Viscosity measurement by the “Oscillating Drop“ method:
Limits of the linear model

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Problems of liquid metal processing

High temperatures of liquid metal

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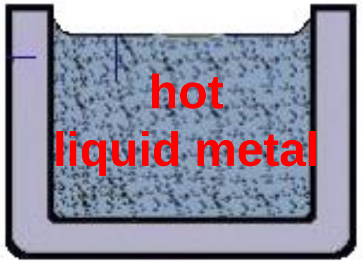
- **Chemical reactions**
- **Mechanical interaction**

with crucible



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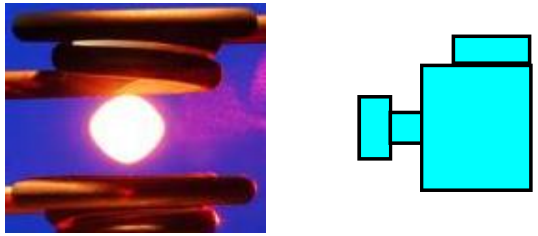
Solution

Contactless

Handling + Measurement



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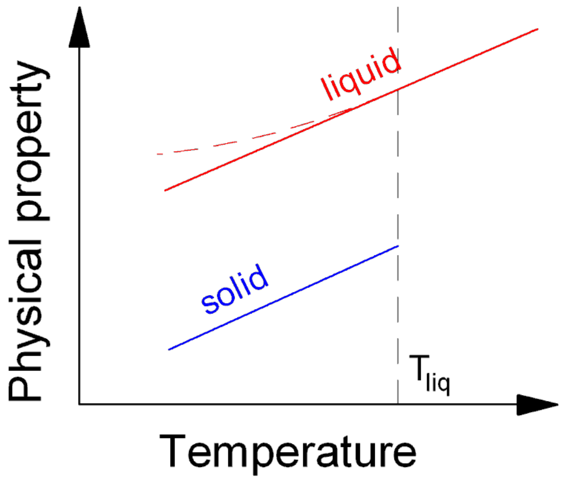
Benefit

Undercooled

temperature range accessible

⇒ **Enlargement** of Temp.-range

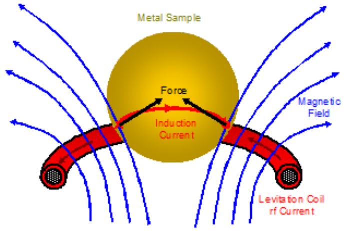
⇒ **Structure** formation in liquids



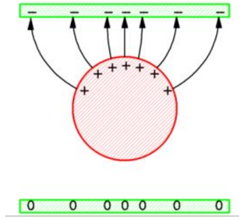
The **"Oscillating Drop"** method:

Contactless measurement of **viscosity** η

1. **Containerless handling** (levitation)



rf-electromagnetic

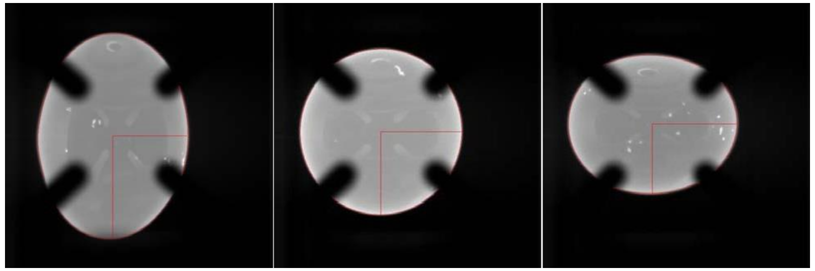


electrostatic

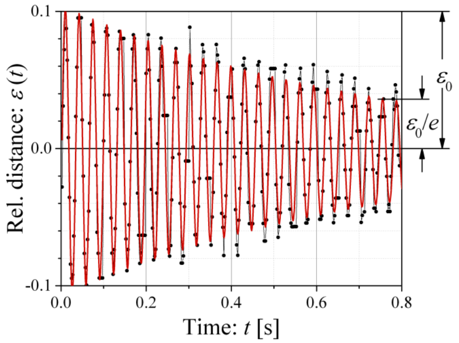
2. **Contactless melting** (induct., laser)

⇛ **Liquid droplet**

3. **Excitation** of **oscillations**



4. **Contactless** (optical, inductive) **detection** of
- damping time: τ
- frequency: ω



5. **Evaluation**: Lamb formula

$$\frac{1}{\tau} = \frac{20\pi}{3}\frac{R}{M}\eta$$

damping time

viscosity

Conditions

- ➢ **Spherical droplet**
- ➢ **Negligible fluid flow** (levitation induced)

⇛

- **Low gravity** (TEMPUS)
- **Electrostatic levitation**

➢ **Narrative**: $\Delta R/R << 1$ ⇒ Everything all right !!

Expt. finding: Lamb formula fails for low viscous liquids !

⇒ **Revision of the basic theory**

Theoretical description

Navier-Stokes eq. for oscillating fluid flow

(Newton's law of motion)

$$\rho \frac{d}{dt}\mathbf{v} = \underbrace{\eta\nabla^2\mathbf{v}}_{\text{Viscous shear force}} - \underbrace{\nabla p(\gamma)}_{\text{Surface tension driving force}}$$

Simplified model

of
Chandrasekhar,
Lamb, Rayleigh

Linearization

$$\rho \frac{\partial}{\partial t}\mathbf{v} = \underbrace{\eta\nabla^2\mathbf{v}}_{\text{Viscous shear force}} - \underbrace{\nabla p(\gamma)}_{\text{Surface tension driving force}}$$

Results: (axially symmetric)

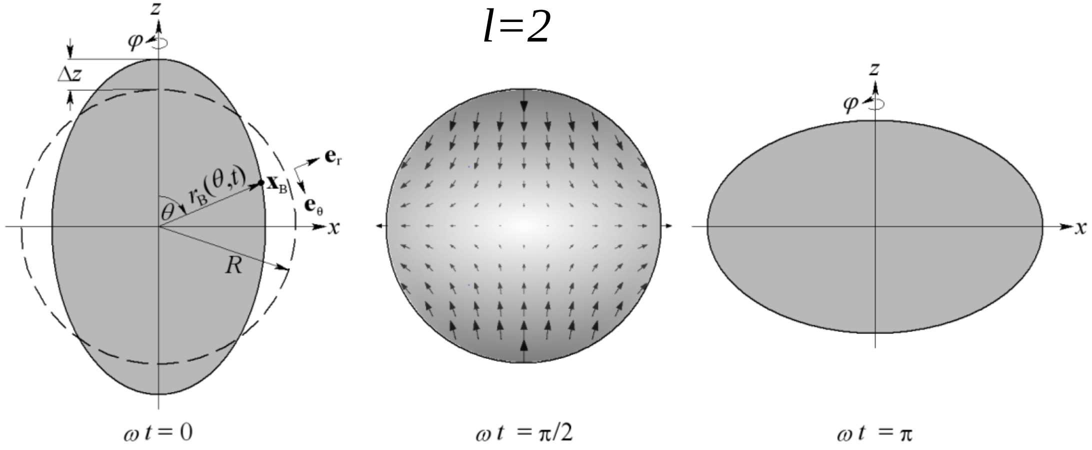
$$\mathbf{v}(\mathbf{x},t) = \sum_{l\geq 2} \frac{\Delta z_l}{R}\,\mathrm{Re}\left[\tilde{\mathbf{w}}_l(\mathbf{x})\, e^{i\omega_l t}\right] e^{-t/\tau_l}$$

$$\omega_l^2 = l(l+2)(l-1)\frac{\gamma}{\rho R^3}$$:Ray.

$$\frac{1}{\tau_l} := (2l+1)(l-1)\frac{\eta}{\rho R^2}$$:Lamb

Condition: $\Delta z_l / R << 1$

$l=2$



Question: How well approximates the **linear Nav.-Stokes** eq. the **experimental** findings ?

Genuine Navier-Stokes eq.

Linear Navier-Stokes eq.

$$\rho \frac{\partial}{\partial t}\mathbf{v} + \underbrace{\rho\,\mathbf{v}\cdot\nabla\mathbf{v}}_{\substack{\text{Non-linearity}\\ \text{Trouble maker}}} = \underbrace{\eta\nabla^2\mathbf{v}}_{\substack{\text{Viscous}\\ \text{shear force}}} - \underbrace{\nabla p(\gamma)}_{\substack{\text{Surface tension}\\ \text{driving force}}}$$

$$\approx$$

$$\rho \frac{\partial}{\partial t}\mathbf{v}_{lin} = \underbrace{\eta\nabla^2\mathbf{v}_{lin}}_{\substack{\text{Viscous}\\ \text{shear force}}} - \underbrace{\nabla p_{lin}(\gamma)}_{\substack{\text{Surface tension}\\ \text{driving force}}}$$

Linear model

Lamb formula (viscosity η) **applicable if:**

$$\frac{O(\textbf{Non}-\textbf{linearity})}{O(\textbf{Visc. shear force})} = \frac{O(\rho\,\mathbf{v}_{lin}\cdot\nabla\mathbf{v}_{lin})}{O(\eta\nabla^2\mathbf{v}_{lin})} << 1$$

!!!

Rayleigh formula (surf. tens. γ) **applicable if:**

$$\frac{O(\textbf{Non}-\textbf{linearity})}{O(\textbf{Surf. tension})} = \frac{O(\rho\,\mathbf{v}_{lin}\cdot\nabla\mathbf{v}_{lin})}{O(\nabla p_{lin}(\gamma))} << 1$$

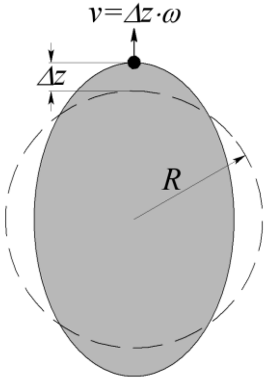
⇓

Question: How to define the **order of magnitude** of a vector field ?

Order of magnitude: First try

Rough estimate:

$$\frac{O(\text{Non}-\text{linearity})}{O(\text{Visc. shear force})} = \frac{O(\rho\,\mathbf{v}_{lin}\cdot\nabla\mathbf{v}_{lin})}{O(\eta\,\nabla^2\mathbf{v}_{lin})} \approx \frac{\rho\,R}{\eta}v_{ch} = Re << 1 \quad (\text{Reynolds No.})$$



with

1.) $\eta = \rho R^2/5\tau$ (Lamb)

2.) $v_{ch} \approx \frac{\partial}{\partial t}(R+\Delta z\sin(\omega t)) \approx \Delta z\,\omega$

$$\boxed{\Rightarrow \quad Re \approx 5\frac{\Delta z}{R}\omega\tau = 10\pi\frac{\Delta z}{R}Ne << 1}$$

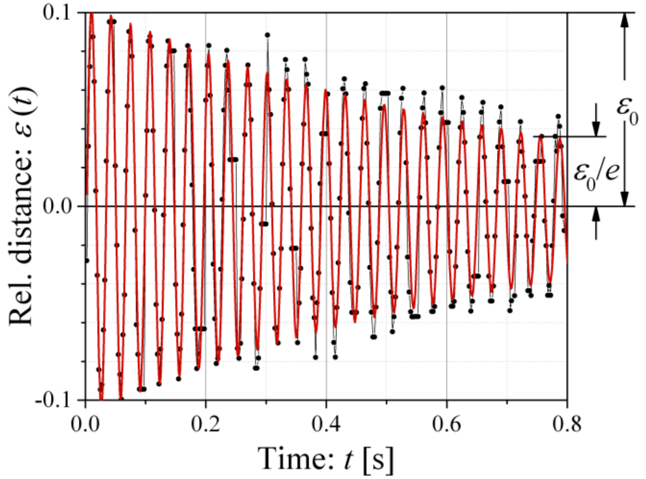
$\omega\tau/2\pi = Ne$: Number of oscillations until the amplitude dropped down to $1/e$

2 Relevant parameters !

$\Delta z/R$: Amplitude (normal.)

Ne: Oscillation number

Expt. result:



$\Delta z/R \approx 0.1$

$Ne \approx 23$

$$\boxed{\Rightarrow \quad Re \approx 70 >> 1 \text{ !!!}}$$

Order of magnitude: Use of analytical results

Order of magnitude
defined by **quadratic mean**

$$O(\mathbf{A}) := \sqrt{\left\langle \left\|\mathbf{A}(\mathbf{x},t)\right\|^2 \right\rangle} := \sqrt{\frac{1}{V(S)}\int_S \left(\frac{1}{\tau}\int_0^\infty \left\|\mathbf{A}(\mathbf{x},t)\right\|^2 dt\right) d^3x}$$

Results from **linear model**:

$$\boxed{\frac{O(\text{non}-\text{linearity})}{O(\text{Visc. shear force})} = \sqrt{\left\langle \left\|\rho\,\mathbf{v}_{lin}\cdot\nabla\mathbf{v}_{lin}\right\|^2 \right\rangle}\Big/\sqrt{\left\langle \left\|\eta\,\nabla^2\mathbf{v}_{lin}\right\|^2 \right\rangle} \sim 2.4\,\frac{\Delta z_2}{R}\sqrt{Ne_2}}\;\boxed{\approx \begin{cases} 1 & \text{for } \Delta z_2/R \approx 0.1\,,\ Ne \approx 23 \\ 0.5 & \text{for } \Delta z_2/R \approx 0.1\,,\ Ne \approx 5 \end{cases}}$$

$\Rightarrow$ **Lamb** formula **applicable** only for **small** values of oscillation number ***Ne*** !

Results from **linear model**:

$$\boxed{\frac{O(\text{non}-\text{linearity})}{O(\text{Surf. ten. force})} = \sqrt{\left\langle \left\|\rho\,\mathbf{v}_{lin}\cdot\nabla\mathbf{v}_{lin}\right\|^2 \right\rangle}\Big/\sqrt{\left\langle \left\|\nabla p_{lin}(\gamma)\right\|^2 \right\rangle} \sim \frac{1}{2}\,\frac{\Delta z_2}{R}}$$

$\Rightarrow$ **Rayleigh** formula **applicable** for **small** values of norm. amplitude **$\Delta z/R$** !

Significance of the viscous **shear force**

Linear Navier-Stokes eq.

$$\rho \frac{\partial}{\partial t}\mathbf{v}_{lin} = \underbrace{\eta \nabla^2 \mathbf{v}_{lin}}_{\text{Viscous shear force}} - \underbrace{\nabla p_{lin}(\gamma)}_{\text{Surface tension driving force}}$$

Results from the **linear model**:

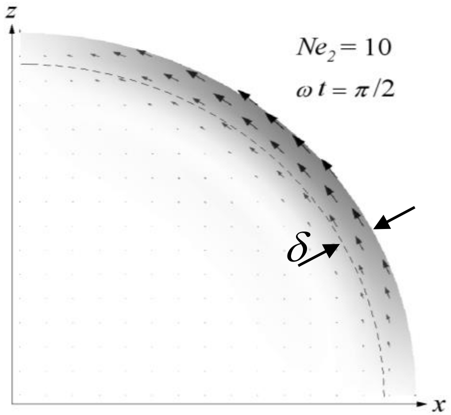
$$\frac{O(\text{Visc. shear force})}{O(\text{Driving force})} = \sqrt{\left\langle \left\| \eta \nabla^2 \mathbf{v}_{lin} \right\|^2 \right\rangle} \Big/ \sqrt{\left\langle \left\| \nabla p_{lin}(\gamma) \right\|^2 \right\rangle} \sim \frac{0.2}{Ne^{3/4}} < \begin{cases} 0.12 & \text{for } Ne \geq 2 \\ 0.02 & \text{for } Ne = 23 \end{cases}$$

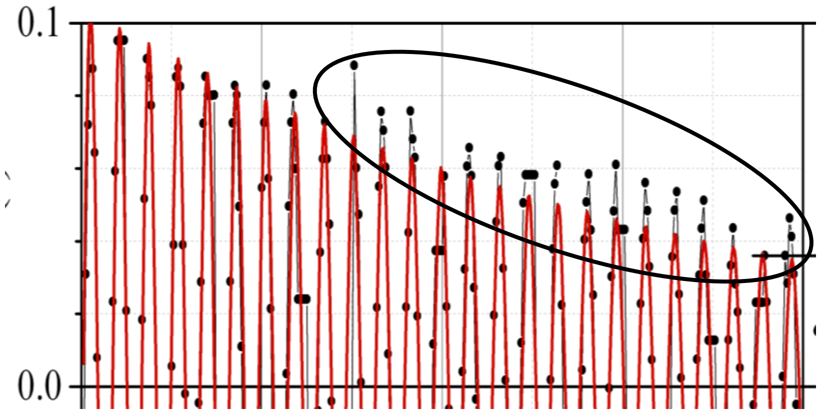
$\Rightarrow$ Visc. **shear force** **<<** γ-induced **driving force**

Shear force field

restricted to small boundary layer (exp. decrease)

$$\delta / R = \frac{1}{\sqrt{5\pi Ne_2}} \approx \begin{cases} 0.2 & \text{for } Ne = 2 \\ 0.04 & \text{for } Ne = 23 \end{cases}$$



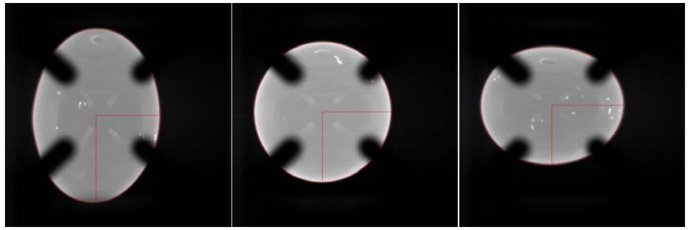


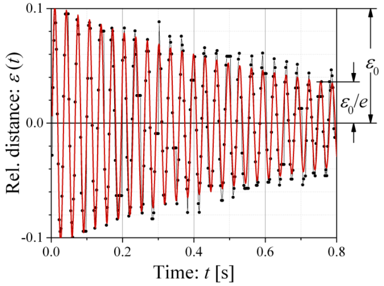
Small **external distortions** result in significant **deviations** from exp. damping behavior !

Summary

- **Oscillating drop** method

  Excitation of free floating liquid droplet oscillations





- **Simplified** theoretical **basis**

  Linear Navier-Stokes eq.

$$\rho \frac{\partial}{\partial t}\mathbf{v} + \mathbf{0} = \underbrace{\eta \nabla^2 \mathbf{v}}_{\text{Viscous shear force}} - \underbrace{\nabla p(\gamma)}_{\text{Surface tension driving force}}$$

$\Rightarrow$ **Requirements**

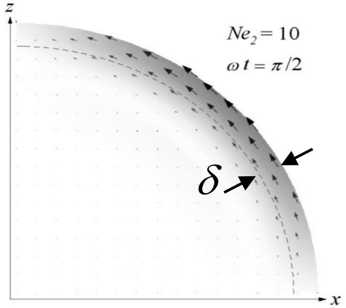
Lamb formula (viscosity η) **applicable if:** $2.4\Delta z/R\sqrt{Ne} << 1 \Rightarrow$ **few** oscillations only

Rayleigh formula (surf. tens. γ) **applicable if:** $0.5\Delta z/R \quad << 1$

- **Shear force**

  Minor significance
  in oscillating fluid flow

$$\frac{O(\text{Visc. shear force})}{O(\text{Driving force})} \sim \frac{0.2}{Ne^{3/4}} < 0.12$$



Accurate viscosity measurement by ODM needs **particular care** !!!