Safe and Stable Adaptive Control for a Class of Dynamic Systems

Johannes Autenrieb[1] and Anuradha Annaswamy[2]

Abstract—Adaptive control has focused on online control of dynamic systems in the presence of parametric uncertainties, with solutions guaranteeing stability and control performance. Safety, a related property to stability, is becoming increasingly important as the footprint of autonomous systems grows in society. One of the popular ways for ensuring safety is through the notion of a control barrier function (CBF). In this paper, we combine adaptation and CBFs to develop a real-time controller that guarantees stability and remains safe in the presence of parametric uncertainties. The class of dynamic systems that we focus on is linear time-invariant systems whose states are accessible and where the inputs are subject to a magnitude limit. Conditions of stability, state convergence to a desired value, and parameter learning are all elucidated. One of the elements of the proposed adaptive controller that ensures stability and safety is the use of a CBF-based safety filter that suitably generates safe reference commands, employs error-based relaxation (EBR) of Nagumo's theorem, and leads to guarantees of set invariance. To demonstrate the effectiveness of our approach, we present two numerical examples, an obstacle avoidance case and a missile flight control case.

I. INTRODUCTION

The field of adaptive control has focused on providing real-time inputs for dynamic systems through parameter learning and control design using a stability framework [1]–[6]. A different direction of research has been growing in the area of safety-critical systems [7]–[9] motivated by the need to provide verifiable guarantees of safe behavior in systems with mixed autonomy. This paper takes a step in combining adaptive control methods with safety-critical methods for a specific class of dynamic systems. In addition to ensuring safety and stability, the proposed adaptive control design also seeks to accommodate magnitude constraints on the control input.

The tools utilized in establishing stability in adaptive control include Lyapunov stability, analytical continuity, and a reference model that establishes a target for the adaptive control system to track. In the case of safety, control barrier functions (CBFs) and the notion of positive invariance are utilized in order to design the exogenous input into the system. The approach we have used in this paper is a careful combination of both sets of tools; a CBF-based filter design is used in order

to design the reference input into the reference model. This in turn is followed by the use of a reference model that utilizes a closed-loop structure [10] and a calibration that acknowledges the possibility of input saturation. With the resulting calibrated closed-loop reference model (CCRM), global boundedness and tracking in the absence of input saturation and a domain of attraction result in the presence of input constraints are established. In all cases, safety of the system is guaranteed due to the combined use of Calibrated CBF and CCRM.

Several approaches have been reported in the literature where CBFs are are used to ensure safety for a plant with parametric uncertainties [11]–[16]. Approaches in [12], [13] use a robust but conservative approach in the choice of how the system is rendered safe. A less restrictive approach is used in [14] where a CBF filter minimizes the risks of controlling an unknown model and a controller learns how to operate the system via a data-driven approach. A drawback of that approach is that during the learning phase, no guarantees of set invariance or bounds can be provided. In [15], the notion of a CBF is expanded as an adaptive CBF which requires the barrier function to exist for all unknown parameters and all adaptive gains in a set. A similar approach is proposed in [16] where L1-adaptive methods are used to update the parameters and the corresponding Control Lyapunov Function (CLF) is assumed to exist for all parameter estimates. While magnitude limits are imposed on the control input in [16], those constraints are imposed in the development of the CLFs, which may not be guaranteed to lead to a feasible solution.

Safety has also been addressed using non-CBF approaches [17]–[19] in the form of state-constraints. In [17], [18], the authors introduced modulation functions that are able to lower the control input such that it never directs the system out of a chosen set of states. In [19] a bounding function on the reference input is imposed to lower the control input and therefore ensure state-constraints. The trade offs between the use of such a bounding function and command following are however only empirically addressed in these papers. In contrast, a CBF-based approach, which is used in this paper, allows a streamlined use of a computational solution within a constrained optimization framework, with a quadratic cost.

In this paper, we propose an adaptive controller that guarantees stability of the closed-loop system in the presence of parametric uncertainties, and safety, both with and without input-saturation. Unlike [15], the computational burden on CBF is significantly reduced by removing the requirements to be satisfied for all unknown parameters. Instead, the CBF

[1] German Aerospace Center (DLR), Institute of Flight Systems, Department of Flight Dynamics and Simulation, 38108, Braunschweig, Germany. (email: johannes.autenrieb@dlr.de)
[2] Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA. (email: aanna@mit.edu)

is designed using a reference model and suitably calibrated to accommodate the presence of adaptation.

Preliminaries and problem statement are presented in Section II. The main contributions, the development of an adaptive controller with stability and safety properties, are presented in Sections III and IV. Section III does not include any magnitude limits on the control input, while Section IV includes these limits. Section V includes numerical simulations.

II. PRELIMINARIES & PROBLEM FORMULATION

A. Preliminaries

We define a nonlinear continuous system

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where $x(t) \in \mathbf{R}^n$. In order to define safety, we consider a continuously differentiable function $h : \chi \to \mathbf{R}$ where $\chi \subset \mathbf{R}^n$, and a set S defined as the zero-superlevel set of h, yielding:

$$S \triangleq \big\{ x(t) \in \chi | h(x(t)) = 0 \big\} \tag{2}$$

$$\partial S \triangleq \big\{ x(t) \in \chi | h(x(t)) \geq 0 \big\} \tag{3}$$

$$int(S) \triangleq \big\{ x(t) \in \chi | h(x(t)) > 0 \big\} \tag{4}$$

The following definitions are introduced [20], [21]:

Definition 1. The set S is positively invariant for the system (1), if for every $x_0 \in S$, it follows $x(t) \in S$ for $x(0) = x_0$ and all $t \in I(x_0) = [0, \tau_{max} = \infty)$.

Definition 2. The set S is weakly positively invariant for the system (1), if among all the solutions of (1) originating in $x_0 \in S$, there exists at least one globally defined solution $x(t)$ which remains inside S for all $t \in I(x_0) = [0, \tau_{max} = \infty)$.

Next, we define the distance from a point to a set:

Definition 3. Given a set $S \subset \mathbf{R}^n$ and a point $y \subset \mathbf{R}^n$, the distance from the point to the set is defined as

$$dist(y, S) = \inf_{w \in S} \|y - w\|_* \tag{5}$$

where $\| \cdot \|_*$ is any relevant norm.

Based on Definition 3, we can formulate the definition of a tangent cone for a closed set.

Definition 4. Given a closed set S, the tangent cone to S at x is defined as:

$$T_S(x) = \left\{ z : \liminf_{\tau \to 0} \frac{dist(x + \tau z, S)}{\tau} = 0 \right\} \tag{6}$$

If S is convex $T_S(x)$ is convex, and "lim inf" can be replaced by "lim". Furthermore if $x \in int(S)$, then $T_S(x) = \mathbf{R}^n$, whereas if $x \notin S$, then $T_S(x) = \emptyset$, since S is defined as a closed set. Therefore $T_S(x)$ is only non-trivial on the boundary of S.

We use Definition 4 to introduce Nagumo's theorem [20]:

Theorem 1. Consider the system defined in (1). Let $S \subset \mathbf{R}^n$ be a closed set. Then, S is weakly positively invariant for the system if and only if (1) satisfies the following condition:

$$f(x(t)) \in T_S(x(t)), \ \text{ for } \ \forall x \in S \tag{7}$$

The theorem states that if the direction of the dynamics defined in (1) for any $x(t)$ at the boundary of the safe set ∂S points tangentially or inside to the safe set S, then the trajectory $x(t)$ stays in S.

Definition 5. A continuous function $\alpha : (-b, a) \to \mathbf{R}$, with $a, b > 0$, is an extended class $\mathcal{K}$ function ($\alpha \in \mathcal{K}$), if $\alpha(0) = 0$ and α is strictly monotonically increasing. If $a, b = \infty$, $lim_{r \to \infty} \alpha(r) = \infty$, $lim_{r \to -\infty} \alpha(r) = -\infty$ then α is said to be a class $\mathcal{K}_\infty$ function ($\alpha \in \mathcal{K}_\infty$).

Definition 6. For the system considered in (1), a continuously differentiable and convex function $h : \mathbf{R}^n \to \mathbf{R}$ is a zeroing barrier function (ZBF) for the set S defined by (3) and (4), if there exist an extended class $\mathcal{K}$ function $\alpha(h(x(t)))$ and a set $S \in \mathbf{R}^n$ such that $\forall x \in S$,

$$\dot{h}(x(t)) \geq -\alpha(h(x(t))) \tag{8}$$

The above definitions lead to a less restrictive version of (7) as it weakens the requirement to that in (8).

We now expand the scope of the problem statement from (1) to those with an affine control input, of the form:

$$\dot{x} = f(x(t)) + g(x(t))u(t) \tag{9}$$

where g is Lipschitz and $u(t) \in \mathbf{R}^m$. We introduce the notion of a Control Barrier Function (CBF) such that its existence allows the system to be rendered safe w.r.t. S [22], [23] and allows a weaker requirement for system safety with a control input $u(t)$, similar to (8).

Definition 7. Let $S \subset \chi$ be the zero-superlevel set of a continuously differentiable function $h : \chi \to \mathbf{R}$. The function h is a zeroing control barrier function (ZCBF) for S, if there exists a class $\mathcal{K}_\infty$ function $\alpha(h(x(t)))$ such that for the system defined in (9) we obtain:

$$\sup_{u \in \mathbf{R}^m} \frac{\partial h}{\partial x} \big[f(x(t)) + g(x)u(t) \big] \geq -\alpha(h(x(t))) \tag{10}$$

for all $x \in S$.

Using the Lie derivative notation, we obtain the following formulation for a ZCBF considering the system defined in (9):

$$\dot{h}(x(t)) = L_f h(x(t)) + L_g h(x(t))u(t) \tag{11}$$

B. Problem Formulation

We consider a linear plant with parametric uncertainties of the form:

$$\dot{x}_p(t) = A_p x_p(t) + B_p \Lambda(R_{u_0}(u(t)) \tag{12}$$

where $x_p(t) \in \mathbf{R}^n$ is a measurable state vector and $u(t) \in \mathbf{R}^m$ is a control input vector. The matrices $A_p \in \mathbf{R}^{n \times n}$ and $\Lambda \in$

$\mathbf{R}^{m \times m}$ are unkown and Λ has only diagonal positive entries. The input matrix $B_p \in \mathbf{R}^{n \times m}$ is known. The control input is assumed to be magnitude limited by u_0, which is represented using the function $R_{u_0}(\cdot)$ as

$$R_{u_0}(u(t)) = \begin{cases} u(t) & \text{if } \|u(t)\| \leq u_0 \\ u_0 \frac{u(t)}{\|u(t)\|} & \text{if } \|u(t)\| > u_0 \end{cases} \tag{13}$$

The objectives are to determine a $u(t)$ for (12) such that the plant state $x_p(t)$ tracks a desired reference $x_d(t)$ and that for any initial condition $x_0 := x(t_0) \in S$, it is ensured that the plant state vector $x_p(t)$ stays within the safe set $S \in \mathbf{R}^n$. This is equivalent to having the control input ensures that there is a CBF with $h(x(t)) \geq 0$ for $\forall t \geq 0$.

III. SAFE ADAPTIVE CONTROL DESIGN WITH OPEN-LOOP REFERENCE MODEL

In this section, we consider a simpler version of the problem statement, where the control input magnitude limit is removed. The main challenge in ensuring positive invariance for the plant in (12), which has uncertainties in A_p and Λ, is the design of a suitable ZCBF, which requires the model to be known, as is evident from (10). We therefore first choose a target system, i.e. a reference model, that the adaptive system can be made to contract towards. This reference model is chosen so that its state approaches the desired reference $x_d(t)$ and simultaneously allows the generation of a suitable ZCBF that ensures safety. We choose an open-loop reference model (ORM) of the form:

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{14}$$

where $x_m(t) \in \mathbf{R}^n$ is the reference model state vector and $r(t) \in \mathcal{R} \subset \mathbf{R}^m$ the reference input vector. The matrix $A_m \in \mathbf{R}^{n \times n}$ is a Hurwitz matrix, and $B_m \in \mathbf{R}^{n \times m}$ has full column rank. It is easy to see that if

$$r(t) = B_m^+(\dot{x}_d(t) - A_m x_d(t)) \tag{15}$$

then $x_m(t)$ approaches $x_d(t)$, where B_m^+ denotes the Moore-Penrose inverse of B_m.

In order to ensure that $x_m(t)$ stays inside a safe set S, rather than choose r as in (15), we use a QP-ZCBF safety filter for the plant in (12) as follows [11]:

$$\min_{r(t) \in \mathcal{R}} \|r - r^*\|_2 \tag{16}$$

s.t.

$$\frac{\partial h}{\partial x}[A_m x_m + B_m r] \geq -\alpha(h(x_m)) + \Delta,$$

where $r^* = B_m^+(\dot{x}_d - A_m x_d)$ and $\Delta > 0$ is a positive constant that introduces a safety buffer. The optimal solution of (16) can be easily determined by using KKT conditions given by:

$$r - r^* - L_{B_m} h(x_m)^T \lambda = 0 \tag{17a}$$

$$\lambda(-L_{A_m x_m} h(x_m) - L_{B_m} h(x_m)r - \alpha(h(x_m)) + \Delta) = 0 \tag{17b}$$

$$-L_{A_m x_m} h(x_m) - L_{B_m} h(x_m)r - \alpha(h(x_m)) + \Delta \leq 0 \tag{17c}$$

with a suitable choice of $\lambda \geq 0$.

In what follows, for ease of exposition, we choose the class $\mathcal{K}$-function $\alpha(h(x)) = \gamma h(x)$, where γ is a positive scalar constant [22], [24].

A. Adaptive control design

Our reference system is now determined using (14) and (16) as

$$\dot{x}_m(t) = A_m x_m(t) + B_m r_s(t) \tag{18}$$

The following assumptions are made regarding the unknown parameters in (12):

Assumption 1. Constant matrices θ_x^* and θ_r^* exist that solve the following:

$$A_m = A_p + B_p \Lambda \theta_x^* \tag{19}$$

$$B_m = B_p \Lambda \theta_r^* \tag{20}$$

Assumption 2. The uncertainty Λ is a diagonal positive definite matrix.

We now propose the adaptive controller for the plant in (12):

$$u(t) = \hat{\theta}_x(t)x_p(t) + \hat{\theta}_r(t)r_s(t) \tag{21}$$

The time-varying parameters in (21) are adjusted using the following adaptive laws:

$$\dot{\hat{\theta}}_x(t) = -\Gamma_x x_p(t)e_x(t)^T PB_p, \quad \Gamma_x > 0 \tag{22}$$

$$\dot{\hat{\theta}}_r(t) = -\Gamma_r r(t)e_x(t)^T PB_p, \quad \Gamma_r > 0 \tag{23}$$

where $e_x(t) = x_p(t) - x_m(t)$ and P is the solution of the Lyapunov equation $A_m^T P + PA_m = -Q$, where $Q > 0$. Both Γ_x and Γ_r are positive definite matrices defined as the adaptive update gains. We further introduce a corresponding output error $e_u(t) = u(t) - u^*(t)$, which will be useful to quantify the safety of the adaptive controller, where $u^*(t)$ represents the ideal control input and is defined as:

$$u^*(t) = \theta_x^* x_p(t) + \theta_r^* r_s(t) \tag{24}$$

In what follows, it will be assumed that the reference input $r_s(t)$ has a bounded derivative.

Theorem 2. The overall closed-loop adaptive system defined by the plant in (12), the control input in (21) and the adaptation laws in (22), (23) has globally bounded solutions for any initial conditions $x_p(t_0)$, $\hat{\theta}_x(t_0)$, and $\hat{\theta}_r(t_0)$ and both the errors $e_x(t)$ and $e_u(t)$ converge to zero as $t \to \infty$.

The proof of the theorem follows from standard adaptive control arguments, since the error dynamics is of the form

$$\dot{e}_x(t) = A_m e_x(t) + B_p \Lambda(\tilde{\theta}_x(t)x_p(t) + \tilde{\theta}_r(t)r_s(t)) \tag{25}$$

where $\tilde{\theta}_x(t) = \hat{\theta}_x(t) - \theta_x^*$, $\tilde{\theta}_r(t) = \hat{\theta}_r(t) - \theta_r^*$ and together they admit a Lyapunov function

$$V(e_x(t), \tilde{\theta}_x(t), \tilde{\theta}_r(t)) = \frac{1}{2}e_x^T(t)Pe_x(t) + \frac{1}{2}\text{Tr}[\tilde{\theta}_x(t)\Gamma_x^{-1}\tilde{\theta}_x^T(t)\Lambda] + \frac{1}{2}\text{Tr}[\tilde{\theta}_r(t)\Gamma_r^{-1}\tilde{\theta}_r^T(t)\Lambda] \tag{26}$$

It is easy to see that $\dot{V} \leq 0$, and $e_x \in \mathcal{L}_2$. As $e_x(t)$ is bounded and has a bounded derivative, an application of the Barbalat's Lemma leads to $\lim_{t\to\infty} e_x(t) = 0$ [1]. From (25), it follows that $e_u(t)$ is an input into an LTI system, with a bounded derivative, whose state is $e_x(t)$; therefore it follows that $\lim_{t\to\infty} e_u(t) = 0$ [25].

B. Safety in the presence of uncertainties

With stability guaranteed from the discussions above, we now derive conditions for the safety of the proposed adaptive controller. The core idea is to render the known reference model to be safe i.e. $h(x_m) \geq 0$ for $\forall t \geq 0$, and to use the adaptive system to make the closed-loop contract towards the reference model, which ensures that $h(x_p) \geq 0$ for $\forall t \geq 0$. The goal is to derive conditions under which

$$\dot{h}_p \geq -\gamma h(x_p(t)) + \Delta \tag{27}$$

We note that the QP-CBF filter ensures that

$$\dot{h}_m = \underbrace{\frac{\partial h}{\partial x}|_{x_m}}_{a_0} \underbrace{[A_m x_m(t) + B_m r_s(t)]}_{a_1} \geq -\underbrace{\gamma h(x_m(t))}_{a_2} + \Delta \tag{28}$$

where $\dot{h}_p = \frac{\partial h}{\partial x}|_{x_p}$ and $\dot{h}_m = \frac{\partial h}{\partial x}|_{x_m}$. To ensure that a ZCBF exists for the adaptive system specified by (12),(21)-(23), we consider

$$\dot{h}_p = \frac{\partial h}{\partial x}|_{x_p}[A_p x_p(t) + B_p \Lambda(\hat{\theta}_x(t) x_p(t) + \hat{\theta}_r(t) r_s(t))] \tag{29}$$

From (12), (19)-(21), (24), (28), and the definition of the errors e_x and e_u, we obtain that

$$\dot{h}_p = \frac{\partial h}{\partial x}|_{x_p}[\underbrace{A_m x_m(t) + B_m r_s(t)}_{a_1}$$
$$+ \underbrace{A_m e_x(t) + B_p \Lambda e_u(t)}_{\bar{e}}] \tag{30}$$

with b_0 under the first bracket.

Algebraic manipulations allow us to rewrite (30) using (28) as

$$\dot{h}_p = b_0[a_1 + \bar{e}]$$
$$\geq -a_2 + \Delta + \underbrace{a_0 \bar{e} + (b_0 - a_0)\bar{e} + (b_0 - a_0)a_1}_{z(t)} \tag{31}$$

Since the goal is to establish safety of the closed-loop adaptive system, we utilize the following two inequalities:

$$|g(x_p(t)) - g(x_m(t))| \leq \kappa_1 |e_x(t)| \tag{32}$$

$$|h(x_p(t)) - h(x_m(t))| \leq \kappa_2 |e_x(t)| \tag{33}$$

where $g(x(t)) = \frac{\partial h}{\partial x}$, κ_1 and κ_2 are Lipschitz constants associated with $g(x_p(t))$ and $h(x_p(t))$, respectively. We note additionally from Theorem 2 that $\|x_p(t)\|$, $\|h(x_p(t))\|$ and $\|\bar{e}\|$ are bounded. Therefore, $|z(t)| \leq z_0$, where z_0 is defined as

$$z_0 = |a_0||\bar{e}| + \kappa_1 |e_x(t)| (|\bar{e}| + |a_1|) \tag{34}$$

Using the lower bound $-z_0$ for $z(t)$, we rewrite (31) as

$$\dot{h}_p \geq -\gamma h(x_p(t)) + \Delta - \bar{F}(|\bar{e}|, |e_x(t)|) \tag{35}$$

where

$$\bar{F}(|\bar{e}|, |e_x(t)|) = \gamma \kappa_2 |e_x(t)| + |a_0||\bar{e}| + \kappa_1 |e_x(t)| (|\bar{e}| + |a_1|) \tag{36}$$

The inequality in (35) implies that safety of the closed-loop adaptive system will be guaranteed after $t \geq t_0 + T$, where T is a finite interval, as $\bar{F}(t) \to 0$ as $t \to \infty$, and therefore $|\bar{F}(t)| \leq \Delta \; \forall t \geq t_0 + T$. This in turn implies that the closed-loop adaptive system will remain safe for all $t \geq t_0$ if

$$h(x_p(t)) \geq h_0 \quad \forall t \geq [t_0, t_0 + T] \tag{37}$$

where $\gamma h_0 \geq F_{\max}$ where

$$F_{\max} = \max_{t \in [t_0, t_0 + T]} \bar{F}(t) \tag{38}$$

As $F(t)$ is bounded, it is clear that such an $F_{\max}$ exists. Condition (37) is satisfied if there is a separation between the period of adaptation and the time at which the system approaches its limit of safety. This property is summarized in the following theorem, where $e_h(t) := h(x_p(t)) - h(x_m(t))$.
Remark: For interested readers, an extended version of the preceding derivation and the derivation presented for the CCRM in Section IV-B can be found in the arXiv version of the presented paper under Ref. [26].

Theorem 3. A ZCBF $h(x)$ exists for all S in $\mathbf{R}^n$ for the overall closed-loop adaptive system defined by the plant in (12) and the adaptation laws in (22) and (23), if (37) is satisfied, with $\gamma h_0 \geq F_{\max}$ where $F_{\max}$ is defined as in (38). Further, the inequality (37) also implies that $\lim_{t\to\infty} e_h(t) = 0$.

The following choice of γ as a function of the safety error may allow the condition (37) to be satisfied for a larger class of ZCBFs:

$$\gamma(e_h(t)) = \gamma_0 e^{-(\epsilon e_h(t))^2} \tag{39}$$

with $\gamma_0 \geq 0$ and $\epsilon \geq 0$ are positive constants. Such a choice allows γ to take on a value that is close to γ_0 as long e_h is small, and $\gamma(e_h(t))$ allows it to become small as $|e_h(t)|$ increases. The rationale for such a choice is that near t_0, when the transients of the adaptive system are yet to settle down, e_h may be large and therefore a conservative choice of γ near zero is prudent; as time proceeds, the adaptive system ensures that x_p approaches x_m, and therefore e_h approaches zero. As this occurs, γ can be relaxed to take on larger values. We denote such a choice of Eq. (39) as an error-based relaxation (EBR). It should be noted that the proposed adaptive controller ensures learning in the form of minimization of all performance errors $e_x(t)$, $e_u(t)$, and $e_h(t)$ to zero.

IV. SAFE ADAPTIVE CONTROL DESIGN WITH CALIBRATED CLOSED-LOOP REFERENCE MODEL

We now consider the adaptive control of (12) subject to the magnitude limit as in (13). In order to accommodate these limits and to improve on the transient performance we propose a calibrated closed-loop reference model (CCRM) of the form

$$\dot{x}_m = A_m x_m(t) + B_m r_s(t) + L e_x(t) + B_p \hat{\Lambda} \Delta u(t) \tag{40}$$

where L is a matrix such that $(A_m - L)$ is Hurwitz, $\Delta u(t) = R_{u_0}(u(t)) - u(t)$ and represents a disturbance due to saturation, and $\hat{\Lambda}$ is an estiamtion of the unknown matrix Λ. The input $r_s(t)$ is the solution of a modified QP-ZCBF filter, which is defined by the following constrained optimization:

$$\min_{r \in \mathcal{R}} (r - r^*)^2 \tag{41}$$

s.t.
$$\frac{\partial h}{\partial x}\left[A_m x_m + B_m r + L R_{e_0}(e_x) + B_p \hat{\Lambda} R_{\Delta u_0}(\Delta u) \right] \geq \\ -\alpha(h(x_m)) + \Delta, \tag{42}$$

where $R_{\Delta u_0}(\Delta u(t))$ and $R_{e_0}(e_x(t))$ represent magnitude limited signals of $\Delta u(t)$ and $e_x(t)$, with suitable limits Δu_0 and e_0, respectively. The KKT condition for the QP-ZCBF safety filter is defined as:

$$r - r^* - L_{B_m} h(x_m)^T \lambda = 0 \tag{43a}$$

$$\lambda(-L_{A_m x_m} h(x_m) - L_{B_m} h(x_m)r - L_L h(x_m) R_{e_0}(e_x) \\ - L_{B_p \hat{\Lambda}} h(x_m) R_{\Delta u_0}(\Delta u) - \alpha(h(x_m)) + \Delta) = 0 \tag{43b}$$

$$- L_{A_m x_m} h(x_m) - L_{B_m} h(x_m)r - L_L h(x_m) R_{e_0}(e_x) \\ - L_{B_p \hat{\Lambda}} h(x_m) R_{\Delta u_0}(\Delta u) - \alpha(h(x_m)) + \Delta \leq 0 \tag{43c}$$

where $\lambda \geq 0$ is chosen so as to ensure feasibility, and $\alpha(h(x(t))) = \gamma(e_h)h(x(t))$, with $\gamma(e_h(t))$ defined as in (39). It should be noted that (43b)-(43c) are well defined for any choice of $e_x(t)$ and $\Delta u(t)$, which are yet to be shown to be bounded.

A. Adaptive control in the presence of magnitude limits

The same adaptive controller as presented in (21)-(23) is utilized here as well with a few modifications. The corresponding error dynamics between the CCRM in (40) and the plant in (12) can be derived to be [27], [28]:

$$\dot{e}_x(t) = (A_m - L)e_x(t) + B_p \Lambda (\tilde{\theta}_x(t)x_p(t) \\ + \tilde{\theta}_r(t)r_s(t)) + B_p \tilde{\Lambda}(t)\Delta u(t) \tag{44}$$

where $\tilde{\Lambda}(t) = \Lambda - \hat{\Lambda}(t)$ is the corresponding estimation error for Λ, and $(A_m - L)$ is Hurwitz. In addition to (22)-(23), we adjust the parameter $\hat{\Lambda}$ as:

$$\dot{\hat{\Lambda}}(t) = \Gamma_\Lambda \Delta u(t) e_x^T(t) P B_p \tag{45}$$

where Γ_Λ is positive definite. Based on (44) the following Lyapunov function candidate V is proposed:

$$V = \frac{1}{2} e_x(t)^T P e_x(t) + \frac{1}{2} \text{Tr}[\tilde{\theta}_x(t)\Gamma_1^{-1}\tilde{\theta}_x^T(t)\Lambda] \\ + \frac{1}{2} \text{Tr}[\tilde{\theta}_r(t)\Gamma_2^{-1}\tilde{\theta}_r^T(t)\Lambda] + \frac{1}{2} \text{Tr}[\tilde{\Lambda}(t)\Gamma_\Lambda^{-1}\tilde{\Lambda}^T(t)]$$

From the error equation in (44) and the adaptive laws in (22), (23), (45) and the fact that $(A_m - L)^T P + P(A_m - L) = -Q_0$, we can show that $\dot{V} = -\frac{1}{2}e_x^T Q_0 e_x \leq 0$ and hence $e_x(t)$ is bounded.

Unlike the previous case, we cannot immediately conclude that $x_p(t)$ is bounded, as both $x_p(t)$ and $x_(t)m$ are affected by $\Delta u(t)$. As a result, additional arguments are needed to establish boundedness. Unlike the previous case, when control inputs are limited in magnitude, one cannot guarantee global boundedness but a domain of attraction result (see [28]–[30]), which is briefly stated below.

We introduce the following definitions:

$$\begin{aligned}
K_{max} &= \max(\sup \|\tilde{\theta}_x\|, \sup \|\tilde{\theta}_r\|), \sup \|\tilde{\Lambda}\|) \\
\beta &= \frac{P_B K_{max}}{\|\theta_x^*\| + K_{max}} \\
a_0 &= \frac{\overline{u}_{min} K_{max}}{\|\theta_x^*\| + K_{max}} \\
x_{min} &= \frac{3 P_B K_{max}(r_{max} + 1) + 3 P_B \|\theta_r^*\| r_{max}}{q_{min} - 3 P_B K_{max}} + \\
& \quad \frac{2 P_B \overline{u}_{max}}{q_{min} - 3 P_B K_{max}} \\
x_{max} &= \frac{P_B a_0}{|q_{min} - 2 P_B K_{max}|} \\
\overline{K}_{max} &= \frac{q_{min} - \frac{\rho}{a_0}(3\|\theta_r^*\| 2\overline{u}_{max}) q_{min}}{3 P_B + \frac{3\rho}{a_0}(r_{max} + 1)|q_{min} - 2 P_B \|\theta_x^*\|\|} \\
& \quad - \frac{2 P_B \|\theta_x^*\|}{3 P_B + \frac{3\rho}{a_0}(r_{max} + 1)|q_{min} - 2 P_B \|\theta_x^*\|\|}
\end{aligned}$$

where

$$q_{min} = \min \text{eig}(Q), \ p_{min} = \min \text{eig}(P)$$
$$p_{max} = \max \text{eig}(P)$$
$$\rho = \sqrt{\frac{p_{max}}{p_{min}}}, \ \overline{u}_{max} = \max_i(u_{max,i})$$
$$\overline{u}_{min} = \min_i(u_{max,i}), \ P_B = |PB_p \Lambda|, \ \lambda_{min} = \min(\text{eig}(\Lambda))$$
$$\gamma_{max} = \max(\text{eig}(\Gamma_x), \text{eig}(\Gamma_r), \text{eig}(\Gamma_\lambda))$$

with all defined norms are Euclidean norms and the matrix P_B is the induced matrix norm, which has the property $|PB_p \Lambda x| \leq P_B |x|$. Based on the introduced variables we can state the following theorem.

Theorem 4. [30] The plant described in (12), with the adaptive feedback controller (21) using the adaptive laws defined in (22) - (45) has bounded trajectories for $\forall t \geq t_0$ if

1) $|x_p(t_0)| < \frac{x_{max}}{\rho}$
2) $\sqrt{V(t_0)} < \overline{K}_{max} \sqrt{\frac{\lambda_{min}}{\gamma_{max}}}$

Since $|x_p(t)| < x_{max}$ for $\forall t \geq t_0$, we can state that the error variable is of the same order as the difference between saturated input $R(u(t))$ and the unsaturated $u(t)$, stated as:

$$\|e_x(t)\| = \mathcal{O}[\sup_{\tau \leq t} \|\Delta u(\tau)\|]$$

We refer to [28]–[30] for details of the proof.

B. Safety in the presence of uncertainties and control input limits

With stability guaranteed from the discussions above, we now derive conditions for the safety of the proposed adaptive

controller with magnitude saturation. We again set $\alpha(h(x)) = \gamma h(x)$, where γ is a positive constant. The goal once again is to ensure safety, that is, for inequality (27) to be satisfied. Unlike (28), we note that the modified QP-ZCBF filter in (40) implies that the following inequality holds:

$$
\begin{aligned}
\dot{h}_m = \frac{\partial h}{\partial x}\big|_{x_m}[A_m x_m(t) + B_m r_s(t) + L R_{e_0}(e_x(t)) \\
+ B_p \hat{\Lambda} R_{\Delta u_0}(\Delta u(t))] \geq -\gamma(e_h(t)) h(x_m(t)) + \Delta
\end{aligned}
\tag{46}
$$

using $\Delta e_x(t) = R_{e_0}(e_x(t)) - e_x(t)$ and $\bar{\Delta}u(t) = R_{\Delta u_0}(\Delta u(t)) - \Delta u(t)$ we can reformulate

$$
\begin{aligned}
\dot{h}_m = \frac{\partial h}{\partial x}\big|_{x_m} [\underbrace{A_m x_m(t) + B_m r_s(t)}_{c_1} + \underbrace{L e_x(t)}_{c_2} \\
+ \underbrace{B_p \hat{\Lambda}(t) \Delta u(t)}_{c_3} + \underbrace{L \Delta e_x(t) + B_p \hat{\Lambda}(t) \bar{\Delta}u}_{S_\Delta}] \\
\geq \underbrace{-\gamma(e_h(t)) h(x_m(t))}_{c_4} + \Delta
\end{aligned}
\tag{47}
$$

We can derive $\dot{h}_p$ using (12) and considering (13) as

$$
\begin{aligned}
\dot{h}_p = \frac{\partial h}{\partial x}\big|_{x_p}[A_p x_p + B_p \Lambda(\hat{\theta}_x(t) x_p(t) \\
+ \hat{\theta}_r(t) r_s(t) + \Delta u(t))]
\end{aligned}
\tag{48}
$$

using (12) and the error between x_p and x_m, we can state

$$
\begin{aligned}
\dot{h}_p = \frac{\partial h}{\partial x}\big|_{x_p}[\underbrace{A_m x_m(t) + B_m r_s(t)}_{d_0} + \underbrace{L e_x(t)}_{c_2} \\
+ \underbrace{B_p \hat{\Lambda}(t) \Delta u(t)}_{c_3} + \bar{e}_\Delta]
\end{aligned}
\tag{49}
$$

with $\bar{e}_\Delta = \bar{e} - L e_x(t) + B_p \tilde{\Lambda}(t) \Delta u(t)$. Algebraic manipulations allow us to rewrite (49) as

$$
\begin{aligned}
\dot{h}_p = d_0[c_1 + c_2 + c_3 + \bar{e}_\Delta] \\
\geq -c_4 + \Delta + \underbrace{c_0 \bar{e}_\Delta + (d_0 - c_0)(c_1 + c_2 + c_3 + \bar{e}_\Delta)}_{w(t)} - c_0 S_\Delta
\end{aligned}
\tag{50}
$$

Noting that the inequality in (50) is very similar to (31), similar relations to (32) and (33) can be employed to derive an inequality

$$
\dot{h}_p \geq -\gamma(e_h(t)) h(x_p(t)) + \Delta - \bar{F}_\Delta(|\bar{e}_\Delta|, |e_x|)
\tag{51}
$$

where

$$
\begin{aligned}
\bar{F}_\Delta(|\bar{e}_\Delta|, |e_x(t)|) = \gamma \kappa_4 |e_x| + |c_0||\bar{e}_\Delta| \\
+ \kappa_3 |e_x| (|c_1| + |c_2| + |c_3| + |\bar{e}_\Delta|) - C_0 S_\Delta
\end{aligned}
\tag{52}
$$

Using similar arguments that utilize the asymptotic convergence of $\bar{F}(t)$ to zero, we obtain once again that the closed-loop adaptive system will remain safe for all $t \geq t_0$ if

$$
h(x_p(t)) \geq h_0 \quad \forall t \geq [t_0, t_0 + T]
\tag{53}
$$

where $\gamma h_0 \geq F_{\max}$, where

$$
F_{\max} = \max_{t \in [t_0, t_0 + T]} \bar{F}(t)
$$

An EBR as in (36) can be introduced as in Section III to allow condition (52) to be satisfied for a large class of $h(x(t))$.

A theorem similar to Theorem 3 can be derived to encapsulate the safety property of the closed-loop adaptive system with magnitude saturation using the controller specified by (21)-(23) and (45). The resulting closed-loop system therefore is stable, safe, and accommodates magnitude constraints on the control input.

V. SIMULATIONS

Two simulation examples are provided in this section to illustrate the properties of the safe and stable adaptive control approach described in this paper that includes a QP-CBF filter and an EBR based damping term γ.

A. Obstacle avoidance

The proposed controller is applied to a simple 2D obstacle avoidance problem in the simulation case here. The used model is defined as:

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

where x and y represent the position states, $\dot{x}$ and $\dot{y}$ are their derivatives. The control inputs being represented by u_1 and u_2. The magnitude limit on both control inputs u_0 was set to 10. The reference model is defined as:

$$
A_m = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B_m = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
$$

To ensure safety in the context of obstacle avoidance, we choose for each obstacle the following CBF constraint:

$$
h_o = (x - x_o)^2 + (y - y_o)^2 - r_o^2 \geq 0
$$

where x_o and y_0 represent the x and y position of the obstacles center and r_o the radius of the circular obstacles.
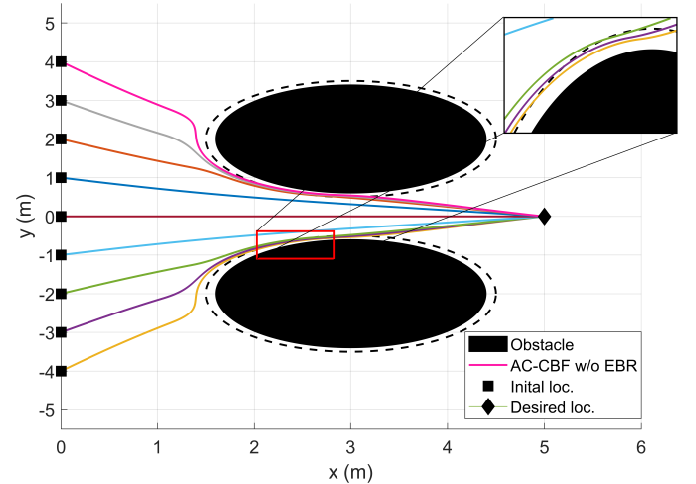

Figure 1: Comparison of state trajectories with different initial states using an AC-CBF without EBR.

Figure 1 shows the state trajectories of the discussed adaptive control framework without the proposed error-based
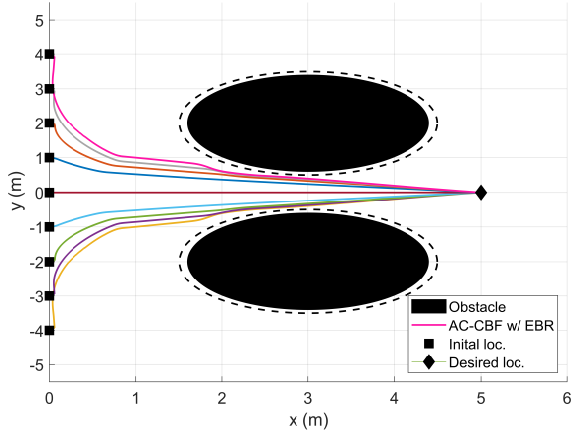
Figure 2: Comparison of state trajectories with different initial states using a AC-CBF with EBR.

Figure 3: Trajectory comparison of state AoA for AC-CBF with EBR and without EBR w.r.t. to the commanded and the maxmimum AoA.

Figure 4: Time history of relaxation parameter γ for the case with EBR and without EBR.

relaxation (EBR) of the CBF constraint for adaptive closed-loop systems with different initial locations. It is apparent that even though the adaptive controller learns the parameter, it cannot ensure safety since it cannot learn fast enough. This leads to a safety violation for most of the shown trajectories. Figure 2 shows the state trajectories for systems with different initial locations using the proposed adaptive controller with EBR of the CBF constraint. It can be seen that the proposed method introduces a helpful conservatism where the model is not yet learned. This renders the system safer and ensures that no trajectory violates the safety constraint even though the model is not learned yet.

B. Missile pitch dynamics

The used missile pitch dynamics model is shared in [31]. The model was modified such that a model mismatch can be considered:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \Delta A_p \begin{bmatrix} Z_\alpha & Z_q \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_\delta \\ M_\delta \end{bmatrix} \lambda_\delta \delta$$
$$= \Delta A_p \begin{bmatrix} -0.8757 & 1 \\ -68.9210 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} -0.1531 \\ -74.2313 \end{bmatrix} \lambda_\delta \delta$$

where α represents the aerodynamic angle of attack (AoA), q the pitch rate and δ the fin deflection. ΔA_p and λ_δ are scalar parameters used to introduce static parameter deviations. For the here regarded case, the parameters were set to $\Delta A_p = 1.2$ and $\lambda_\delta = 0.6$. The regarded linear dynamics are modeled at Mach 0.8 and an altitude of 4000 ft, with a trim angle of attack of 6 degrees. The magnitude limits on the control input are set to 10 degrees for δ. The reference model of the system was chosen by defining a nominal closed-loop response, using a conventional LQR technique to define a suitable feedback controller [32]. The reference model is defined as:

$$A_m = \begin{bmatrix} -0.8707 & 0.9927 \\ -65.5877 & -3.5903 \end{bmatrix}, B_m = \begin{bmatrix} 0.1395 & -0.0364 \\ 68.2893 & -17.7947 \end{bmatrix}$$

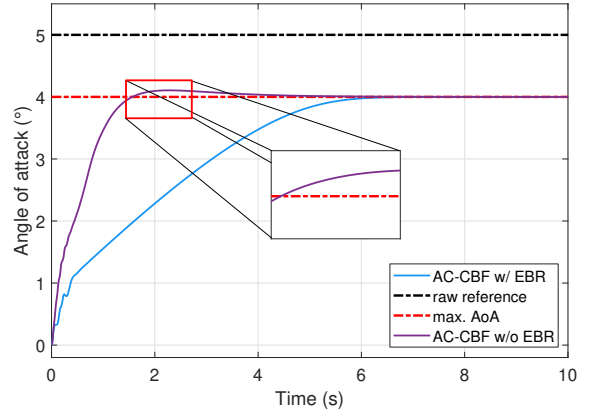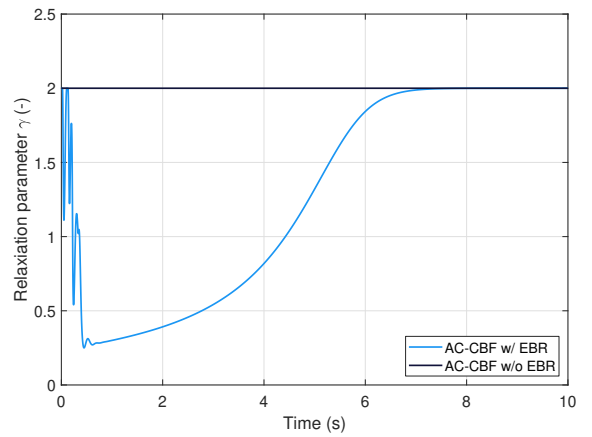For the here presented simulation case, it was chosen that the

maximum missile's AoA is limited by an arbitrary value. To ensure that, we choose the following CBF constraint:

$$h_\alpha = \alpha_{max} - \alpha \geq 0$$

Figure 3 compares the closed-loop response of the suggested adaptive controller in both versions, with and without the error-based relaxation parameter. In this case, the desired AoA is set to 5 degrees, but the maximum allowable state was set to 4 degrees. It can be seen that the controller without the EBR violates the defined maximum AoA constraint. The controller with EBR is able to cope better with the problem of uncertainties and uses the added conservatism during the learning to converge to the defined maximum value when the confidence in the model allows it. Figure 4 shows how the relaxation parameter $\gamma(e)$, for the adaptive controller with EBR, decreases at the beginning when the model is the least known. During the learning, the value increases and indicates a higher confidence in the operation within the safe set. Figure 5 compares the time history of the control input for the proposed adaptive controller with and without EBR, which shows that
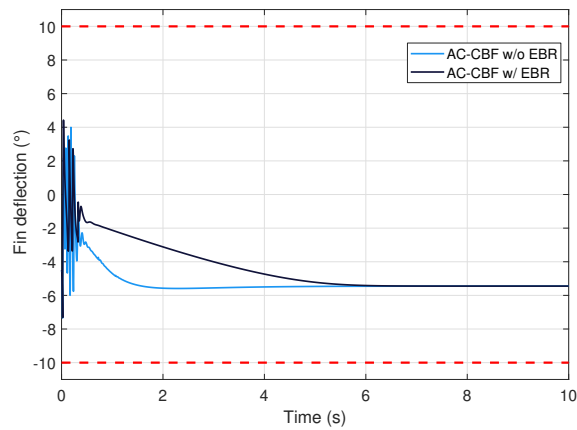
5065

Figure 5: Comparison of the time history of the fin deflection δ for the adaptive controller with EBR and without EBR.

the control input always stays well within its limits. We expect that the oscillations can be improved further by deploying rate limits and extending the proposed approach in this paper along the lines of [28].

VI. CONCLUSIONS

This paper takes a step in combining adaptive control methods with safety-critical methods for a specific class of dynamic systems. In addition to ensuring safety and stability, the proposed adaptive control design also seeks to accommodate magnitude constraints on the control input. The proposed approach employs a combination of classical adaptive control, closed-loop reference model that is calibrated to accommodate input saturation, a quadratic programming based CBF filter, and an error-based relaxation of the damping characteristics of the CBF. The resulting combination is shown to lead to global boundedness without input constraints, and a domain of attraction result with input constraints. In both cases, conditions for the existence of a control barrier function are derived. Numerical results validate the analytical derivations.

REFERENCES

[1] K. S. Narendra and A. M. Annaswamy, *Stable adaptive systems*. NJ, USA: Dover Publications (original publication by Prentice-Hall Inc., 1989), 2005.

[2] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Prentice-Hall, 1996.

[3] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*. Prentice-Hall, 1989.

[4] J.-J. E. Slotine and W. Li, *Applied nonlinear control*. Prentice hall Englewood Cliffs, NJ, 1991.

[5] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and adaptive control design*. John Wiley & Sons, Inc., 1995.

[6] K. J. Åström and B. Wittenmark, *Adaptive control*. Courier Corporation, 2013.

[7] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2016.

[8] Q. Nguyen and K. Sreenath, "Exponential control barrier functions for enforcing high relative-degree safety-critical constraints," in *2016 American Control Conference (ACC)*. IEEE, 2016, pp. 322–328.

[9] W. Xiao and C. Belta, "Control barrier functions for systems with high relative degree," in *2019 IEEE 58th conference on decision and control (CDC)*. IEEE, 2019, pp. 474–479.

[10] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, "On adaptive control with closed-loop reference models: Transients, oscillations, and peaking," *IEEE Access*, vol. 1, pp. 703–717, 2013.

[11] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, "Robustness of control barrier functions for safety critical control**this work is partially supported by the national science foundation grants 1239055, 1239037 and 1239085." *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 54–61, 2015, analysis and Design of Hybrid Systems ADHS. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S2405896315024106

[12] M. Jankovic, "Robust control barrier functions for constrained stabilization of nonlinear systems," *Automatica*, vol. 96, pp. 359–367, 2018. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0005109818303509

[13] J. Buch, S.-C. Liao, and P. Seiler, "Robust control barrier functions with sector-bounded uncertainties," *IEEE Control Systems Letters*, vol. 6, pp. 1994–1999, 2022.

[14] A. Taylor, A. Singletary, Y. Yue, and A. Ames, "Learning for safety-critical control with control barrier functions," in *Proceedings of the 2nd Conference on Learning for Dynamics and Control*, ser. Proceedings of Machine Learning Research, A. M. Bayen, A. Jadbabaie, G. Pappas, P. A. Parrilo, B. Recht, C. Tomlin, and M. Zeilinger, Eds., vol. 120. PMLR, 10–11 Jun 2020, pp. 708–717. [Online]. Available: https://proceedings.mlr.press/v120/taylor20a.html

[15] A. J. Taylor and A. D. Ames, "Adaptive safety with control barrier functions," in *2020 American Control Conference (ACC)*, 2020, pp. 1399–1405.

[16] Q. Nguyen and K. Sreenath, "L1 adaptive control barrier functions for nonlinear underactuated systems," in *2022 American Control Conference (ACC)*, 2022, pp. 721–728.

[17] E. Lavretsky and R. Gadient, "Robust adaptive design for aerial vehicles with state-limiting constraints," *Journal of Guidance, Control, and Dynamics*, vol. 33, no. 6, pp. 1743–1752, 2010. [Online]. Available: https://doi.org/10.2514/1.50 101

[18] D. P. Wiese, A. M. Annaswamy, J. A. Muse, M. A. Bolender, and E. Lavretsky, "Sequential loop closure based adaptive output feedback," *IEEE Access*, vol. 5, pp. 23 436–23 451, 2017.

[19] J. Muse, *A Method For Enforcing State Constraints in Adaptive Control*. [Online]. Available: https://arc.aiaa.org/doi/abs/10.2514/6.2011-6205

[20] M. Nagumo, "Über die lage der integralkurven gewöhnlicher differentialgleichungen," *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, vol. 24, pp. 551–559, 1942.

[21] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[22] A. D. Ames, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs with application to adaptive cruise control," in *53rd IEEE Conference on Decision and Control*, 2014, pp. 6271–6278.

[23] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.

[24] J. Zeng, B. Zhang, and K. Sreenath, "Safety-critical model predictive control with discrete-time control barrier function," in *2021 American Control Conference (ACC)*, 2021, pp. 3882–3889.

[25] A. M. Annaswamy, A. Guha, Y. Cui, S. Tang, and J. E. Gaudio, "Online algorithms and policies using adaptive and machine learning approaches," *arXiv preprint arXiv:2105.06577*, 2021.

[26] J. Autenrieb and A. M. Annaswamy, "Safe and stable adaptive control for a class of dynamic systems," *arXiv preprint arXiv:2309.05533*, 2023.

[27] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, "Closed-loop reference models for output-feedback adaptive systems," in *2013 European Control Conference (ECC)*, 2013, pp. 365–370.

[28] J. E. Gaudio, A. M. Annaswamy, M. A. Bolender, and E. Lavretsky, "Adaptive flight control in the presence of limits on magnitude and rate," *IEEE Access*, vol. 10, pp. 65 685–65 702, 2022.

[29] S. Karason and A. Annaswamy, "Adaptive control in the presence of input constraints," *IEEE Transactions on Automatic Control*, vol. 39, no. 11, pp. 2325–2330, 1994.

[30] M. Schwager and A. Annaswamy, "Direct adaptive control of multi-input plants with magnitude saturation constraints," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005, pp. 783–788.

[31] K. A. Wise, "Singular value robustness tests for missile autopilot uncertainties," *Journal of Guidance, Control, and Dynamics*, vol. 14, no. 3, pp. 597–606, 1991.

[32] R. Kalman, "Contribution to the theory of optimal control," *Boletín de la Sociedad Matemática Mexicana*, vol. 5, 1960.