OGP LIMITS LIMIT SWAPPING IN QAOA

DPG Berlin 2024| Mark Goh

Combinatorial Optimization Problem

- Informally, Find the optimal configuration that would minimize Hamiltonian.

Combinatorial Optimization Problem

- Informally, Find the optimal configuration that would minimize Hamiltonian.

- Formally, The problem is defined as

$$\min H(z), \qquad z = (z_1, \ldots, z_n) \in \{\pm 1\}^n \tag{1}$$

where the maximum is achieved for configuration z^*.

Combinatorial Optimization Problem

- Informally, Find the optimal configuration that would minimize Hamiltonian.

- Formally, The problem is defined as

$$\min H(z), \qquad z = (z_1, \ldots, z_n) \in \{\pm 1\}^n \tag{1}$$

  where the maximum is achieved for configuration z^*.

- NP-Hard Problem (e.g travelling salesman problem, Max-q-XORSAT, ground state (energy) of spin glass).

Overlap Gap Property (informal) [Gamarnik, 2021]

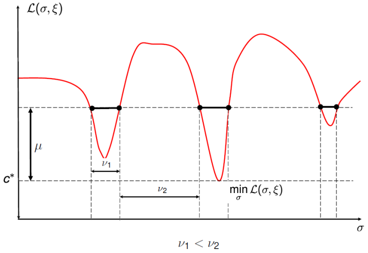
- Overlap Gap Property (OGP) is a known obstruction for combinatorial optimization problem with various variations (C-OGP, m-OGP).

- Informally, solutions are clustered and 2 solutions are either very close or very far from each other.

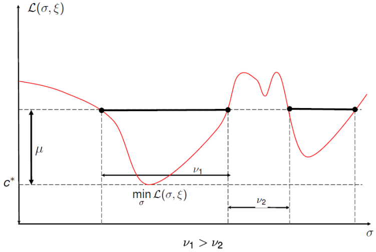
Overlap Gap Property (informal) [Gamarnik, 2021]

- Overlap Gap Property (OGP) is a known obstruction for combinatorial optimization problem with various variations (C-OGP, m-OGP).

- Informally, solutions are clustered and 2 solutions are either very close or very far from each other.



OGP present



No OGP

q-spin model and Max-q-XORSAT

- Hamiltonian of q-spin glass

$$H_q(z) = -\frac{1}{\sqrt{N^{(q-1)}}} \sum_{j<k<\cdots<q} J_{jk\ldots q} z_j z_k \ldots z_q, \qquad J_{jk\ldots q} \sim_{iid} \mathcal{N}(0, 1/n^{(q-1)}).$$

q-spin model and Max-q-XORSAT

- Hamiltonian of q-spin glass

$$H_q(z) = -\frac{1}{\sqrt{N^{(q-1)}}} \sum_{j<k<\dots<q} J_{jk\dots q} z_j z_k \dots z_q, \qquad J_{jk\dots q} \sim_{iid} \mathcal{N}(0, 1/n^{(q-1)}).$$

- Hamiltonian of Max-q-XORSAT

$$H^q_{XOR}(z) = -\sum_{(i_1,\dots,i_q)\in E} \frac{1}{2}(1 + J_{i_1 i_2 \dots i_q} z_{i_1} z_{i_2} \dots z_{i_q}), \qquad J_{i_1 i_2 \dots i_q} \in \{+1, -1\}.$$

Quantum Approximate Optimization Algorithm

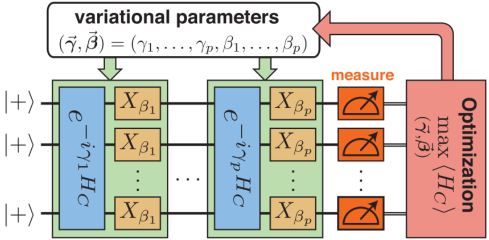
Components of QAOA

1. Each layer of QAOA has:

   $e^{-i\gamma H_C}$ with cost function H_C

   $e^{-i\beta B}$ where $B = \sum_{i=1}^{n} X_i$

2. As $p \to \infty$, QAOA gets exact result (provided spectral gap does not close).



Circuit diagram of QAOA [Zhou et al., 2020]

QAOA performance [Basso et al., 2022a]

- Averaging over the disorder/randomness ($\mathbb{E}_J$), for any p and parameters $(\boldsymbol{\gamma}, \boldsymbol{\beta})$

$$\lim_{N\to\infty} \mathbb{E}_J \left[\langle \boldsymbol{\gamma}, \boldsymbol{\beta} | H_q/N | \boldsymbol{\gamma}, \boldsymbol{\beta} \rangle \right] = V_p^{(q)}(\boldsymbol{\gamma}, \boldsymbol{\beta}) \tag{2}$$

QAOA performance [Basso et al., 2022a]

- Averaging over the disorder/randomness ($\mathbb{E}_J$), for any p and parameters $(\boldsymbol{\gamma}, \boldsymbol{\beta})$

$$\lim_{N\to\infty} \mathbb{E}_J \left[\langle \boldsymbol{\gamma}, \boldsymbol{\beta} | H_q/N | \boldsymbol{\gamma}, \boldsymbol{\beta} \rangle \right] = V_p^{(q)}(\boldsymbol{\gamma}, \boldsymbol{\beta}) \tag{2}$$

- On any D-regular q-uniform hypergraph with girth $> 2p+1$, for any p and parameters $(\boldsymbol{\gamma}, \boldsymbol{\beta})$

$$\frac{1}{|E|} \langle \boldsymbol{\gamma}, \boldsymbol{\beta} | C^q_{XOR} | \boldsymbol{\gamma}, \boldsymbol{\beta} \rangle = \frac{1}{2} + \nu_p^{[q]}(D, \boldsymbol{\gamma}, \boldsymbol{\beta}) \sqrt{\frac{q}{2D}}, \tag{3}$$

$$\nu_p^{[q]}(\boldsymbol{\gamma}, \boldsymbol{\beta}) = \lim_{D\to\infty} \nu_p^{[q]}(D, \boldsymbol{\gamma}, \boldsymbol{\beta}) \tag{4}$$

Asymptotic equivalence + limitation

- For any p and parameters $(\boldsymbol{\gamma}, \boldsymbol{\beta})$, we have equivalence of performance on dense and sparse graphs [Basso et al., 2022b]:

$$V_p^{(q)}(\boldsymbol{\gamma}, \boldsymbol{\beta}) = \sqrt{2}\nu_p^{[q]}(\sqrt{q}\boldsymbol{\gamma}, \boldsymbol{\beta})$$

Asymptotic equivalence + limitation

- For any p and parameters $(\boldsymbol{\gamma}, \boldsymbol{\beta})$, we have equivalence of performance on dense and sparse graphs [Basso et al., 2022b]:

$$V_p^{(q)}(\boldsymbol{\gamma}, \boldsymbol{\beta}) = \sqrt{2}\nu_p^{[q]}(\sqrt{q}\boldsymbol{\gamma}, \boldsymbol{\beta})$$

Conjecture for Sherrington–Kirkpatrick Model

$$\lim_{N\to\infty} \lim_{p\to\infty} \stackrel{?}{=} \lim_{p\to\infty} \lim_{N\to\infty}$$

Asymptotic equivalence + limitation

- For any p and parameters $(\boldsymbol{\gamma}, \boldsymbol{\beta})$, we have equivalence of performance on dense and sparse graphs [Basso et al., 2022b]:

$$V_p^{(q)}(\boldsymbol{\gamma}, \boldsymbol{\beta}) = \sqrt{2}\nu_p^{[q]}(\sqrt{q}\boldsymbol{\gamma}, \boldsymbol{\beta})$$

Conjecture for Sherrington–Kirkpatrick Model

$$\lim_{N\to\infty} \lim_{p\to\infty} \stackrel{?}{=} \lim_{p\to\infty} \lim_{N\to\infty}$$

- For Max-q-XORSAT on a random Erdös–Rényi hypergraph, for every even $q \geq 4$, there exists a value $E_{OGP} > E_{min}$ such that QAOA is limited if $p \sim \mathcal{O}(\log(N))$ [Chou et al., 2022]:

$$\langle \boldsymbol{\gamma}, \boldsymbol{\beta} | C_{XOR}^q / N | \boldsymbol{\gamma}, \boldsymbol{\beta} \rangle \geq E_{OGP}.$$

Conjecture and main result

Conjecture

The OGP for Max-q-XORSAT is a monotonic increasing property of graphs in general and not just for Erdös–Rényi

Conjecture and main result

Conjecture

The OGP for Max-q-XORSAT is a monotonic increasing property of graphs in general and not just for Erdös–Rényi

Theorem

The OGP also holds for Max-q-XORSAT on a random regular hypergraph and thus has the same limitation mentioned before.

Conjecture and main result

Conjecture

The OGP for Max-q-XORSAT is a monotonic increasing property of graphs in general and not just for Erdös–Rényi

Theorem

The OGP also holds for Max-q-XORSAT on a random regular hypergraph and thus has the same limitation mentioned before.

- Numerical results support + random Erdös–Rényi hypergraph and random regular hypergraph similar in large degree limit

Conjecture and main result

Conjecture

The OGP for Max-q-XORSAT is a monotonic increasing property of graphs in general and not just for Erdös–Rényi

Theorem

The OGP also holds for Max-q-XORSAT on a random regular hypergraph and thus has the same limitation mentioned before.

- Numerical results support + random Erdös–Rényi hypergraph and random regular hypergraph similar in large degree limit

Spectrum for $n = 40$ Max-4-XORSAT with degree 10.

Corollaries

Corollary 1
If conjecture is true, then the performance of QAOA on the pure q-spin glass for even $q \geq 4$ converges to E_{OGP} as $p \to \infty$ and is strictly greater than the optimal value, i.e. the Parisi value, using recent algorithm [Basso et al., 2022b].

Corollaries

Corollary 1

If conjecture is true, then the performance of QAOA on the pure q-spin glass for even $q \geq 4$ converges to E_{OGP} as $p \to \infty$ and is strictly greater than the optimal value, i.e. the Parisi value, using recent algorithm [Basso et al., 2022b].

Corollary 2

If a problem exhibits the OGP, then the swapping of limits results in a sub-optimal solution.

Corollaries

Corollary 1
If conjecture is true, then the performance of QAOA on the pure q-spin glass for even $q \geq 4$ converges to E_{OGP} as $p \to \infty$ and is strictly greater than the optimal value, i.e. the Parisi value, using recent algorithm [Basso et al., 2022b].

Corollary 2
If a problem exhibits the OGP, then the swapping of limits results in a sub-optimal solution.

Corollary 3
By failure of reduction to Quantum adiabatic algorithm, the spectral gap closes in the thermodynamic limit when the OGP is present.

What's next?

1. Sherrington–Kirkpatrick (2-spin glass) model unlikely to have OGP, does limit swapping still holds?

What’s next?

1. Sherrington–Kirkpatrick (2-spin glass) model unlikely to have OGP, does limit swapping still holds?

2. OGP likely to exists for all $q \geq 3$. Current proofs only for even $q \geq 4$ and for $q \gg 1$.

Summary

- Assuming monotonicity of the OGP, Max-q-XORSAT on random regular uniform hypergraph also exhibits OGP.
- OGP implies
  1. Suboptimal performance of QAOA even if $p \to \infty$ for current methods.
  2. Optimising QAOA under limit swapping is suboptimal.
  3. Spectral gap closes during adiabatic evolution for $n \to \infty$.

Thanks for listening! Any questions?

References I

- Basso, J., Farhi, E., Marwaha, K., Villalonga, B., and Zhou, L. (2022a).
  The quantum approximate optimization algorithm at high depth for maxcut on large-girth regular graphs and the sherrington-kirkpatrick model.
  Schloss Dagstuhl - Leibniz-Zentrum für Informatik.

- Basso, J., Gamarnik, D., Mei, S., and Zhou, L. (2022b).
  Performance and limitations of the QAOA at constant levels on large sparse hypergraphs and spin glass models.
  In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE.

References II

- Chou, C.-N., Love, P. J., Sandhu, J. S., and Shi, J. (2022).

  Limitations of local quantum algorithms on random max-k-xor and beyond.

- Gamarnik, D. (2021).

  The overlap gap property: A topological barrier to optimizing over random structures.

  Proceedings of the National Academy of Sciences, 118(41).

References III

Zhou, L., Wang, S.-T., Choi, S., Pichler, H., and Lukin, M. D. (2020).
Quantum approximate optimization algorithm: Performance, mechanism, and implementation on near-term devices.
Physical Review X, 10(2).