Direct Computation of Higher Order Mode Components from Successively Linearized Geometrically Nonlinear Beam Theory

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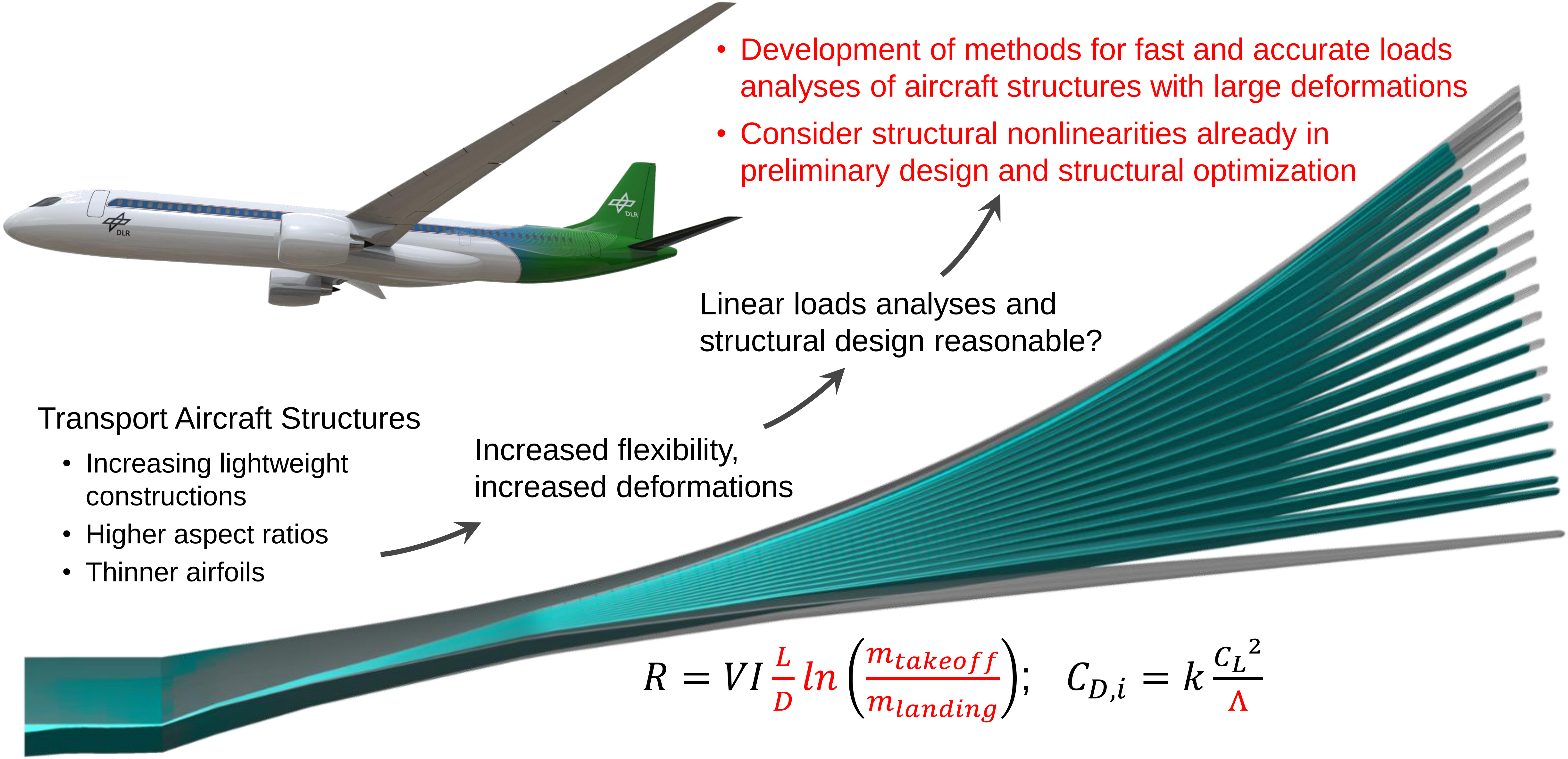
Large Deformations in Aircraft Design



- Development of methods for fast and accurate loads analyses of aircraft structures with large deformations
- Consider structural nonlinearities already in preliminary design and structural optimization

Linear loads analyses and structural design reasonable?

Transport Aircraft Structures

- Increasing lightweight constructions
- Higher aspect ratios
- Thinner airfoils

Increased flexibility, increased deformations

$$R = V I \frac{L}{D} \ln\left(\frac{m_{takeoff}}{m_{landing}}\right); \quad C_{D,i} = k\frac{{C_L}^2}{\Lambda}$$

Extended Modal Approach

An improved structural method for geometrically nonlinear aeroelastic analyses was proposed

Linear modal approach

$$\boldsymbol{u} = \boldsymbol{\Phi}\,\boldsymbol{q}$$

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{D}\dot{\boldsymbol{q}} + \boldsymbol{\Omega}^2\boldsymbol{q} = \boldsymbol{Q}$$

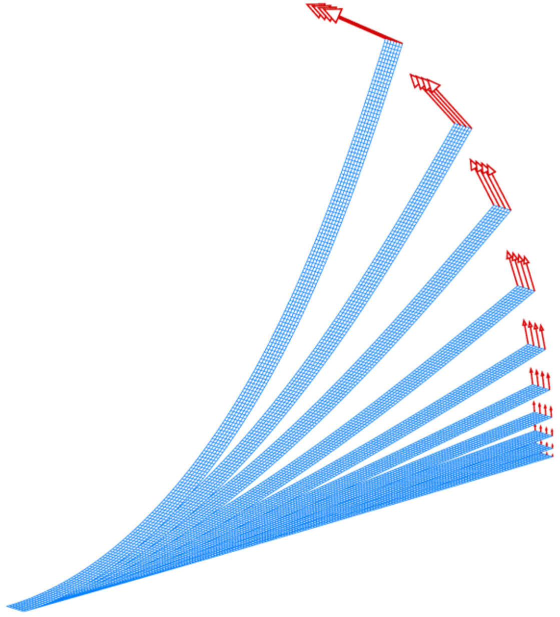
Nonlinear extensions

$$\boldsymbol{u} = \boldsymbol{\Phi}\,\boldsymbol{q} + \tilde{\boldsymbol{\Phi}}(\boldsymbol{q}^2, \boldsymbol{q}^3, \boldsymbol{q}^4) \qquad (1)$$

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{D}\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}, \boldsymbol{q}^2, \boldsymbol{q}^3, \boldsymbol{f}) = \boldsymbol{Q} \qquad (2)$$

1. Higher-order modal components yield geometrical nonlinearities
2. Nonlinear stiffness terms for nonlinear force-displacement relations

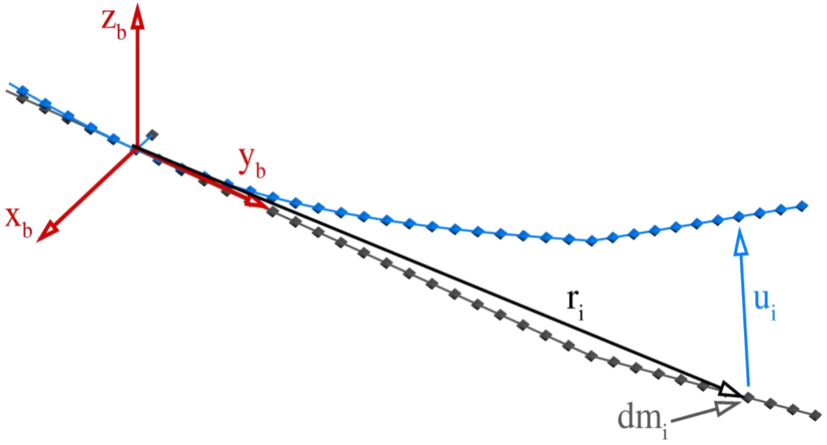
- Assumes nonlinear nodal displacement field is still composed of modes
- Developed for moderately large deflections, $\mathcal{O}\ (< 30\%)$
- Low computational costs (few DOFs: modal space, no iterative solution)
- Applicable to complex/arbitrary FE models

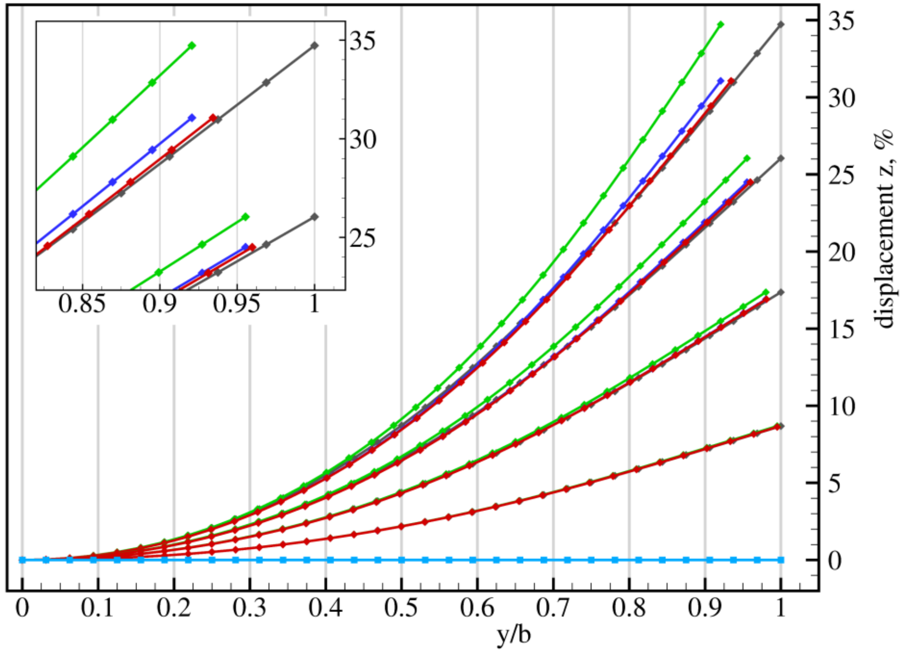


Ritter, M. R., **An Extended Modal Approach for Nonlinear Aeroelastic Simulations of Highly Flexible Aircraft Structures**, Dissertation, Technische Universität Berlin, 2018

Extended Modal Approach

- Geometrically nonlinear displacements are represented by *higher-order mode components*
- They are higher order tensors and enable the geometric “coupling” of the individual modes





$$\boldsymbol{u}(\boldsymbol{q}) = {}^{p}\boldsymbol{\Phi}_0 \; q_p + {}^{p}\boldsymbol{\Phi}_1^{i} \, q_p q_i + {}^{p}\boldsymbol{\Phi}_2^{ij} \; q_p \, q_i q_j + {}^{p}\boldsymbol{\Phi}_3^{ijk} \; q_p q_i q_j q_k$$

mode components: linear quadratic cubic quartic/fourth order

The mode itself becomes a function of the amplitude $(\boldsymbol{q})$:

$${}^{p}\boldsymbol{\Phi}(\boldsymbol{q}) = {}^{p}\boldsymbol{\Phi}_0 \; + \; 2\,{}^{p}\boldsymbol{\Phi}_1^{i} \, q_i + 3\,{}^{p}\boldsymbol{\Phi}_2^{ij} \, q_i q_j \; + \; 4\,{}^{p}\boldsymbol{\Phi}_3^{ijk} \, q_i q_j q_k$$

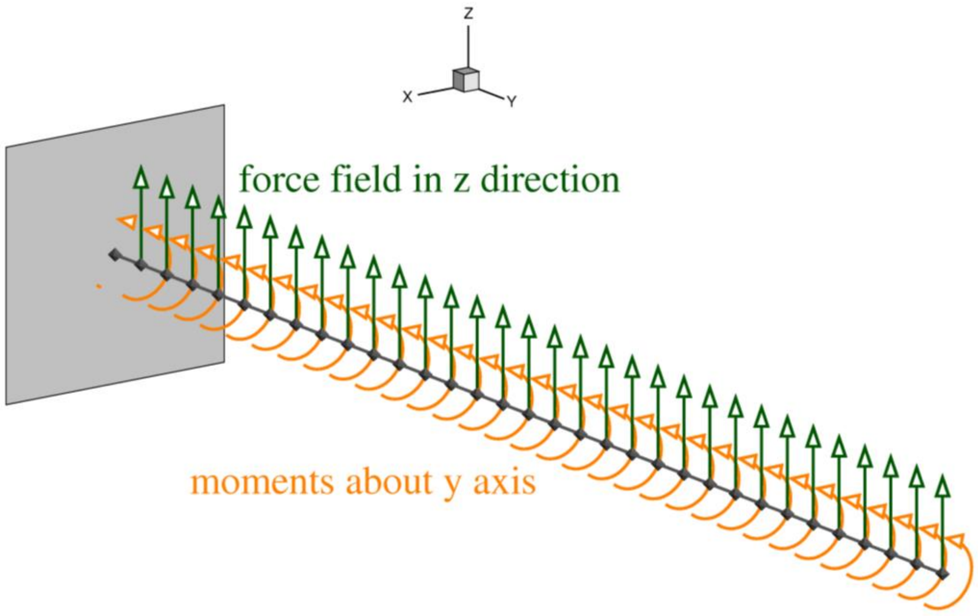
Extended Modal Approach

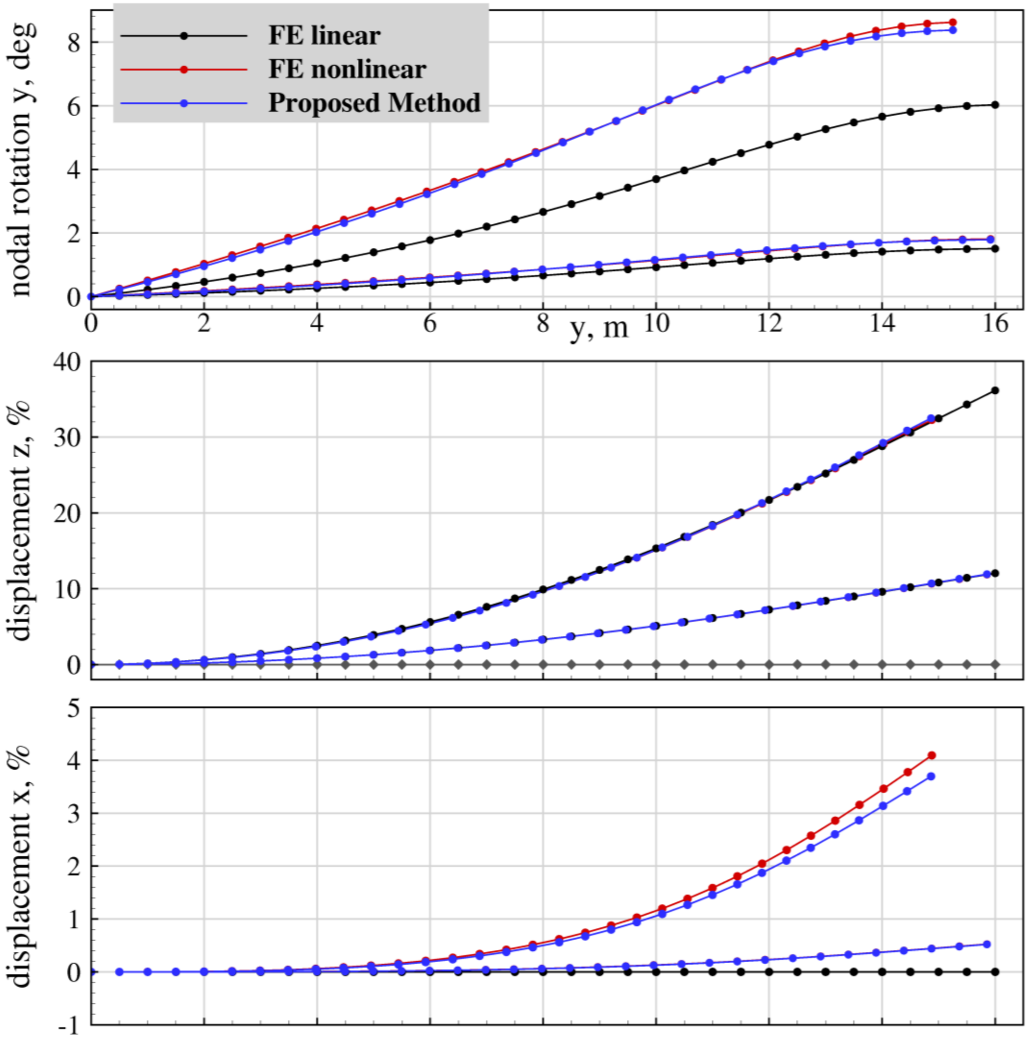
Nonlinear (static) governing equation: Linear stiffness term becomes a function of the applied loads

$$\delta V = \delta \boldsymbol{u}^T \boldsymbol{f}$$

$$Q^p = {}^p\boldsymbol{\Phi}_0^T \boldsymbol{f} + {}^p{\boldsymbol{\Phi}_1^i}^T \boldsymbol{f}\, q_i$$

$$\left({}^p G_1^i - {}^p{\boldsymbol{\Phi}_1^i}^T \boldsymbol{f}\right) q_i + {}^p G_2^{ij} q_i q_j + {}^p G_3^{ijk} q_i q_j q_k = {}^p\boldsymbol{\Phi}_0^T \boldsymbol{f}$$





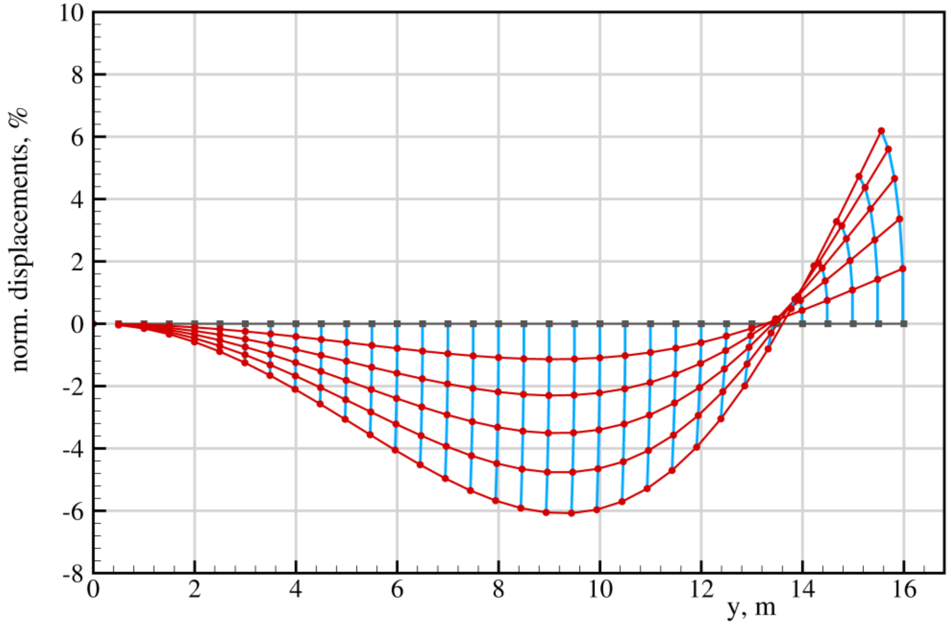
Extended Modal Approach

- A difficulty with the method is the computation of the higher order stiffness terms and mode components
- So far, an identification process was used:
  - Define a set of force fields and run a series of nonlinear, static simulations (MSC Nastran SOL 400)

$${}^{ab}\boldsymbol{F}_{ij} = \boldsymbol{K} \cdot \left({}^{a}s_i\, \boldsymbol{\phi}^i + {}^{b}s_j\, \boldsymbol{\phi}^j \right) \qquad \boldsymbol{K} = \text{(linear) stiffness matrix, } s = \text{scaling factor}$$

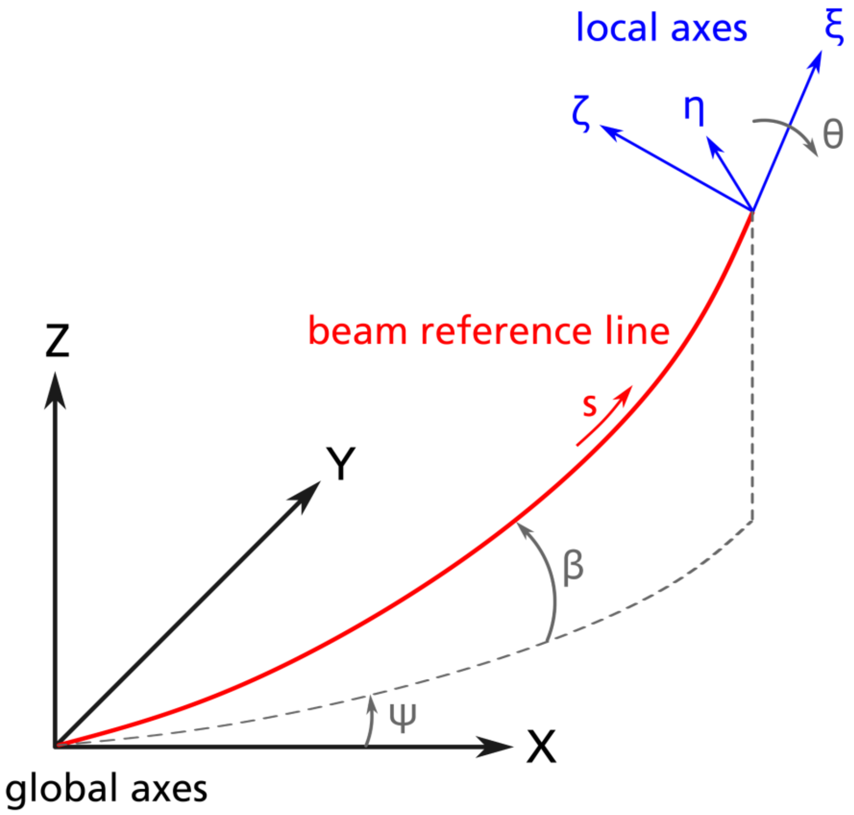
  - Use polynomials to fit the (nonlinear) displacements
  - Higher order stiffness terms and mode components correspond to the polynomial coefficients
- Advantages of this process:
  - Can be used with nearly all kinds of FE models
- Drawbacks:
  - Complicated process which requires lots of experience
  - Computationally costly for large FE models (GFEMs)
  - Structure must be clamped

→ A more “direct” method would be nice



Geometrically Exact Beam Theory (GEBT)

Consider the displacement-based, geometrically exact beam theory



- Loads - strain/curvature relations

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & \mathrm{SYM} & & & E_{55} & E_{56} \\ & & & & & E_{66} \end{bmatrix} \begin{bmatrix} \epsilon \\ \gamma_{\zeta\eta} \\ \gamma_{\xi\zeta} \\ \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}$$

- Compatibility equations

$$\frac{d\theta}{ds} = \omega_\xi - sin\theta\, tan\beta\, \omega_\eta - cos\theta\, tan\beta\, \omega_\zeta \qquad \frac{dx}{ds} = (1+\epsilon)\, cos\beta\, cos\psi$$

$$\frac{d\beta}{ds} = -cos\theta\, \omega_\eta + sin\theta\, \omega_\zeta \qquad \frac{dy}{ds} = (1+\epsilon)\, cos\beta\, sin\psi$$

$$\frac{d\psi}{ds} = \frac{sin\theta}{cos\beta}\omega_\eta + \frac{cos\theta}{cos\beta}\omega_\zeta \qquad \frac{dz}{ds} = (1+\epsilon) sin\beta$$

Minguet, P. and Dugundji, J., **Experiments and analysis for composite blades under large deflections. I - Static behavior**, 1990, AIAA Journal, Vol. 28, No 9

“Linearized” GEBT

The compatibility equations are linearized to obtain cubic, quadratic, and linear expressions for the orientations of the beam nodes (Euler angles) and displacements

Compatibility equations for Euler angles (cubic)

$$\frac{d\theta}{ds} = \omega_\xi - (\theta - \frac{1}{6}\theta^3)\,(\beta + \frac{1}{3}\beta^3)\omega_\eta - (1 - \frac{1}{2}\theta^2)\,(\beta + \frac{1}{3}\beta^3)\omega_\zeta$$

$$\frac{d\beta}{ds} = -(1-\theta^2)\,\omega_\eta + (\theta - \frac{1}{6}\theta^3)\,\omega_\zeta$$

$$\frac{d\psi}{ds} = \frac{\theta - \frac{1}{6}\theta^3}{1-\beta^2}\omega_\eta + \frac{1-\theta^2}{1-\beta^2}\omega_\zeta$$

Compatibility equations (quadratic)

$$\frac{d\theta}{ds} = \omega_\xi - \theta\,\beta\omega_\eta - (1 - \frac{1}{2}\theta^2)\,\beta\,\omega_\zeta$$

$$\frac{d\beta}{ds} = -(1-\theta^2)\,\omega_\eta + \theta\,\omega_\zeta$$

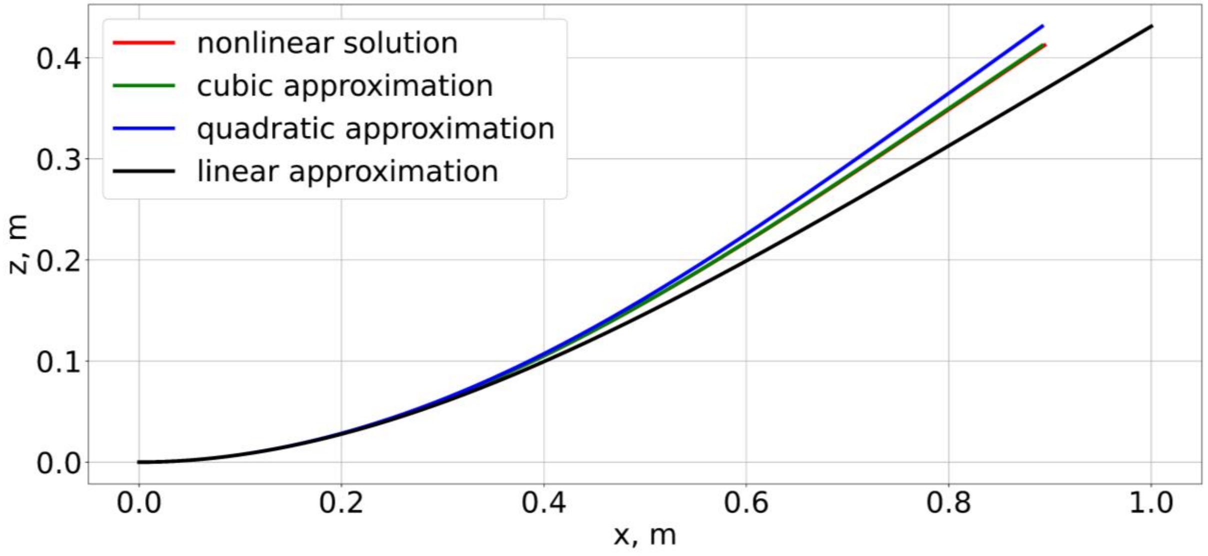
$$\frac{d\psi}{ds} = \frac{\theta}{1-\beta^2}\omega_\eta + \frac{1-\theta^2}{1-\beta^2}\omega_\zeta$$

Compatibility equations (linear)

$$\frac{d\theta}{ds} = \omega_\xi$$

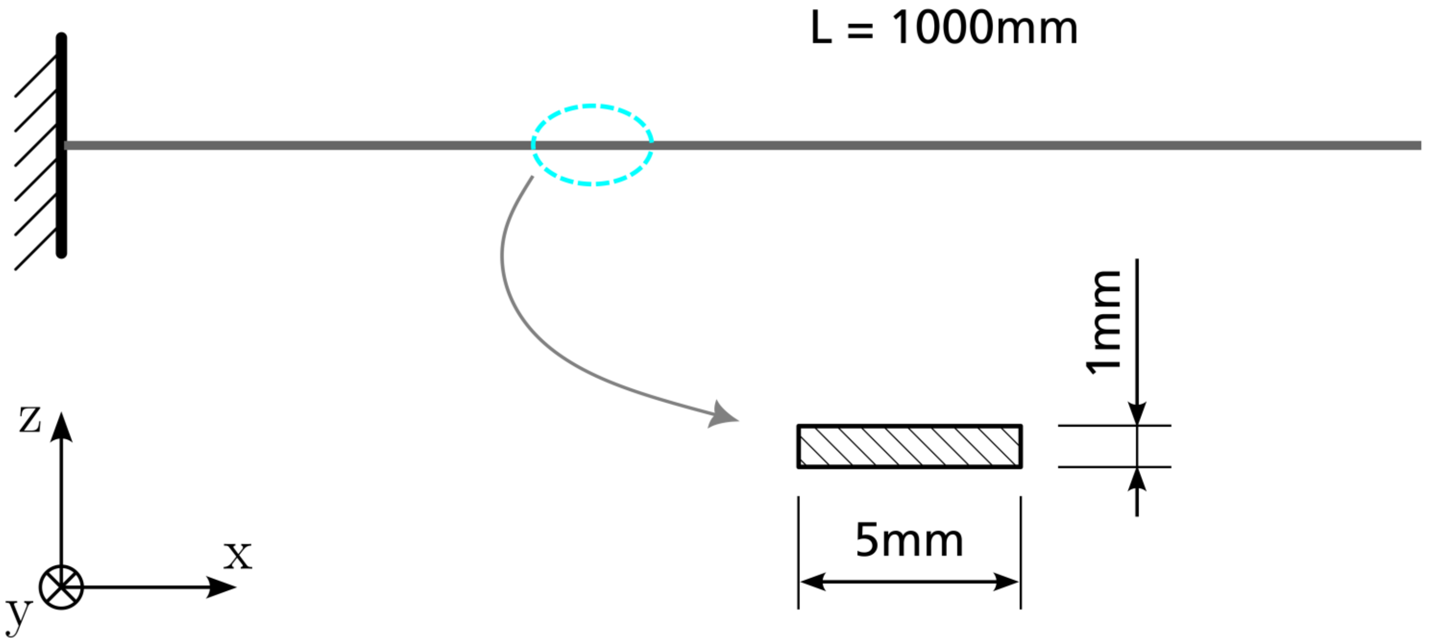
$$\frac{d\beta}{ds} = -\omega_\eta$$

$$\frac{d\psi}{ds} = \omega_\zeta$$



“Linearized” GEBT

- The Extended Modal Approach is based on mode shapes with higher order components and couplings between individual modes
- We will compute higher order mode components for beams using the successively linearized constitutive equations and the curvature functions of the individual modes
- Let’s consider a simple example (homogeneous 1m beam) with an analytical description and solution



“Linearized” GEBT

- Dynamic governing equation of motion

$$EI\frac{\partial^4 w(s,t)}{\partial s^4} + \rho A\frac{\partial^2 w(s,t)}{\partial t^2} = 0$$

- The mode shapes and frequencies are eigensolutions to this equation, they represent standing waves
- For a clamped-free beam, the displacements (in the z direction) are given as

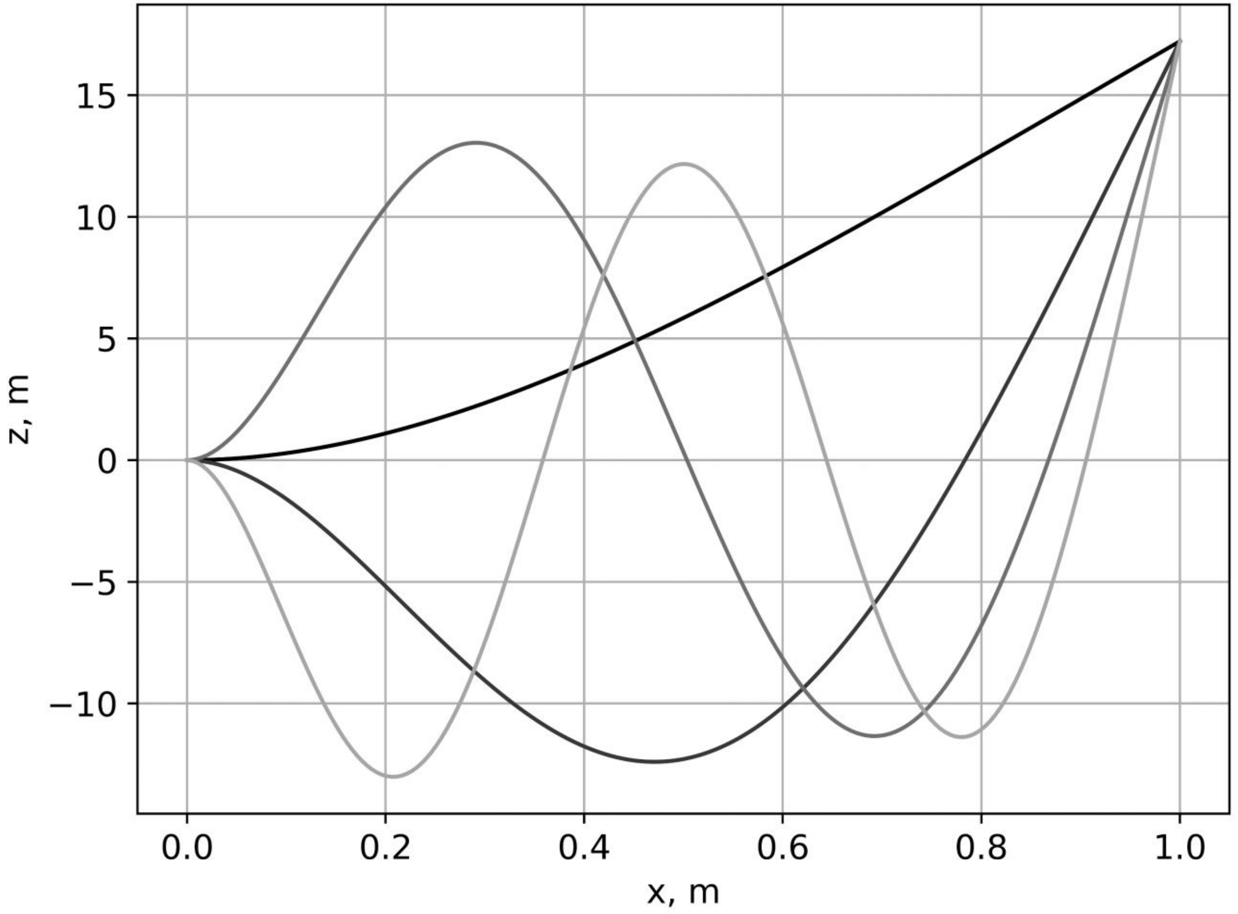
$$w(s) = 0.5\left[\left(-cos(k_n\, s)+cosh(k_n\, s)\right)+\frac{cos(k_n\, L)+cosh(k_n\, L)}{sin(k_n\, L)+sinh(k_n\, L)}\left(sin(k_n\, s)-sinh(k_n\, s)\right)\right]$$

- k_n denotes the roots of the corresponding oscillation function ($k_1 \approx 1.875104$, $k_2 \approx 4.694091$, etc.)

$$cos(k_n L)\, cosh(k_n L) + 1 = 0 \qquad\qquad \omega^2 = \sqrt{\frac{EI\, k_n^4}{\rho A}}$$

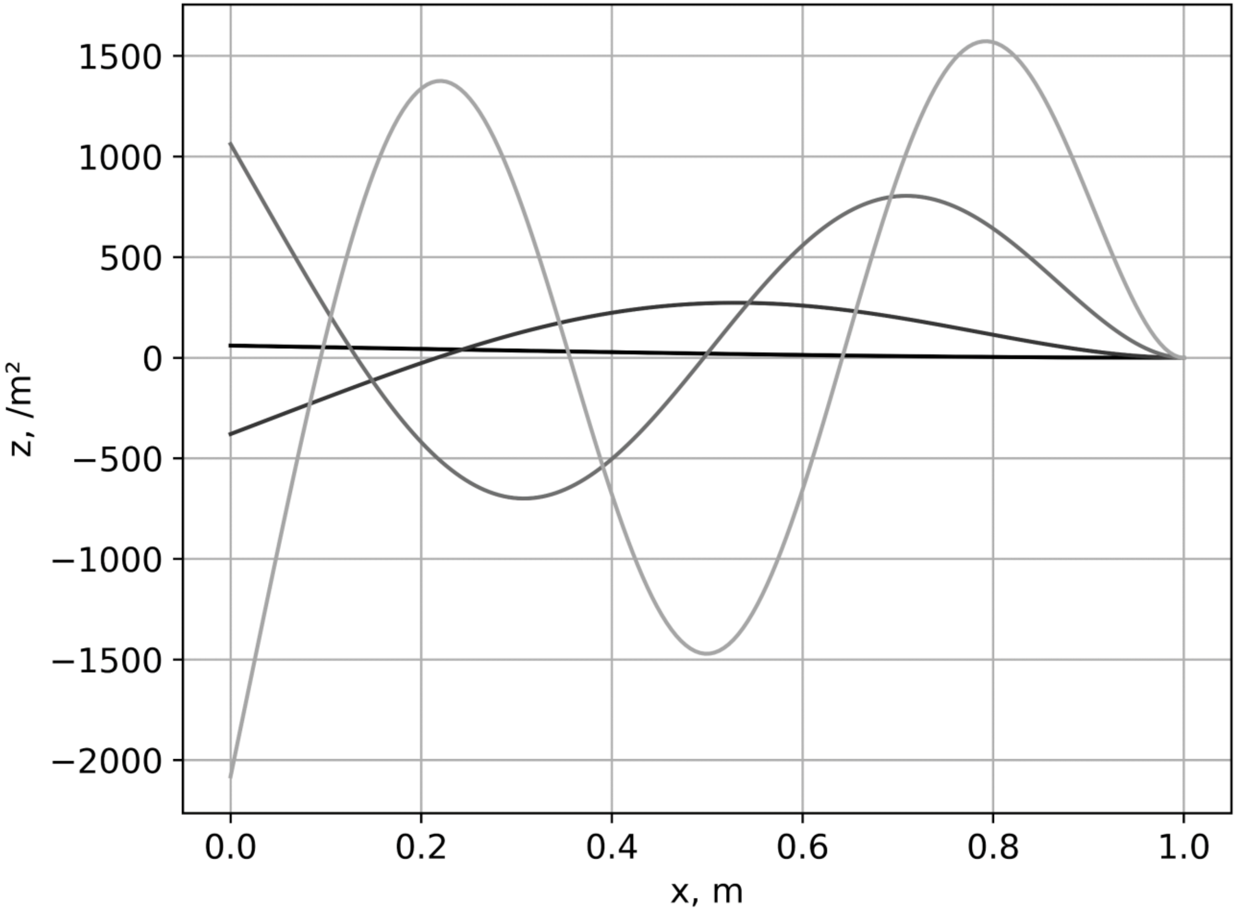
“Linearized” GEBT

- First four out-of-plane modes of clamped-free 1m beam, scaled to generalized mass of unity



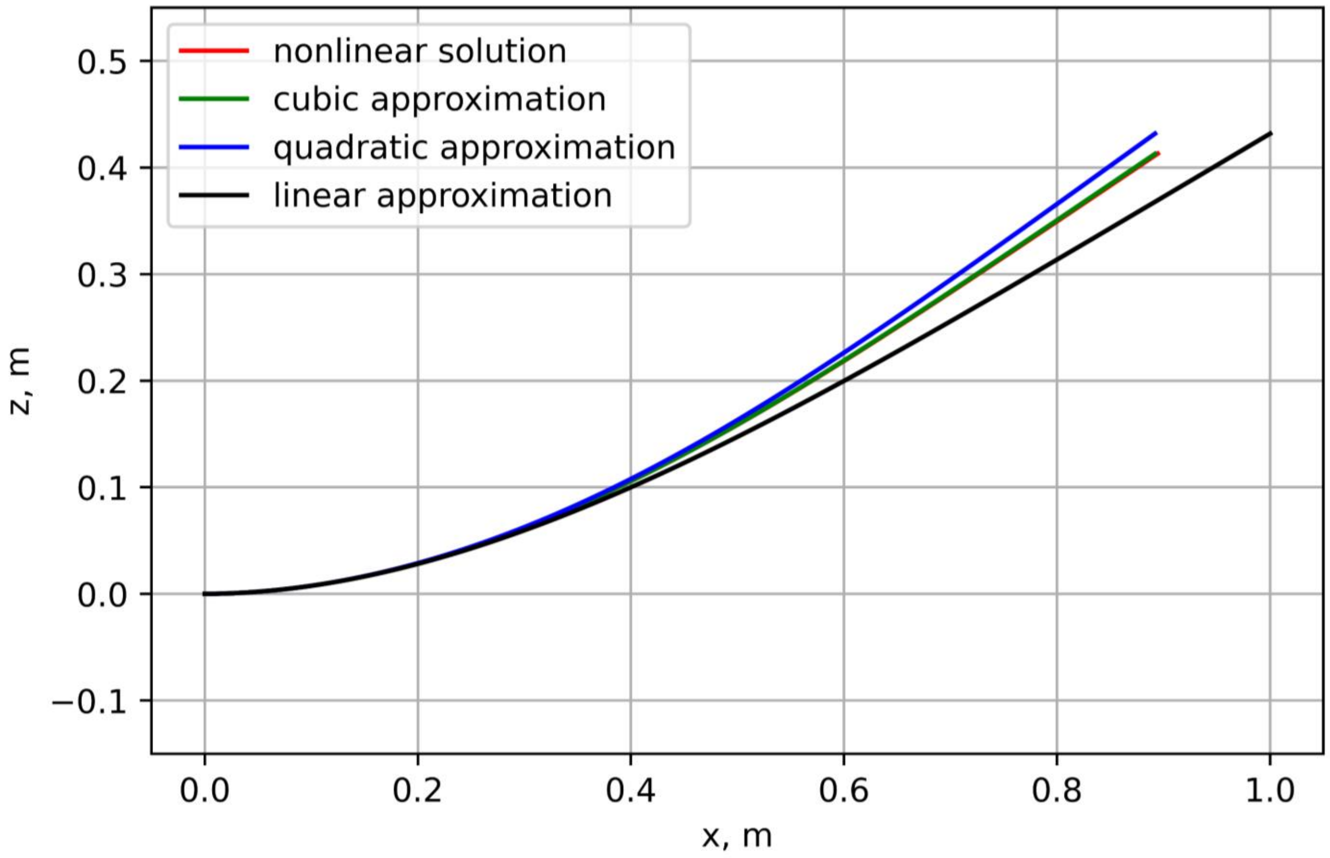
“Linearized” GEBT

- Corresponding curvature functions $\omega_\eta = \dfrac{\partial^2 w(s)}{\partial s^2}$



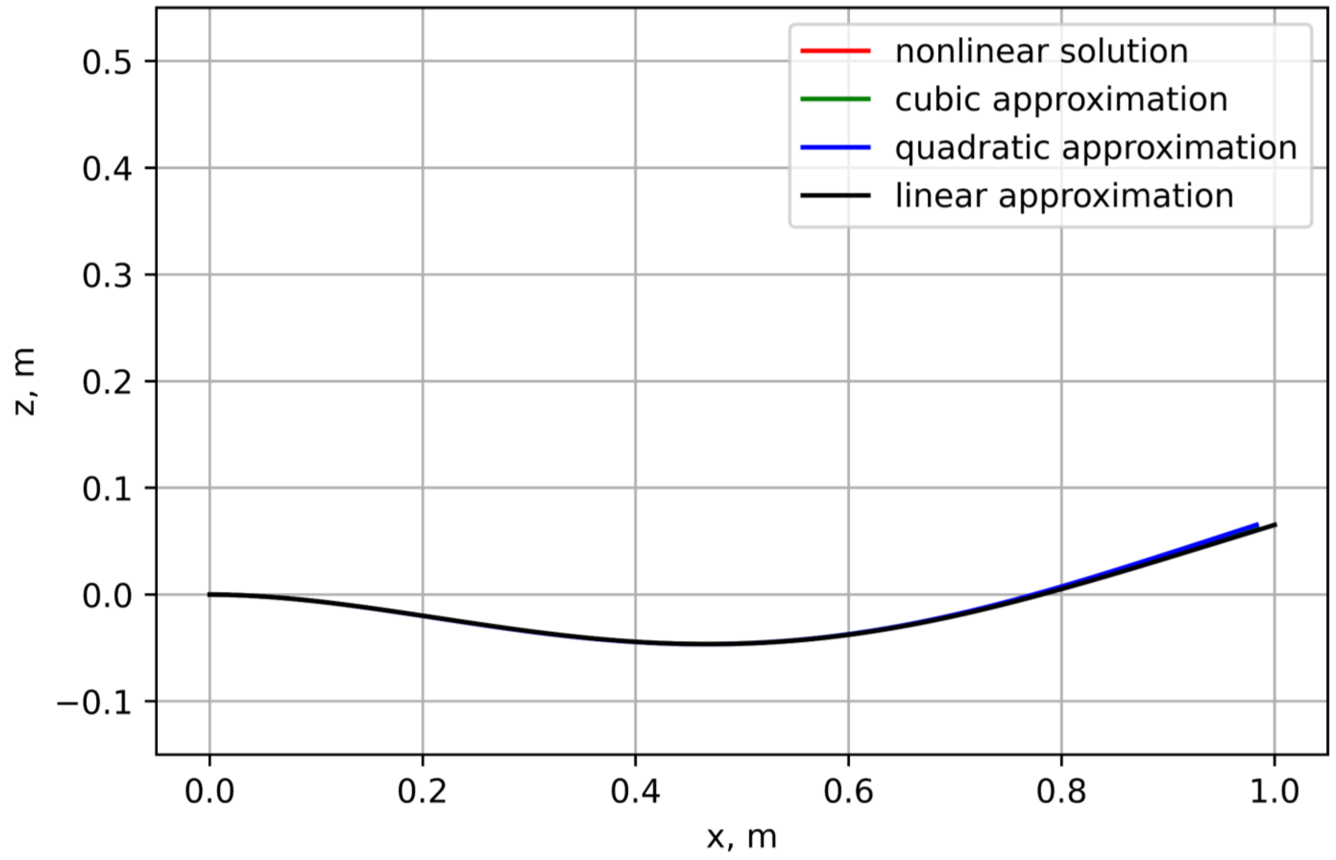
“Linearized” GEBT

- Linear, quadratic, cubic, and nonlinear displacement fields of 1st bending mode curvature function



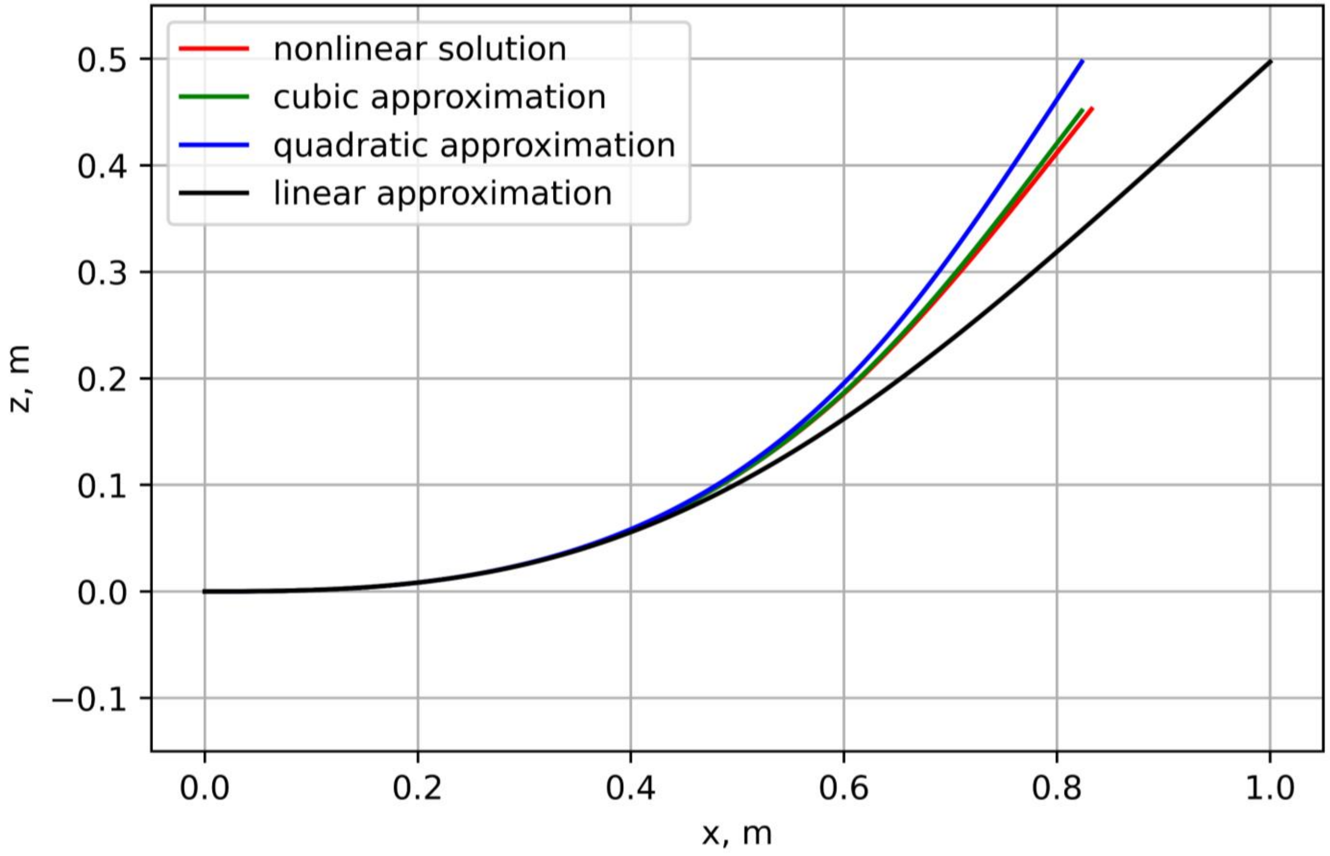
“Linearized” GEBT

- Linear, quadratic, cubic, and nonlinear displacement fields of 2nd bending mode curvature function



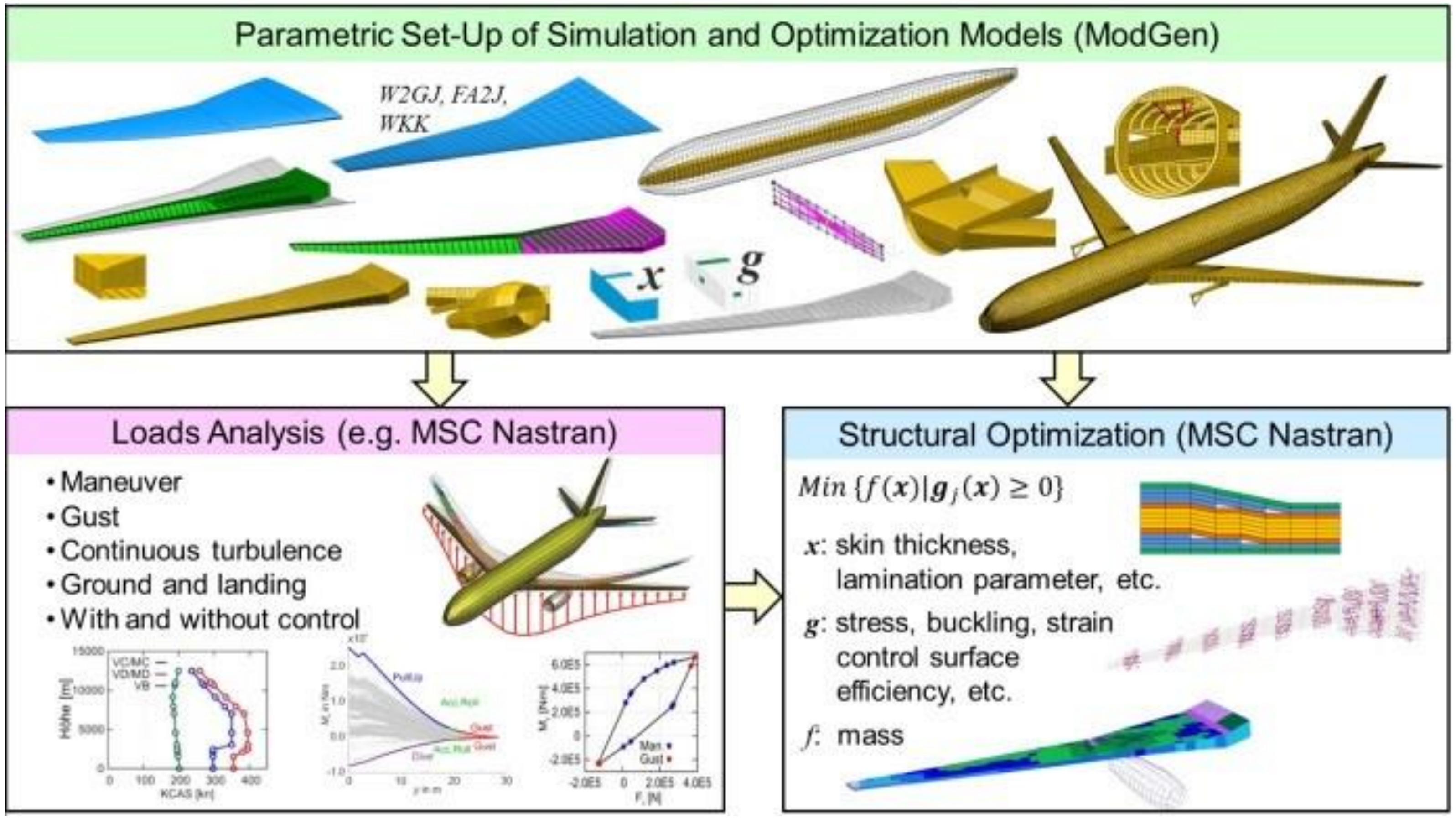
“Linearized” GEBT

- Linear, quadratic, cubic, and nonlinear displacement fields of 1st and 2nd bending modes curvature functions



Loads Analysis and Structural Optimization

- Consider DLR’s aircraft design and structural optimization process (MONA, cpacs-MONA)
- Structural optimization uses GFEM, loads analyses use statically condensed models
- The loads process is to be extended to account for nonlinearities (large deflections)



Parametric Set-Up of Simulation and Optimization Models (ModGen)

Loads Analysis (e.g. MSC Nastran)

- Maneuver
- Gust
- Continuous turbulence
- Ground and landing
- With and without control

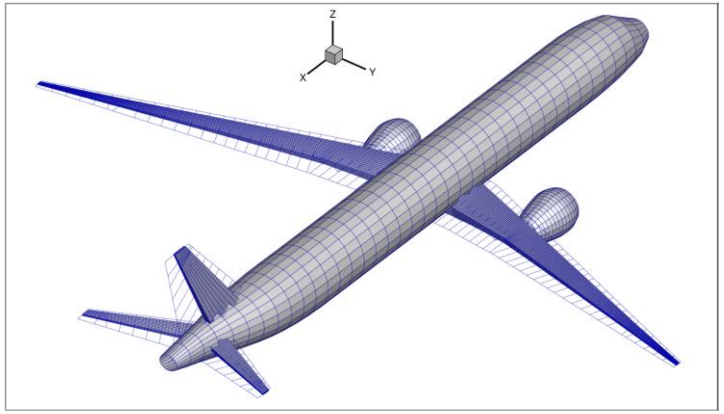
Structural Optimization (MSC Nastran)

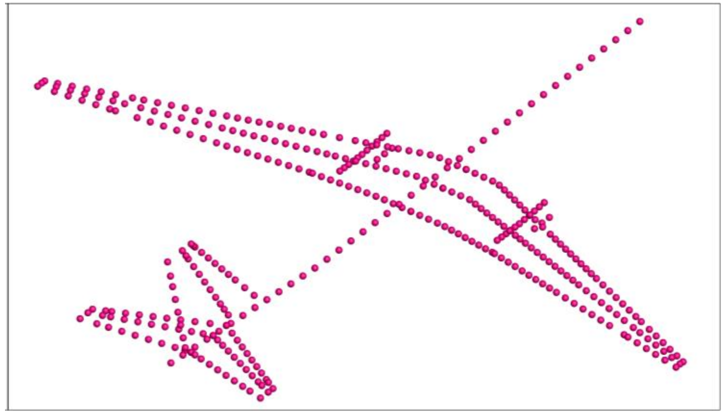
$Min\,\{f(\boldsymbol{x})\,|\,\boldsymbol{g}_j(\boldsymbol{x}) \geq 0\}$

$\boldsymbol{x}$: skin thickness, lamination parameter, etc.

$\boldsymbol{g}$: stress, buckling, strain control surface efficiency, etc.

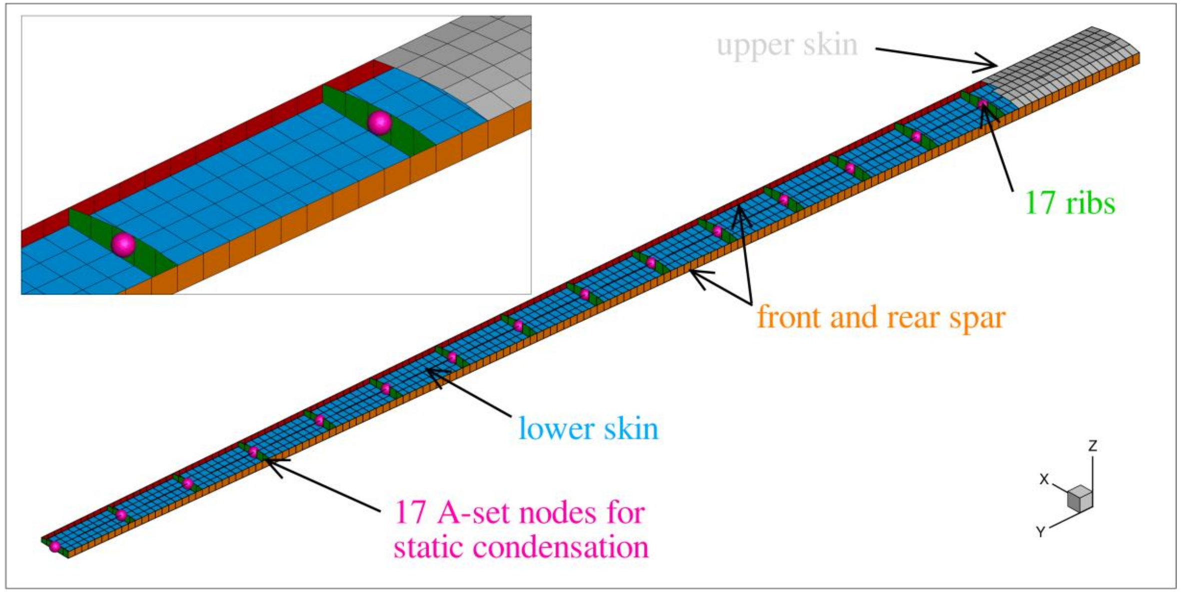
f: mass

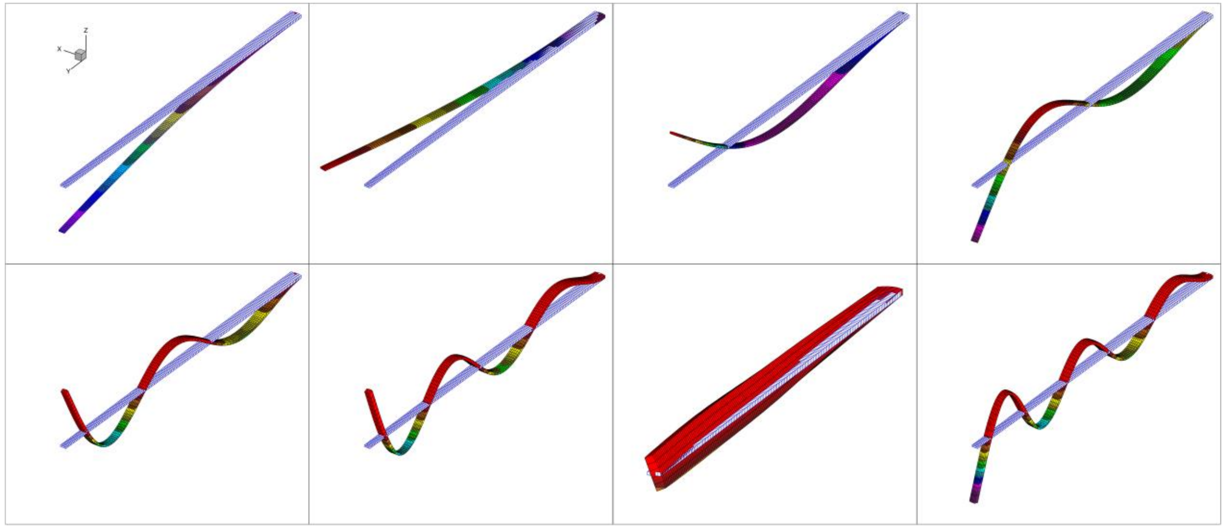




Higher Order Mode Components for Condensed Models

- To use the extended modal approach in the loads analysis, the higher order mode components of the condensed FE model are required
- The linearized GEBT equations can be used to calculate them from the curvatures of the individual (and coupled) mode shapes by integration along the beam reference line
- Consider the slender composite wingbox (scwb) test case as example



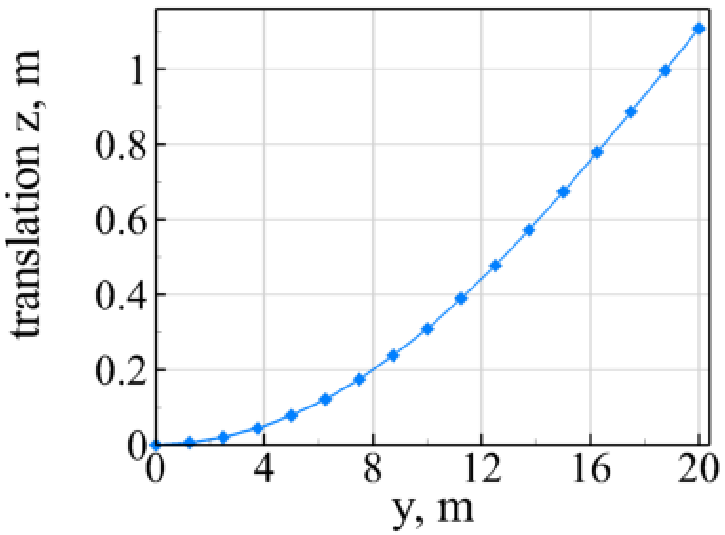


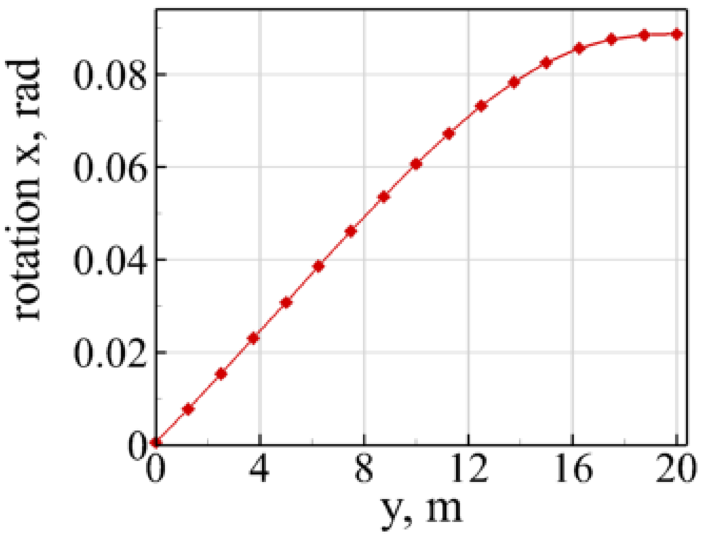
Slender Composite Wingbox (SCWB) test case

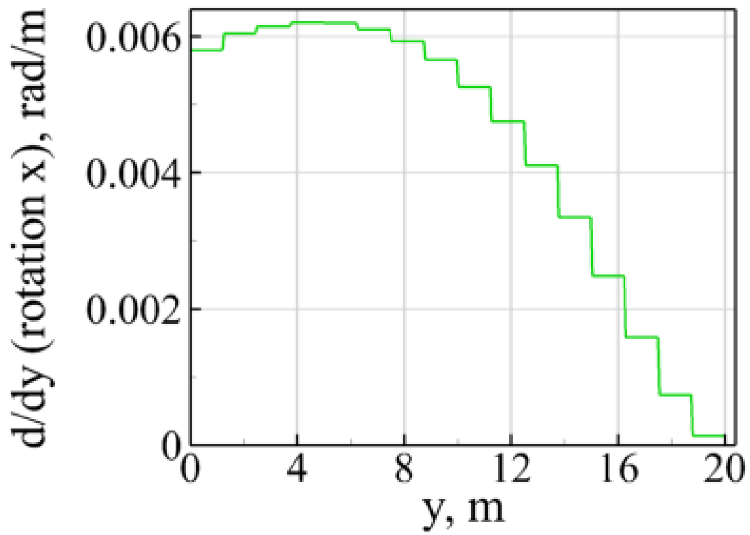
span = 20m, AR = 25

Application: SCWB

- Translational and rotational DOFs (of the mode shapes) for the <u>condensed model</u> are computed by MSC Nastran SOL 103
- The rotational DOFs (modal rotations) are splined and differentiated along the span to obtain the curvatures
- Note: this step is not required if you obtain the curvatures (or stresses/strains) directly from a FE solver
- 1st out-of-plane bending mode:

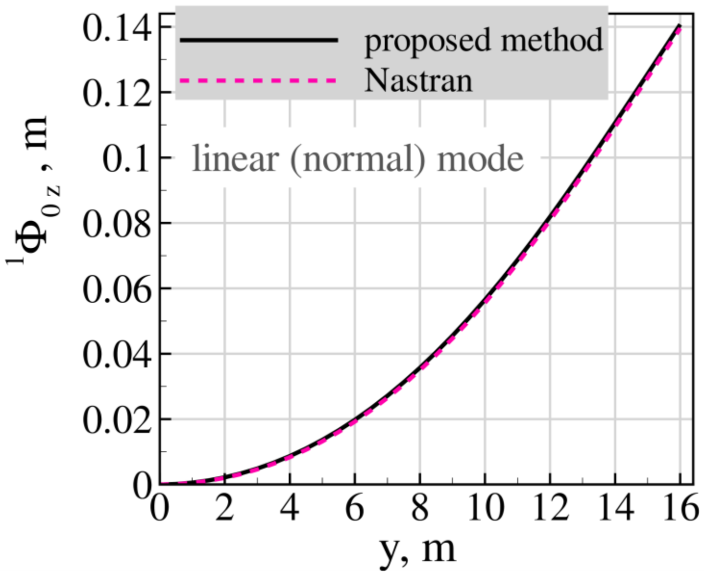


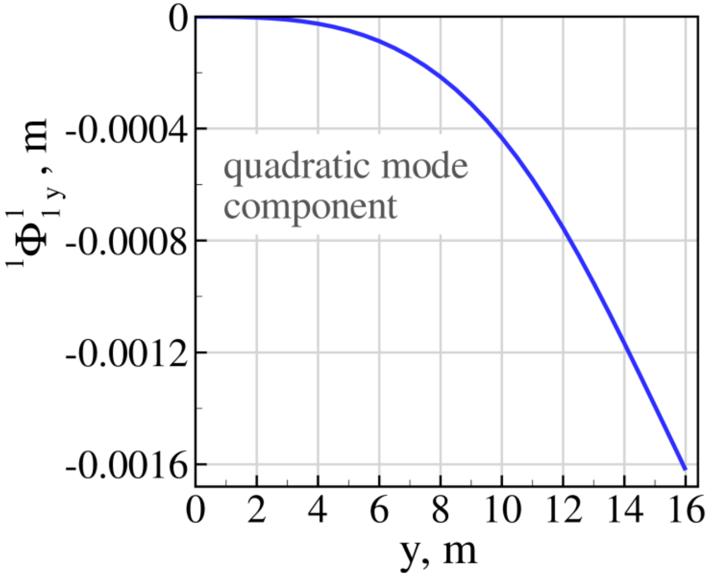


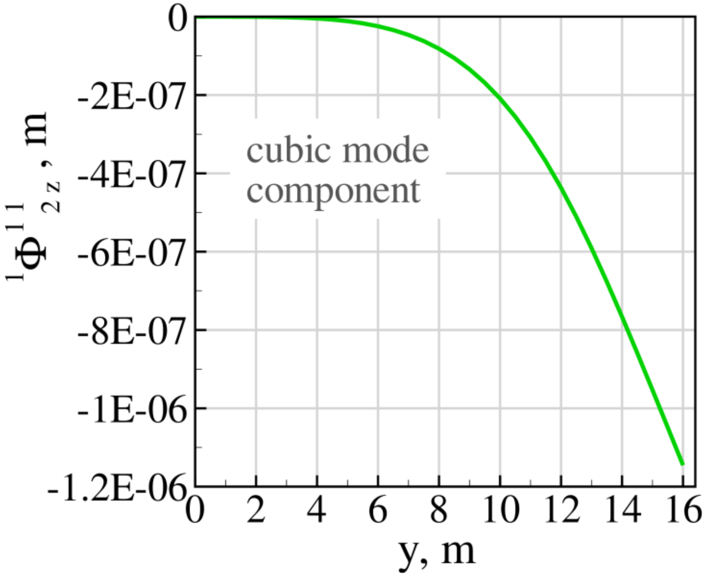


Application: SCWB

- Mode components of 1st out-of-plane bending mode:



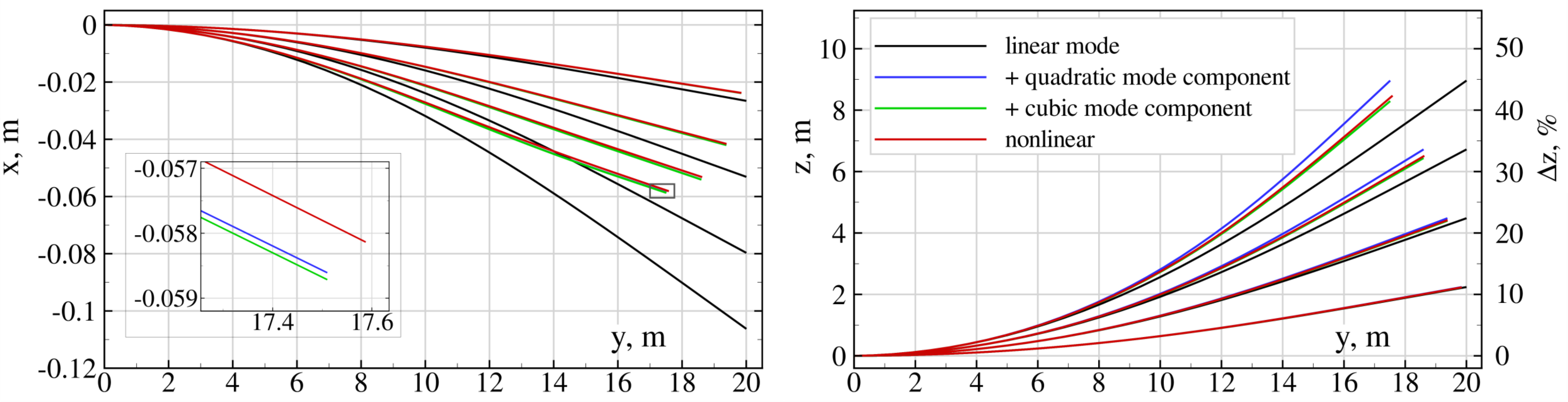




Application: SCWB

- 1st out-of-plane bending mode: Nonlinear, cubic, quadratic, and linear displacement fields
- Scaled by generalized coordinate to excite large deflections

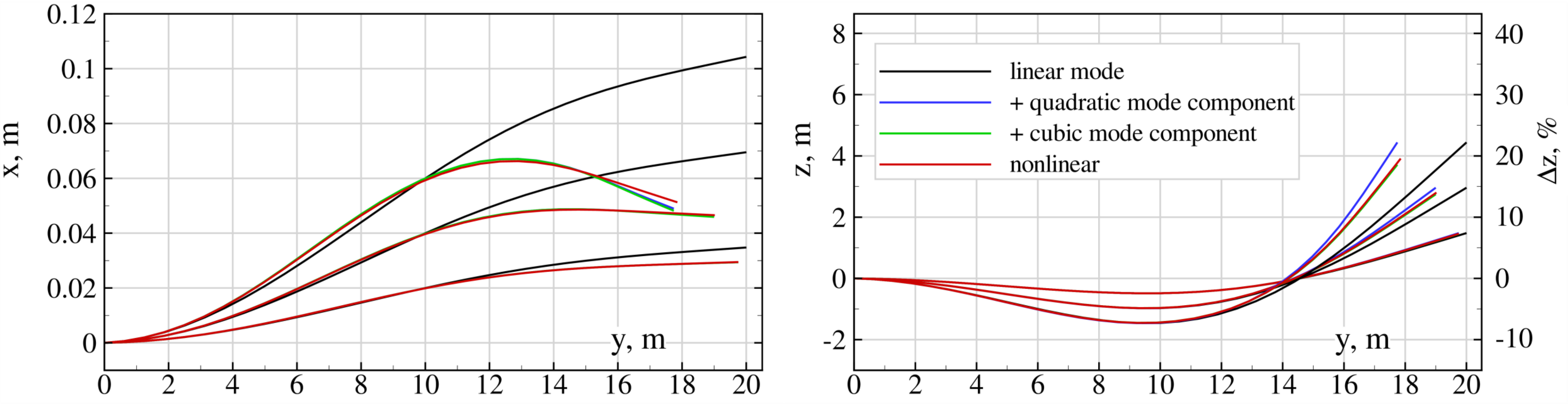
$$\boldsymbol{u}(\boldsymbol{q}) = {}^{1}\boldsymbol{\Phi}_0 \; q_1 + {}^{1}\boldsymbol{\Phi}_1^{1} \, q_1 q_1 + {}^{1}\boldsymbol{\Phi}_2^{11} \; q_1 \, q_1 q_1$$



Application: SCWB

- 2^{nd} out-of-plane bending mode: Nonlinear, cubic, quadratic, and linear displacement fields
- Scaled by generalized coordinate to excite large deflections

$$\boldsymbol{u}(\boldsymbol{q}) = {}^{3}\boldsymbol{\Phi}_0 \ q_3 + {}^{3}\boldsymbol{\Phi}_1^3 \, q_3 q_3 + {}^{3}\boldsymbol{\Phi}_2^{33} \ q_3 \, q_3 q_3$$



Summary and Outlook

- The presented approach is an alternative way to “directly” compute higher order mode components for the extended modal approach, i.e. a (costly) identification process is avoided
- The curvatures of the structure‘s mode shapes are integrated by cubic, quadratic, and linear expressions „borrowed“ from GEBT to obtain (nonlinear) nodal rotations and displacements of those modes
- The whole approach can be seen as a way to „move“ the integrations of the compatibility equations of the GEBT to a preprocessing step independent of the actual nonlinear structural solution
- Likewise, no discretization (FE) of the equations of the GEBT is required
- See also the SciTech 2025 paper (https://elib.dlr.de/211807/):

Markus Ritter, **Direct Computation of Higher Order Mode Components From Successively Linearized Geometrically Nonlinear Beam Theory**, AIAA SciTech Forum, Orlando, Florida, 2025