3} + \frac{Z_{\mathbb{Z}^2,3}(0)}{4\pi}\,\frac{1}{N^3} + \alpha\,\frac{1}{N^7}$$

$$(\ \alpha \approx 0.1441459732\ )$$



arXiv:2509.26274

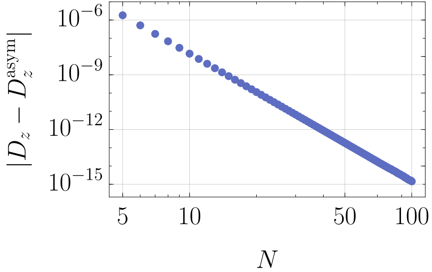
pip install epsteinlib

Conclusions

Efficient computation of:

- *n*-dimensional lattice sums in *d*-dimensional systems
- including any partiall derivatives
- for any geometry
- beyond machine-precision

N	D_z
2	0.4222 0496 3454 0001 7334 0215 3861 9290
5	0.3390 8248 7690 9804 5966 0589 1398 9491
10	0.3340 5219 1717 7249 3459 0417 7785 8568
50	0.3333 3908 4315 3283 8944 2139 2339 3160
100	0.3333 3405 2206 1043 3571 9460 7573 5189



Reuse your code + apply correction

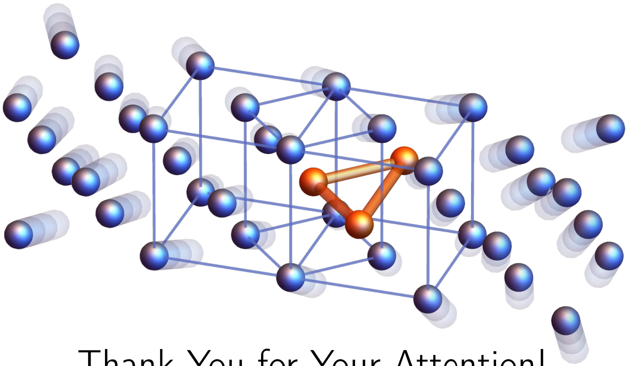
arXiv:2509.26274

pip install epsteinlib

Outlook

Next steps:

- Rigorous discussion of set zeta derivatives
- Include set zeta derivatives in EpsteinLib
- Investigations into stability of matter

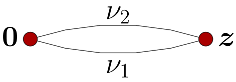


Thank You for Your Attention!

Generalization to many-body lattice sums

Two-body zeta function

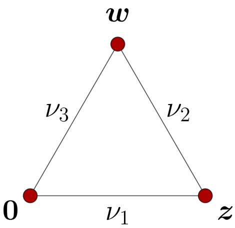
$$Z_{\Lambda,\nu_1+\nu_2}\begin{vmatrix}0\\0\end{vmatrix} = \sum_{\boldsymbol{z}\in\Lambda}{}' \frac{1}{|\boldsymbol{z}|^{\nu_1}}\frac{1}{|\boldsymbol{z}|^{\nu_2}} = \zeta_\Lambda^{(2)}(\nu_1,\nu_2)$$



Buchheit et al. (2025)

Three-body zeta function

$$\zeta_\Lambda^{(3)}(\nu_1,\nu_2,\nu_3) = \sum_{\substack{\boldsymbol{z}\in\Lambda\\ \boldsymbol{w}\in\Lambda}}{}' \frac{1}{|\boldsymbol{z}|^{\nu_1}}\frac{1}{|\boldsymbol{w}-\boldsymbol{z}|^{\nu_2}}\frac{1}{|\boldsymbol{w}|^{\nu_3}}$$



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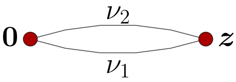
Generalization to many-body lattice sums

Two-body zeta function

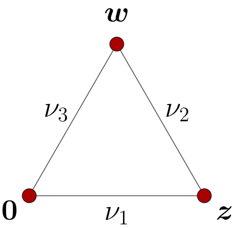
$$Z_{\Lambda,\nu_1+\nu_2}\left|\begin{matrix}0\\0\end{matrix}\right| = \sum_{\boldsymbol{z}\in\Lambda}{}' \frac{1}{|\boldsymbol{z}|^{\nu_1}}\frac{1}{|\boldsymbol{z}|^{\nu_2}} = \zeta_\Lambda^{(2)}(\nu_1,\nu_2)$$



Buchheit et al. (2025)

Three-body zeta function

$$\zeta_\Lambda^{(3)}(\nu_1,\nu_2,\nu_3) = \mathrm{Vol}(\Lambda)\int_{\mathrm{BZ}} Z_{\Lambda,\nu_1}\left|\begin{matrix}0\\\boldsymbol{y}\end{matrix}\right| Z_{\Lambda,\nu_2}\left|\begin{matrix}0\\\boldsymbol{y}\end{matrix}\right| Z_{\Lambda,\nu_3}\left|\begin{matrix}0\\\boldsymbol{y}\end{matrix}\right| \mathrm{d}\boldsymbol{y}$$



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