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TREATMENT OF SINGULAR INTEGRALS FOR BOUNDARY ELEMENT METHODS IN HYDRO- AND AERODYNAMICS - A SHORT REVIEW

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ABSTRACT

Boundary element methods are widely employed in engineering and research to solve partial differential equations in the form of boundary integral equations numerically. In hydro- and aerodynamics, these methods provide fast solution to potential flow problems, especially useful in early design phases of ships or aircraft, e.g. for the computation of ship motion and added wave resistance or of aircraft loads and acoustic analyses. However, the fundamental solution of the underlying boundary integral equations as well as their derivatives are characterized by an increasingly singular nature, so the assembly of the required boundary integral operators is non-trivial. Therefore, suitable treatment of the singular integrals is crucial for the boundary element method, since it requires the evaluation of the singular boundary integrals in the near- and self-influence regimes. In this paper, we classify the different methods for singular integration, detail the theory behind these techniques, give examples of existing approaches and sort them according to the presented classification. The present review for singular integration methods for Laplace boundary element methods aims to give an overview of existing frameworks and the related theory, intended as a starting point for choosing appropriate methods by considering the advantageous characteristics or identifying fields of further research.

Keywords: Boundary Element Method, Hydrodynamics, Aerodynamics, Singular Integration

1. INTRODUCTION

The boundary element method (BEM) is a numerical approach for solving boundary integral equations (BIEs). It has been established as one of the classical computational methods in engineering and applied across a wide range of industry levels including small engineering services as well as globally delivering ship yards and aerospace corporations.

Specifically, for solving partial differential equations (PDEs) in unbounded media (exterior problems), the BEM is attractive because the behavior of the solution in the farfield can be treated effectively, e.g. with suited radiation conditions. Also, the reduction of the dimension of the discretization domain by one order compared to the domain of the underlying PDE is advantageous in many applications. Furthermore, various choices in the representation of solution and geometry, the formulation, and the discretization can be made for the BEM allowing to derive tailored methods for the targeted field of application. On the other hand, the appearance of fully populated matrices, the limited applicability to PDEs with fundamental solution and the requirement to evaluate singular integrals have to be addressed when using BEMs.

In the framework of the BEM, two main discretization methodologies exist, the Collocation and the Galerkin method. The Collocation method enforces that the solution fulfills the BIE exactly at the integration points, but not between these. The Galerkin on the other hand minimizes the integral error of the BIE. The occurrence of double integrals increases the computational effort for the Galerkin form, but the weighting of the BIE allows to reduce the continuity requirements for the solution and geometry approximation functions compared to the Collocation approach [1–3]. Also the solution evaluation can be done in different ways, by the direct or indirect method [4, 5]. The indirect method evaluates jump relations of the quantities of interest, whereas the direct formulation gives the physical quantities on the boundary.

The use of BIEs for the solution evaluation of PDEs in BVPs relies on the divergence theorem and the existence of a fundamental solution for the PDE. The divergence theorem gives the basis for evaluating domain quantities by boundary values. The existence of a fundamental solution for the PDE is a key property

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for the applicability of BIEs and is proven to exist for linear partial differential operators with constant coefficients (see e.g. [5], Malgrange-Ehrenpreis theorem).

Within the scope of this paper, we focus on the Laplace equation

$$\Delta\phi = \nabla \cdot (\nabla\phi) = \sum_{i=1}^{d} \frac{\partial^2 \phi}{\partial x_i^2} = 0 \quad , \tag{1}$$

for the velocity potential ϕ with the Nabla operator ∇ denoting the vector of partial derivatives $(\partial/\partial x)$ in the coordinates of x. This equation can be used to model motions of waves, ships and airplanes in an inviscid and incompressible fluid, and an irrotational flow. The Green function for the Laplace equation, e.g. [5, 6], is

$$G(x, y) = -1/(2\pi)\log(|x-y|) \qquad \text{for } \Omega \in \mathbb{R}^2 \tag{2}$$

$$= \frac{1}{4\pi|x-y|} \qquad \text{for } \Omega \in \mathbb{R}^3 \tag{3}$$

where the x and y denotes the source and target point coordinates.

Applying Green's vector identities to the divergence theorem, integral equations for the direct and indirect evaluation of the quantity, its derivative or both can be derived. For direct version and the Dirichlet BVP this reads [5]

$$\phi(x) = \int_\Gamma G(x, y)\frac{\partial\phi(y)}{\partial n_y}\,\mathrm{d}S_y - \int_\Gamma \frac{\partial G(x, y)}{\partial n_y}\phi(y)\,\mathrm{d}S_y \ , \tag{4}$$

where $\phi(y)$ represents the Dirichlet data given on the boundary Γ. By differentiation with respect to the normal direction, the corresponding representation for the Neumann BVP is derived as

$$\frac{\partial\phi(x)}{\partial n_x} = \int_\Gamma \left[\frac{\partial}{\partial n_x} G(x, y)\frac{\partial\phi(y)}{\partial n_y} - \frac{\partial^2 G(x, y)}{\partial n_x \partial n_y}\phi(y)\right]\mathrm{d}S_y \ , \tag{5}$$

where $\partial\phi(y)/\partial n_y$ is specified as a Neumann BC on Γ.

By moving the quantities to the boundary, the BIEs can be derived from the integral equations, Eqs. (4) and (5), and formulated by means of boundary integral operators (BIOs) and trace operators [4, 5]. The trace operators for the Dirichlet and Neumann data characterize the approach to the boundary which is referred to as limiting process. The BIOs are related to the fundamental solution and its derivatives and describe a mapping of the quantities between function spaces (for BIEs Sobolev spaces) related to the Dirichlet and Neumann data.

Based on the above, BEMs solve the BIE numerically by discretizing the boundary of interest into elements. On these elements, basis functions are used to approximate the surface geometry and the solution on the surface. The characteristics of the basis functions (e.g. continuity and order) determines the approximation properties of the solver: For example, low-order methods typically use linear geometry approximations and constant singularity distributions, while iso-geometric methods choose splines for both, see e.g. Beer et al. [7].

The integration over singular kernel functions, e.g. the integrands in Eqs. (4) and (5), is crucial for solving BIEs and represents a mathematical challenge. Foremost, the singular nature of the integrals complicate the evaluation in the near- and self-influence regimes. Therefore, specifically for higher derivatives and non-smooth boundaries careful treatment is necessary.

In this paper, methods for singular integral evaluation are summarized and discussed. The focus is set on the Laplace equation, the governing PDE for potential flow, that have relevant applications in aero- and hydrodynamics. After introducing the mathematics related to singular integration, we classify the methods for singular integration, give an overview on representatives of the groups, and compare the approaches by identified relevant characteristics. Moreover, we discuss them in the context of application and procedure, and identify advantageous characteristics that lead to a favorable choice of the methods in view of the application. The methods used in hydro- and aerodynamics are detailed and references in the literature are given in Sec. 4. Finally, we draw conclusions and close by outlining future topics of interest.

2. SINGULAR INTEGRALS

In the following, we assume definite integrals which can be subdivided into two sub-classes: proper and improper integrals. The proper integrals are Riemann integrable, i.e. they can be calculated by a finite sum of rectangular stripes over the entire interval of the kernel function. If integrals have either unbounded integration intervals or contain unbounded integrands, they cannot be evaluated via the Riemann sum as no rectangular stripe can be determined to the pole of the singularity. This class of integrals are referred as improper (Riemann) integrals.

In the context of BIEs, the latter type of improper integrals appear, also referred to as singular integrals. They can be calculated by excluding a vanishing small neighborhood around the singularity from the integration interval [a,b] which yields the limits

$$\int_a^b h(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_a^{c-\varepsilon} h(x)\,\mathrm{d}x + \lim_{\beta\to 0}\int_{c+\beta}^b h(x)\,\mathrm{d}x \tag{6}$$

with the function $h(x)$ having an unbounded value at $x \to c$, which is contained in the interval [8]. Apart from the singular point $h(c)$, the function is assumed to be at least bounded or even continuous. If the limits in Eq. (6) can be determined, e.g. by considering residual evaluation methods, the improper integral exists and converges.

Considering a symmetric region with radius ε around the singularity, introduces the concept of the Cauchy Principal Value (CPV) and the Hadamard Finite Part (HFP). These integrals might even converge if the limits in Eq. (6) do not exist. The Cauchy Principal Value is given by [9]

$$\text{c.p.v.}\int_a^b h(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\left[\int_a^{\mu-\varepsilon} h(x)\,\mathrm{d}x + \int_{\mu+\varepsilon}^b h(x)\,\mathrm{d}x\right]. \tag{7}$$

The derivative of the CPV in the direction of the variable of integration yield the Hadamard Finite Part

$$\text{h.f.p.}\int_a^b z(x,t)\,\mathrm{d}t = \frac{d}{\mathrm{d}x}\,\text{c.p.v.}\int_a^b z(x,t)\,\mathrm{d}t \tag{8}$$

where the integrand z has to be $C^{1,\alpha}$ in $[a, b]$ and Leibnitz rule is considered for integration [10]. In the CPV, the values in the singular neighborhood cancel out each other due to the point symmetry of the kernel function $\mathcal{O}(r^{-1})$. In contrast, in the HFP a remaining term $\propto \varepsilon^{\alpha+n}$, also referred as finite part, is employed to cancel exactly the values in the singular region considered in the improper integral parts of the HFP. This can be seen from expressing the HFP as

$$\text{h.f.p.}\int_a^b \frac{1}{m}\,\mathrm{d}x = \lim_{\varepsilon\to 0}\left[\int_a^{\mu-\varepsilon}\frac{1}{m}\,\mathrm{d}x + \int_{\mu+\varepsilon}^{b}\frac{1}{m}\,\mathrm{d}x + p(\varepsilon)\right] \quad (9)$$

with $m = (\mu - x)^{\alpha+n+1}$ and the singular point $\mu \in [a, b]$. On the right-hand side, the last function $p(\varepsilon)$ represents the contribution that cancels out the singularity of the two first improper integrals.

The mathematical frameworks of the CPV and the HFP regularize strongly and hypersingular singular kernels, respectively, to weakly singular integrals. These can be treated by standard numerical integration methods and are used in various methods detailed hereafter.

3. METHODS FOR SINGULAR INTEGRAL EVALUATION

In this section, an overview over four classes of singular integration is given by presenting methodologies of representative examples from literature. Namely, these are the analytic, semi-analytic, and numeric integration as well as regularization methods. The common target of the methods is to remove or reduce the singularity of the kernel functions so that standard methods can be used for integration.

3.1 Analytic integration

For the three-dimensional Laplace equation, analytical integration of the potential kernels is possible but generally restricted to simple geometries, most commonly planar polygons. Additionally to yielding exact results, analytic integration is often computationally more efficient than numeric integration [11]. The singular self-term integration is commonly treated by special case paths in the implementation, usually by computing these explicitly as Cauchy Principle Values. This section concentrates on the description of methods for the three-dimensional Laplace kernel. Although most integration methods mentioned here are linked to specific panel method implementations, details about the implementations are not given here but in Sec. 4.1.

Hess and Smith expressed integrals for constant source distributions over planar quadrangels by a superposition of independent integrals over the edges of the quadrangle [12]. In a later extension of the work, integrals for constant vorticity distributions have been added. This extension allows for the computation of lifting bodies [13].

Webster published an extension which allowed the integration of linear source distributions over triangles [14]. A decade later, Newman generalized this to linear distributions for doublets and sources for planar quad- and triangular elements [15]. He defined sectors from infinite extensions of the edges in a way that the differences between these sectors yields the panel. The integral is then evaluated for each vertex in terms of the included solid angle. After rearranging the resulting equations, the integral can effectively be evaluated as a sum of integrals over the panel edges again. Hence, his results for constant distributions are consistent with prior derivations, e.g. by Hess and Smith. Results were given explicitly for potential integrals up to linear distributions. Furthermore, Newman proposed an approach how the more general case of high-order distributions could be derived, but did not carry out the derivation or give the resulting equations. Another method proposed by Fata breaks down integrals of linear potentials over flat triangles into generic integrals over their edges [16]. This work includes integrals up to what is labeled as "quadruple-layer". Carley solved the integrals for linear sources and doublets on triangular elements utilizing a local coordinate system in the panel plane with the projection of the field point in the origin [11]. Subsequently, sub-triangles are formed by combining each edge of the original triangle with the field point projection. The basic integrals are then solved on these sub-triangles using polar coordinates, where the singular case of a field point in the panel plane is treated explicitly as a special case. All of the aforementioned methods work in component-notation in a coordinate system in the panel plane and, hence require a coordinate transformation of the actual panel and field point and - in case of velocities - an additional transformation of the result. In contrast, a method for planar triangles and quadrangles up to linear distributions published by Suh uses vector notation so that no coordinate transformations are required [17]. This method is based on applying Stokes' formula and vector operations on the integrand to decompose the panel integral into line integral over panel edges. It is worth noting that most methods presented here are algebraically equivalent, since they solve the same integrals analytically and should give exact and, hence, equivalent results. However, they differ in the algorithmic execution, especially in the special cases arising and being treated [11].

Integration of high-order potential distributions is much more complicated, particularly on curved surfaces. For this reason, analytical integration methods proposed for these cases generally contain some form of approximation of the actual integral over the curved surface elements. Some methods approximate the curved geometry by piecewise flat sub-elements and then perform an analytical integration of quadratic distributions over these sub-elements [18, 19]. Alternatively, the high-order integration method presented by Hess uses a truncated expansion form of potentials for which analytical integration can be performed on curved elements [20]. Wang proposed to map the curved panel onto a flat panel by projection of the vertices into a tangent plane [21]. Although the edges of the triangle projected into the tangent planes are generally curved [22], this method is based on a linear connection between the projected vertices. The integration is then performed analytically, where polynomial approximations are utilized for terms linked to the geometry-mapping in the resulting integral.

3.2 Semi-analytic integration

The combination of analytic treatment and numerical integration appears typically in methods where the limiting process is explicitly considered. In the region of the singularity, e.g. Taylor series expansion of the relevant quantities around the singular point can be used for the numerical evaluation. In addition with regularizing transformations up to hypersingular kernels can be treated.

The series expansion is usually performed on the discrete level by an adding-and-subtraction procedure. Performing the limit process after the discretization allows to take the basis functions for solution and geometry (including curved boundaries) approximation explicitly into account for the derivation of the regularized integrals.

In the framework of the Collocation BEM, semi-analytic methods have been developed for strongly singular integrals in 3D by Guiggiani & Gigante [23]. For hypersingular kernels in Collocation BEM, a method for 2D has been proposed in Guiggiani [24] and the extension to 3D has been presented by Guiggiani et al. in [25].

For 3D Galerkin BEM and piecewise smooth surfaces, Hackbusch & Sauter [1] have considered a series expansion of the singular integral kernel. They treated its polynomial part analytically and the singular part numerically by a regularizing transformation. To reduce the singular order, the triangular elements have been mapped to a parametric reference element. The resulting weakly singular integrals have been treated with tensor product Gauß quadratures after transformation with Duffy coordinates. The tools used in their work have been pioneering steps for other approaches that applied regularizing quadratures to the Galerkin discretization, see Sec. 3.3.

Bonnet & Guiggiani [2] have applied the semi-analytic approach to 2D Galerkin BEM for hypersingular kernels and curved boundaries. The integration of the double integrals have been performed simultaneously on the inner and outer integral. On the weighted BIE (continuous level), a (first) Taylor series expansion of the density functions has been used to extend the integral equation that allows to separate the singular integration domain from the region of regular integration. After introducing the discretization, the self and adjacent case specific regularization for the singular integration domain part have been derived by considering the (second) Taylor series expansion of the kernel around the singularity and regularizing coordinate transforms. Due to the applied limiting process, additional free terms appear. These free terms can be bounded or unbounded and require special analysis, see [2, 26]. Their occurrence depend e.g. on surface discontinuities and for their treatment, explicit derivation of the Green function specific terms as well as cancellation conditions for the unbounded terms are required, and have been analyzed for the 2D Galerkin BEM in [2].

3.3 Numeric integration

Integration rules, also referred to as quadratures (or cubatures if dimension of integration domain is larger than one), are the key tool for performing numerical integration. By providing integration weights at specific integration points on a reference element in parametric space, integral values can be found by multiplying with function values at the integration points. Davis & Rabinowitz [9] have summarized techniques for numerical integration including examples of corresponding algorithms. In the context of singular integration, regularizing quadratures are widely used as their application is practical: The mathematical complexity of computing singular integrals is quasi hidden and their implementation should be straight forward (as the set of weights and points for specific cases can be integrated theoretically in any algorithm), see e.g. [27]. Also, the convergence behavior with h- and/or p-refinement can be derived explicitly for these purely numeric methods.

The basic idea of regularizing quadratures is the use of suited transformations on a reference element that result in weakening the kernel singularity. The transformations are usually based on a CPV and/or HFP formulation of the singular kernel functions. The obtained at most weakly singular analytic kernels on 2D (Collocation approach) or 4D (double integrals in Galerkin approach) can be numerically evaluated by applying Duffy or polar coordinate transformations and tensor products of standard (1D) quadratures, see Sec. 3.2.

The use of the CPV and HFP concept indicates that the singular region is extracted from the integration domain instead of performing the limiting process (e.g. by TSE in local coordinates around the singular point) explicitly. In consequence, no additional unbounded free term appears due to the limiting process but the assumptions on domain and kernel functions are relatively strict and allow less possibilities in the adaption to specific cases, e.g. surface discontinuities.

Schwab & Wendland [28] and Kieser et al. [29] have introduced relevant methodologies for the numerical evaluation of singular integration in 3D Collocation BEM (also relevant for inner integration for Galerkin BEM) and error estimates have been derived for the different singularity orders and weakly singular integration methods, in [28] for the h-version Collocation BEM and in [29] for the hp-version of the Collocation BEM.

The expansion of up to hypersingular integrals and its treatment as HFP integral has been described in Kieser et al. [29]. An asymptotic expansion for general hypersingular kernel functions and the related assumptions has been introduced. Based on this expansion, the HFP have been defined. The regularizing transformation of the analytic finite part have been explicitly evaluated by using Taylor series expansion and local coordinates. Additional 1D integrals have to considered in the kernel expansion if the boundary is not smooth or if the source point is on the boundary. This might be seen as analogy to the free terms arising from the first TSE applied on the continuous level in the semi-analytic approach of [2].

Schwab & Wendland [28] have applied quadrature methods for the expanded and discretized kernel without deriving an explicit regularized analytic expression for the strongly singular case but instead choosing separate quadrature rules for the radial and circumferential direction. Moreover, the authors state that "For weakly singular integrals it is shown that Duffy's triangular coordinates lead always to a removal of the kernel singularity" ([28], p. 343).

Sauter & Lage [30] have extended the cubature methods for Galerkin BEM to hypersingular BIOs by introducing regularizing transformations and suited decomposition of the integration domains. This contribution have completed the set of previous works on regularizing cubatures for Galerkin BEM up to strongly singular integrals, see [27, 31, 32], to a "fully implicit" and "black-box method" and "with respect to the order of the rule", the methods are "exponentially convergent" and "uniformly stable" ([30], p. 224). The transformation rules aim to map the curved surface triangles to reference domains in parametric space on that the order of singularity is at least reduced to weakly singular order. For the resulting kernel function, analytic formulations are accessible and modified Gauß tensor rules can be employed for numerical integration. The domain decomposition of the kernel functions allows a proper application of the transformations to avoid degenerating of the reference domains for elements with strongly varying size (high aspect ratio). The method makes use of an expansion of the hypersingular kernel function, for which regularized forms in the three adjacent cases can be derived.

For h, p- and hp- version of the Galerkin BEM in 2D, Diligenti & Aimi [33] and Aimi et al. [34, 35]) and in 3D Aimi & Diligenti [36] have introduced methods for hypersingular kernel evaluation. For the kernel and case specific treatment, various methods have been used: foremost modified, high-order weights for 1D quadrature based on Gauß Legendre quadrature have been derived by using the three term recursion relation. The basic assumption in [36] have been the regular triangulation for the geometry and basis functions up to arbitrary degree for solution approximation, as well as a linear mapping to a reference triangle in parametric space. With the expansion of the hypersingular kernel function and its formulation in the HFP sense, the finite part has been derived in local coordinates explicitly. Singularity smoothing and coordinate transformation have been used for further regularizations.

3.4 Regularization techniques

Regularization methods convert singular integrals into a more regular form by application-based approaches. The resulting integrals are at most of weakly singular order and can be integrated by applying transformations and standard quadrature methods or treated analytically. Contrary to other singular integration techniques, regularization methods are typically applied on continuous functions, i.e. on BIE level and before discretization. In what follows, we summarize some of these techniques with focusing on applications in the framework of the Laplace equation.

The simple solutions are application-specific assumptions (e.g. rigid body motion [37, 38], hydrostatic pressure mode), usually based on physical constraints, that allow to exclude certain singular cases from the numerical treatment.

In the context of hydrodynamic applications, the desingularized BIE methods [39], for a review see Cao & Beck [40], have been applied, specifically suitable for BEMs in that the standard Laplace Green function is considered including the ones using the mixed-Eulerian-Lagrangian (MEL) method, see Sec. 4. The non-singular BIEs are derived by using two sets of boundary layers, the original water and body boundary, and an auxiliary boundary, on that the integration and solution evaluation points are distributed separately from each other ensuring that the singular cases cannot occur. The BEM implementation is realized by using two meshes with a certain spacing. The comparatively lower accuracy and possible convergence issues due to ill conditioned BIOs can be regarded as disadvantages. The ill conditioned BIOs might appear because the spacing between the surface meshes is not adapted to the discretization (can be improved by using a definition of the distance depending on the discretization) or because the BIOs are less dominated by the diagonal entries, which increase the condition number so that more solver iterations are required [40].

The intersection of body and free surface meshes is treated by considering both the free surface and body collocation points at the intersection points/curves. This method is referred as double node technique, see e.g. [41], and gives higher stability and accuracy, see e.g. [42].

The singularity subtraction method was applied by Hwang & Huang [43] in the context of hydrodynamic applications and for smooth body geometries and coinciding collocation and integration point locations. A modified BIE has been obtained by adding the conventional BIE for the total velocity potential and adding a second BIE obtained from the Gauß flux theorem multiplied with a constant velocity potential value, see also Landweber & Macagno [44]. In this new BIE formulation and assuming a continuous Dirichlet/Neumann datum of the velocity potential in 2D/3D, the integrand becomes zero in the self-influence case and the singularity vanishes. The obtained desingularized form of the BIE contains only regular integrals, evaluable by standard quadratures, so that implementation effort and computational costs can be reduced.

In 2D and 3D Galerkin BEMs, integration by parts can be used to reduce the singular order before discretization effectively. In contrast to other regularization approaches, integration by parts can be regarded a relatively general approach that has been applied for different applications e.g. plasticity and elasticity as indicated in the review on symmetric Galerkin BEM in Bonnet et al. [45]. As one example of application, the use of integration by parts in addition to Stokes' theorem leads to a representation of the hypersingular boundary integral operator in bilinear form by the surface curl and the weakly singular kernel, see e.g. [5] and [4].

3.5 Discussion

After presenting the methods and diving into the approaches, we now aim to compare the different classes of methods for singular integration qualitatively. We evaluate the methods based on their accuracy, computational cost, and implementation effort. Furthermore, we take into account, whether they have any restrictions concerning applicability to certain problems, e.g. concerning geometries or singularity orders and summarize the main outcomes in Tab. 1.

The analytic methods for singular integration in principle yield exact results and are computationally highly efficient. Usually, the computational costs are mainly driven by evaluation of trigonometric functions linked to the geometric configurations. The identification and treatment of special cases can be challenging for some algorithms and influences the implementation effort considerably. Compared to other methods, these additional cost are regarded as moderate. However, analytic integration is only available for relatively simple geometries, i.e. planar polygons, and up to quadratic order of elemental potential distributions.

More flexibility in the application for different problems is achieved by employing semi-analytic approaches due to their partly numerical treatment of the integral. In addition with transformations and series expansions, this causes more implementation effort in particular for singular cases. In general the methods offer high accuracy as the limiting process is performed explicitly so that the values in the self and near regime can be captured accurately. Nevertheless, free terms that can appear due to the limiting process, are often sensitive, e.g. with regard to discontinuities in function spaces, and require special treatment. In comparison with analytic integration, these methods are algorithmically more complex and, hence, computationally more expensive.

The purely numeric treatment of singular integrals is the most general approach of the here presented methods regarding application to different classes of geometries and potential distributions. In addition to quadratures themselves, which are relatively simple to apply, other lower-level functions are required, e.g. transformation and case specific treatment which affect the implementation effort directly. The implementation effort can be reduced considerably by using available implementations, e.g. libraries, for these lower-level functions. Usually, error estimates with h- and/or p-refinement are available for the numeric integration methods. Note, that for weighted quadratures applied to Galerkin formulations, the interdependency of inner and outer integral can have a negative impact on the overall accuracy, see [4]. The quadrature order primarily influences the efficiency and the accuracy of the numeric integration methods and an appropriate choice of the number of integration points allows to find a sweep point of these opposite features. In Galerkin BEMs, the simultaneous integration over test and trial function space possibly improves the efficiency in numeric and semi-analytic approaches in comparison to analytic integration. Beyond the scope of this paper, it is worth noting, that semi-analytic and numerical integration methods offer the additional advantage that they can also be applied to other, non-Laplace, kernel types.

The regularization approaches are comparatively efficient, due to not using computationally more involved singular integration methods. These methods are usually tailored to specific problem and assumed to be easier to implement compared to other singular handling methods. Either because complicated singular cases are avoided, or because resulting weak singularities can be treated by standard approaches, e.g. Duffy transformation and standard quadratures. Usually, the methods are applied before discretization and thus have only a limited dependency on the geometry. Compared to the singular integration methods, the method depending approximation error can be high. For some methods, additional simulation parameters might be used to control the convergence of the error (e.g. distance parameter in desingularized BIE for hydrodynamics).

4. APPLICATIONS IN HYDRO- AND AERODYNAMICS

In this chapter, the application of BEMs in the fields of hydro- and aerodynamic engineering are outlined and reviewed according to the techniques used for singular integration.

4.1 Aerodynamic applications

This section gives a brief overview over the history of BEM methods in aerodynamics concentrating on three-dimensional panel methods, i.e. methods which incorporate bodies with a thickness. During the history of these methods, the development of singularity handling initially was closely linked to the development of panel methods as well as special software solutions. Hence, focus will be put on the advances in panel methods here, while more detail concerning the singularity handling was given in Sec. 3.1.

Pioneering the solution of three-dimensional panel methods problems in the late 1960s, the so-called Hess-Code allowed the computation of flow around non-lifting bodies using elementwise constant sources and Neumann boundary condition at element-central collocation points of quadrangular elements [12]. This work was then extended to flows around lifting bodies with constant vorticity on panels [13]. It is worth noting that in this method the vorticity resides on the surface of the lifting bodies, while vorticity or doublets were usually distributed body-internally on the camber-plane of the wing at the time, see e.g. [19, 46]. Furthermore, singularity distributions were integrated over planar quadrangles, but the quadrangles connecting nodes in a watertight mesh are usually bilinear and not planar. While this sounds problematic at first, Hess clarified that the boundary conditions are only enforced in the panel centers, i.e. collocation points, and hence, he body will "leak" elsewhere [12]. Later, the SOUSSA code also used constant source and doublet distributions. However, it introduced and popularized the so-called Morino boundary condition, i.e. a Dirichlet boundary condition [47].

During the 1980s different approaches for panel methods with high-order singularity distributions have been developed, which can be considered a second generation of the development [6]. PAN AIR [18], as well as the European counterpart HISSS [19], used an approximation of curved geometry by piecewise flat sub-elements for geometric modeling. Singularity distributions with up to quadratic basis functions were integrated exactly on these elements, which resembles an approximate integration over the underlying curved panel. These codes offered the solution of Dirichlet and Neumann boundary conditions. The high-order Hess II code utilized a truncated expansion form of potentials for which analytical integration can be performed instead [20]. While methodically different, this also resembles an approximate solution of the singular integrals of quadratic potential distributions over curved panels.

High-order codes offer multiple advantages, such as better convergence and a more accurate local fulfillment of boundary conditions. The improvements of boundary condition fulfillment are caused by the continuous potential distribution. Moreover, it was show that they can applied to supersonic problems, e.g. [18, 48]. However, the derivation of the required singular integration methods are more involved. In addition, the software development proved considerably more complex.

VSAERO [49] was one of the first developments stepping back to constant distribution low-order codes. Also due to the advances in computer technology at the time, this code allowed smaller companies to utilize panel methods commercially [6]. Consequently, low-order codes became popular again especially for subsonic applications and remain so to this day. The main reason for this popularity is that they are methodically relatively simple and, hence, require only moderate implementation effort in comparison. They yield results, which are widely considered accurate enough for their applications. The presentation of low-order methods in textbooks, e.g. [6], further influenced the development of codes, e.g. [50]. As another example in use today, UPM [46] is a panel method specifically tailored for sim-

TABLE 1: CHARACTERISTICS OF METHODS (+: ADVANTAGE, ∘: NEUTRAL, -:DISADVANTAGE) WITH RESPECT TO SEC. 3.5 AND RELATED EXEMPLARY REFERENCES (SEE ALSO SEC. 3.1-SEC. 3.4).

Method	Implementation effort	Applicability	Accuracy	Efficiency	Exemplary references
Analytic	∘	-	exact	+	[11, 12, 15]
Semi-Analytic	∘	+	+	∘	[1, 2, 25]
Numeric	+	+	+	∘	[29, 30, 36]
Regularization	+	-	∘	+	[40, 43, 45]

ulation of helicopters and works similar to the low-order version of [19]. Nevertheless, research and development of high-order methods using linear panels [51] and also curved panels [22] is still ongoing.

4.2 Hydrodynamic applications

The BEM in hydrodynamics is used in the framework of the potential flow assumption, modeled by the Laplace equation Eq. (1), for the analysis of wave-body and wave-wave interaction in the view of linear and nonlinear dynamics. By converting the water-body interaction problem into a BVP, the BEM can be efficiently used and the advantage of dimension reduction is significant.

The BIEs related to Eq. (4), Eq. (5) or combinations of both can be used to determine the main quantities of interest, the velocity potential and the velocity at the body surface, and (when solving for the free surface) the velocity at the free surface. In this regard, the boundary conditions at the free surface ($z = \eta$) are essential to capture the dynamics of waves. It reads in linear (nonlinear) form, see e.g. Mei et al. [52],

$$\frac{\partial \phi}{\partial t} = -g\eta - \frac{1}{2}(\nabla\phi)^2 \quad \text{and} \quad \frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} \tag{10}$$

with the velocity potential ϕ, the surface elevation η and the gravitational acceleration g.

In the linear regime of dynamics, the superposition principle is valid, see e.g. Clauss et al. [53]. This allows to analyze the wave and body dynamics frequency-wise and using spectra to represent the correlated energy of wave and body motion in frequency domain.

For nonlinear wave-body interaction, the superposition principle is not valid as nonlinear interactions between the wave components appear and time domain methods have to be considered for resolving the nonlinear motions.

It follows the specification of the, in our opinion, three main variants of BEMs for hydrodynamic applications. Supplementary, the reviews in [54–57] can be used as additional sources.

Free surface Green function (FSGF) methods use tailored Green functions. These fulfill the free surface (see Eq. (10)) boundary condition, as well as the radiation condition by construction, see e.g. [58–60]. Consequently, only the discretization of the body surface needs to be considered (except for QTF analysis). Linear wave-body interaction problems analyzed in frequency domain for the evaluation of hydrodynamic coefficients are in focus and typically, (low-order) panel methods have been employed whose development had been significantly influenced by the aerodynamic codes described in Sec. 4.1. The singular integration is usually treated with analytic integration or regularization methods.

Rankine source (RS) approaches are used for the modeling of both linear and nonlinear wave-body interaction, see e.g. [61–64]. Contrary to FSGF methods, the free space Green function and its derivatives require the explicit consideration of the free surface BCs, i.e. the equation is solved on body and water surface requiring the discretization of both domains. Crucial for these methods is the radiation condition (see e.g. Dawson's methods [65]). In addition to body surface desingularization, the singularities at the free surface have usually been treated by regularization techniques, e.g. by defining the evaluation points in a distance to the boundary.

Boundary integral equation methods for marine hydrodynamics (BIEMH) represent high-order BEMs for the analysis of nonlinear wave-wave and wave-body interaction and typically considered for numerical wave tank modeling, see e.g. [66–71]. For the analysis of e.g. overturning waves, the MEL approach, [72], is used and allows to track the water particles due to considering the Lagrangian form of the boundary condition in an Eulerian BIE frame work. For the treatment of singularities in the neighborhood of the intersection of water and body surface, the double node technique has been employed in the hydrodynamic regularization methods. Here, the methods suited for integration over trimmed surfaces in isogeometric BEMs, see for an overview Marussig & Hughes [73], could be considerable choices and alternatives e.g. for the double node technique.

4.3 Singular integration in hydrodynamic softwares

The singular integration for some selected 3D FSGF codes are shortly outlined hereafter. These programs are used for the hydrodynamic analysis of ships and offshore structures in the maritime industry and research sector and have been continuously developed over the past thirty-five years. For the low-order method in WAMIT [74–77], the analytic integration of Newman [15], see Sec. 3.1 is applied in addition with the method proposed in [75] for logarithmic expansion terms occurring in the FSGF formulation. The integration in the high-order method of WAMIT [74] is done separately for the test and trial function space. For the inner integration, the strong singularity related terms are treated either in the CPV sense in the self-influence case or by patch subdivision in the near-field regime. The outer integration and all regularized integrals are numerically integrated by the Gauß-Legendre quadrature. The panel method panMARE [50] uses analytic treatment for singular integration. In HYDROSTAR [59], the approximation of the singular kernels by Chebyshev basis polynomial have been used for singular integration. In NEMOH [60, 78], the singular integration is treated by the method of Delhommeau [79]. Pre-computable basis integrals have been identified to reconstruct the singular FSGFs, and the application of Lagrange interpolation to the tabulated data has provided an efficient method for non-far-field integration. In [54], the methods of Delhommeau and Newman with other singular integration approaches and have shown that the efficiency is comparable, but the accuracy varies between the methods.

5. CONCLUSIONS

In this paper, we provided an overview on approaches for singular integration in BEMs used in the context of hydro- and aerodynamics. After detailing the mathematical background of singular integrals, we classified the singular integration methods and used exemplary references to outline the group specific concepts. Subsequently, we compared them qualitatively according to efficiency, accuracy, feasibility and applicability, indicating the advantageous characteristics as well as the drawbacks of the methods. In the following, we analyzed the singularity handling within well known BEM software solutions for hydro- and aerodynamics. This analysis clearly indicates that mainly analytic and regularization approaches are used within the considered community codes. Furthermore, it was shown that panel methods remain relevant and are widely applied for industry related research and development. These methods are relatively easy to implement and offer sufficient accuracy at low computational cost, which is an especially useful combination for early design phases. Nevertheless, nonlinear dynamics can be regarded only with limited accuracy by this methods. To capture the nonlinear phenomena of the underlying physics with an improved consistency, high-order methods are required. In this context, the use of alternative techniques for singular integration shortly outlined in this paper, could be reviewed concerning possible improvements induced by their advantageous characteristics. In future work, a quantitative comparison of selected methods for relevant test cases could prove to be valuable, specifically regarding computational efficiency and accuracy. To achieve this, the implementation of these methods within a unified test framework would be required.

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