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Extension of Doak's momentum potential theory for multi-species and reacting flows

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Abstract: This work extends Doak's momentum potential theory to multi-chemical-component and reactive, time-stationary fluctuating flows. Additional mixture-related components are found to be superimposed on the canonical vortical, acoustic, and thermal parts of momentum fluctuations and total fluctuating enthalpy. These extended relations are used to develop a time-averaged model that relates the acoustic power radiated to the far-field with clearly defined vortical, acoustic, thermal, and compositional near-field sources. The resulting model is designed to offer a more general and comprehensive way to describe the noise generated within combustion chambers.

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1. Introduction

Doak's momentum potential theory of energy flux carried by momentum fluctuations (MPT)[1] constitutes a general framework for the identification of sound generated aerodynamically. The generality of this theory is one of its strengths. Indeed, the MPT makes no strong assumptions except for a time-stationary fluctuating flow. That said, the distinguishing feature of the MPT is represented by a clear identification and separation of vortical (intended as "hydrodynamic" or "turbulent"), acoustic, and thermal fluctuations. Furthermore, what sets the MPT apart from similar flow decomposition methodologies[2,3] is its explicit emphasis on the concept of a *local fluctuating equilibrium*, i.e., a local balance between fluctuating quantities only.

After being successfully applied to produce new insights into the acoustics of turbulent jets,[4–7] the MPT was recently proposed as a general framework for combustion noise applications[8] and applied to analyze the dynamics of a lab-scale gas turbine combustor in stable operating conditions.[9] In these previous studies, the clear separation of vortical, acoustic, and thermal effects was used to highlight the relation of the acoustic properties of the combustor with turbulent and thermal dynamics. However, the original MPT formulation misses a description of compositional and combustion phenomena and, therefore, cannot fully describe flows in combustor systems. Hence, the theory requires an extension to fully describe the dynamics of such flows as well.

This limitation is overcome in this study by extending the model to the motion of multi-chemical-component continua. The strategy for such an extension had already been broadly proposed, first by Doak[1] and then by D'Aniello *et al.*[8,10] and Brokof *et al.*[11] In this study, the basis laid in Refs. 8, 10, and 11 is built on. More specifically, new components related to the dynamics of the mixtures, ideally characterizing a multi-chemical-component flow, are derived and included. The model extension should not be considered just as a mere "theoretical result." Indeed, in the literature, it is reported that compositional effects can play a substantial role in the generation of the combustion noise.[12] Moreover, these effects need to be carefully distinguished from similar contributions due to the thermal fluctuations.[10,13] In this sense, the proposed model extension offers a more general and comprehensive framework, since it enables one to describe the generation of combustion noise more fully.

As proposed in previous studies,[8–10] an MPT for combustion acoustics may consist of three fundamental steps: First, the momentum fluctuations vector needs to be decomposed, through a Helmholtz decomposition, into vortical, acoustic, and thermal components; second, a similar decomposition needs to be obtained for the total fluctuating enthalpy (TFE), whose components represent a kind of flow state variables for the respective turbulent, acoustic, and thermal dynamics; finally, the TFE fluxes due to the momentum fluctuations should be related to clearly separated vortical, acoustic, and thermal sources in a time-averaged model. In particular, this model is assumed to offer a more general and comprehensive framework to describe the generation of the combustion noise.

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The above-described steps are formally extended in the following by inclusion of the newly obtained terms related to the dynamics of the mixtures. All details about the derivation of such new terms are reported in Sec. 2. A discussion of the physical interpretation of the extended MPT model follows in Sec. 3. The most important results are finally summarized, and conclusions are drawn in Sec. 4.

2. MPT extension to multi-component and reactive flows

2.1 Decomposition of the momentum fluctuations

The foundation for the extension of the MPT model to multi-chemical component continua has already been globally charted out by Doak.[1] This extension is based on the supposed existence of a constitutive relation for the density, ρ, of the following type:

$$\rho = \rho(p, s, Y_1, \ldots, Y_N), \tag{1}$$

which relates the density ρ to local values of static pressure p and specific entropy s, as well as to the quantities $Y_1, \ldots, Y_N$, which represent suitable thermodynamic variables for N species, which ideally characterize the multi-chemical-component flow. In the following, these are going to be considered, without loss of generality, as the species mass fractions. As in the original MPT, the momentum density, $\rho\mathbf{u}$, can be decomposed, with the help of a Helmholtz decomposition, as the sum of solenoidal and irrotational components, i.e.,

$$\rho\mathbf{u} = \mathbf{B} - \nabla\psi, \quad \text{with } \nabla\cdot\mathbf{B} = 0, \tag{2}$$

with $\mathbf{u}$ representing the local velocity vector. $\mathbf{B}$ and $\nabla\psi$ are the solenoidal and irrotational momentum density components, respectively. For a time-stationary fluctuating flow,[1] the continuity equation can then be written, by using Eq. (2), as

$$\nabla^2\psi' = \partial\rho'/\partial t, \tag{3}$$

where the superscript $'$ indicates fluctuating quantities. Furthermore, the following relation can be obtained for the mean values:

$$\nabla^2\overline{\psi} = 0 \Rightarrow \overline{\rho\mathbf{u}} = \overline{\mathbf{B}}. \tag{4}$$

As a consequence, the momentum fluctuations vector, $(\rho\mathbf{u})'$, is defined as

$$(\rho\mathbf{u})' = \mathbf{B}' - \nabla\psi'. \tag{5}$$

When assuming a multi-chemical-component continuum, Eq. (3) can be rewritten by using the differential form of Eq. (1), i.e.,

$$d\rho = \rho_p dp + \rho_s ds + \sum_k^N \rho_{Y_k} dY_k. \tag{6}$$

Equation (3) then becomes

$$\nabla^2\psi' = \rho_p \partial p'/\partial t + \rho_s \partial s'/\partial t + \sum_k^N \rho_{Y_k} \partial Y_k'/\partial t, \tag{7}$$

so that the scalar potential ψ' can be expressed as the sum of clearly defined acoustic, thermal, and mixture components, i.e.,

$$\psi' = \underbrace{\psi_A'}_{acoustic} + \overbrace{\psi_T'}^{thermal} + \underbrace{\sum_k^N \psi_{Y_k}'}_{mixture}. \tag{8}$$

These potentials can be obtained by solving the following Poisson's equations:

$$\nabla^2\psi_A' = \rho_p \partial p'/\partial t, \quad \nabla^2\psi_T' = \rho_s \partial s'/\partial t, \quad \nabla^2\psi_{Y_1}' = \rho_{Y_1}\partial Y_1'/\partial t, \ \ldots, \ \nabla^2\psi_{Y_N}' = \rho_{Y_N}\partial Y_N'/\partial t. \tag{9}$$

To solve the system defined by Eqs. (3) and (9), the $N+2$ derivatives ρ_p, ρ_s, $\rho_{Y_1}, \ldots, \rho_{Y_N}$ must be determined. To this end, thermodynamics constitutive relations can be used. For this purpose, the following Gibbs relation is employed:

$$dh = TdS + \frac{1}{\rho}dp + \sum_{k=1}^N \frac{\mu_k}{W_k} dY_k, \tag{10}$$

where μ_k and W_k are the chemical potential and the molecular weight of the kth species, respectively. The specific enthalpy h is defined as

$$h = h_{\text{sens}} + \sum_{k=1}^{N} \Delta h_{f_k}^0 Y_k = \int_{T_0}^{T} c_p dT + \sum_{k=1}^{N} \Delta h_{f_k}^0 Y_k, \tag{11}$$

where h_{sens} and $\Delta h_{f_k}^0$ represent sensible enthalpy and species enthalpy of formation. c_p is the specific heat at constant pressure, which can be expressed as the sum of the species-specific contributions, as

$$c_p = \sum_{k=1}^{N} c_{p_k} Y_k. \tag{12}$$

By combining Eqs. (10) and (11), the following differential relation is obtained:

$$c_p dT = TdS + \frac{1}{\rho} dp + \sum_{k=1}^{N} \left(\frac{\mu_k}{W_k} - \Delta h_{f_k}^0 \right) dY_k. \tag{13}$$

As suggested by Brokof *et al.*,[11] the dependence on the temperature T in Eq. (13) can be eliminated by using the ideal gas law, which yields

$$p = \rho T R_m = \rho T \sum_{k=1}^{N} \frac{R}{W_k} Y_k \Rightarrow dp = TR_m d\rho + \rho R_m dT + \rho T \sum_{k=1}^{N} \frac{R}{W_k} dY_k$$
$$\iff dT = \frac{1}{\rho R_m} dp - \frac{T}{\rho} d\rho - T \sum_{k=1}^{N} \frac{R}{R_m W_k} dY_k, \tag{14}$$

where R and R_m are the universal and mixture gas constants, respectively. The combination of Eqs. (13) and (14) leads, hence, to

$$d\rho = \frac{1}{c^2} dp - \frac{\rho}{c_p} dS + \frac{\rho}{c_p T} \sum_{k=1}^{N} \left(\Delta h_{f_k}^0 - \frac{\mu_k}{W_k} - c_p T \frac{W}{W_k} \right) dY_k, \tag{15}$$

where c and $W = R/R_m$ are the local speed of sound and total molecular weight of the mixture, respectively. Finally, a comparison of Eqs. (15) and (6) yields the formulations for the derivatives ρ_p, ρ_s, ρ_{Y_k} as follows:

$$\rho_p = \left. \frac{\partial \rho}{\partial p} \right|_{s, Y_k} = \frac{1}{c^2}, \tag{16}$$

$$\rho_s = \left. \frac{\partial \rho}{\partial s} \right|_{p, Y_k} = -\frac{\rho}{c_p}, \tag{17}$$

$$\rho_{Y_k} = \left. \frac{\partial \rho}{\partial Y_k} \right|_{p, s, Y_{i \neq k}} = \frac{\rho}{c_p T} \left(\Delta h_{f_k}^0 - \frac{\mu_k}{W_k} - c_p T \frac{W}{W_k} \right). \tag{18}$$

Once these terms are known, the system in Eq. (9) can be solved, and the momentum fluctuations can be written, by combining Eqs. (5) and (8), as the sum of four clearly defined vortical, acoustic, thermal, and mixture components, as follows:

$$(\rho \mathbf{u})' = \underbrace{\mathbf{B}'}_{\textit{vortical}} \overbrace{-\nabla \psi'_A}^{\textit{acoustic}} \underbrace{-\nabla \psi'_T}_{\textit{thermal}} \overbrace{-\sum_{k=1}^{N} \nabla \psi'_{Y_k}}^{\textit{mixture}}, \tag{19}$$

where $-\sum_k^N \nabla \psi'_{Y_k}$ is the additional compositional term of the momentum fluctuations. For each of the species, a single contribution is added to the vortical, acoustic, and thermal components of the canonical MPT model.

2.2 Decomposition of the TFE

Equation (19) represents the first, important result of the MPT model extension. In fact, based on this relation, a similar decomposition in vortical, acoustic, thermal, and mixture components can be generally found for all quantities considered in the MPT model.

An important example thereof is represented by the TFE. Defining the stagnation enthalpy as

$$H = h + \frac{\mathbf{u} \cdot \mathbf{u}}{2}, \tag{20}$$

the TFE is its fluctuating part, H', that is,

$$H' = H - \overline{H}. \tag{21}$$

In Doak's original formulation,[1] the TFE is decomposed in the canonical vortical, acoustic, and thermal components. These definitions can be extended to the motion of multi-chemical-component continua by considering the specific enthalpy as a function of additional independent thermodynamic variables for the mixtures. Similar to Eq. (1), such a relation in its most general form is

$$h = h(p, s, Y_1, \dots, Y_N), \tag{22}$$

whose differential form was already introduced in Eq. (10).

Using Eq. (19), the temporal derivative of the TFE in a time-stationary fluctuating flow can be expressed as

$$\begin{aligned}\frac{\partial H'}{\partial t} &= \frac{\partial}{\partial t}\left[h + \frac{1}{\rho^2}\frac{\mathbf{m}\cdot\mathbf{m}}{2}\right]' = \frac{\partial h'}{\partial t} + \left[\frac{\mathbf{m}}{\rho^2}\frac{\partial \mathbf{m}}{\partial t} - \frac{\mathbf{m}\cdot\mathbf{m}}{\rho^3}\frac{\partial \rho}{\partial t}\right]' \\ &= \frac{\partial h'}{\partial t} + \left[\frac{\mathbf{m}}{\rho^2}\cdot\frac{\partial}{\partial t}\left(\mathbf{B}' - \nabla\psi'_A - \nabla\psi'_T - \sum_{k=1}^{N}\nabla\psi_{Y_k}{}'\right) - \frac{\mathbf{m}\cdot\mathbf{m}}{\rho^3}\left(\rho_p\frac{\partial p'}{\partial t} + \rho_s\frac{\partial s'}{\partial t} + \sum_{k=1}^{N}\rho_{Y_k}\frac{\partial Y'_k}{\partial t}\right)\right]',\end{aligned} \tag{23}$$

with $\mathbf{m} = \rho\mathbf{u}$. Expanding the term $\partial h'/\partial t$ with the chain rule by means of Eq. (22), the TFE can be decomposed as

$$H' = \underbrace{H'_B}_{\textit{vortical}} + \overbrace{H'_A}^{\textit{acoustic}} + \underbrace{H'_T}_{\textit{thermal}} + \overbrace{\sum_{k=1}^{N} H'_{Y_k}}^{\textit{mixture}}, \tag{24}$$

where vortical, acoustic, thermal, and mixture TFE components are expressed by the following relations:

$$H'_B = \left[\int \frac{\mathbf{m}}{\rho^2}\cdot\frac{\partial \mathbf{B}'}{\partial t}dt\right]', \tag{25}$$

$$H'_A = \left[\int\left\{\left(h_p - \frac{\mathbf{m}\cdot\mathbf{m}}{\rho^3}\rho_p\right)\frac{\partial p'}{\partial t} - \frac{\mathbf{m}}{\rho^2}\cdot\frac{\partial}{\partial t}(\nabla\psi'_A)\right\}dt\right]', \tag{26}$$

$$H'_T = \left[\int\left\{\left(h_s - \frac{\mathbf{m}\cdot\mathbf{m}}{\rho^3}\rho_s\right)\frac{\partial s'}{\partial t} - \frac{\mathbf{m}}{\rho^2}\cdot\frac{\partial}{\partial t}(\nabla\psi'_T)\right\}dt\right]', \tag{27}$$

$$H'_{Y_k} = \left[\int\left\{\left(h_{Y_k} - \frac{\mathbf{m}\cdot\mathbf{m}}{\rho^3}\rho_{Y_k}\right)\frac{\partial Y'_k}{\partial t} - \frac{\mathbf{m}}{\rho^2}\cdot\frac{\partial}{\partial t}(\nabla\psi_{Y_k}{}')\right\}dt\right]'. \tag{28}$$

By neglecting higher order fluctuating terms and using the values of Eq. (10) for the derivatives h_p, h_s, and h_{Y_k}, these expressions read

$$H'_B \approx \bar{c}\,\overline{\mathbf{M}}\cdot\frac{\mathbf{B}'}{\bar{\rho}}, \tag{29}$$

$$H'_A \approx (1 - \overline{\mathbf{M}\cdot\mathbf{M}})\frac{p'}{\bar{\rho}} - \frac{\bar{c}\overline{\mathbf{M}}}{\bar{\rho}}\cdot\nabla\psi'_A, \tag{30}$$

$$H'_T \approx \frac{\bar{c}^2}{\gamma - 1}\left[1 + (\gamma - 1)\overline{\mathbf{M}\cdot\mathbf{M}}\right]\frac{s'}{c_p} - \frac{\bar{c}\overline{\mathbf{M}}}{\bar{\rho}}\cdot\nabla\psi'_T, \tag{31}$$

$$H_{Y_k}{}' \approx \left\{\left[1 + (\gamma - 1)\overline{\mathbf{M}\cdot\mathbf{M}}\right]\bar{\mu}_k - \Delta h^0_{f_k}(\gamma - 1)\overline{\mathbf{M}\cdot\mathbf{M}}\right\}Y'_k - \frac{\bar{c}\overline{\mathbf{M}}}{\bar{\rho}}\cdot\nabla\psi_{Y_k}{}', \tag{32}$$

with $\bar{c}$, $\overline{\mathbf{M}}$, and γ representing mean speed of sound, mean Mach number vector, and specific heat ratio, respectively. Equations (29)–(32) define the vortical, acoustic, and thermal components of the canonical MPT model as well as the new mixture components of the TFE in the extended MPT framework.

2.3 Time-averaged model of TFE fluxes due to momentum fluctuations

As argued in Sec. 1, a key aspect in the MPT formulation is the definition of a local fluctuating equilibrium, that is, a balance involving fluctuating quantities only. This is represented, in Doak's original MPT, by a time-averaged model for the TFE fluxes due to momentum fluctuations.

To derive such a time-averaged model in the extended framework, the total energy conservation[14] should first be considered. In a multi-chemical-component and reactive flow, when neglecting body forces and external thermal energy addition, this can be written in terms of the stagnation enthalpy, after time-averaging, as

$$\nabla \cdot (\overline{H\rho\mathbf{u}}) = \nabla \cdot (\overline{\underline{\mathbf{S}} \cdot \mathbf{u}} + \lambda \overline{\nabla T}), \tag{33}$$

where $\underline{\mathbf{S}}$ represents the shear stress tensor and λ the heat conduction coefficient. The last relation can be combined with the momentum-conservation equations,[15]

$$\frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\Omega} \times \mathbf{u}) + \nabla H - \left(T\nabla s + \frac{1}{\rho}\nabla \cdot \underline{\mathbf{S}} + \sum_{k=1}^{N} \frac{\mu_k}{W_k} \nabla Y_k \right) = 0, \tag{34}$$

to obtain

$$\nabla \cdot (\overline{H}\,\overline{\mathbf{B}}) + \overline{\mathbf{B}} \cdot \left[\overline{\left(\frac{\Omega}{\rho}\right)' \times (\rho \boldsymbol{u})'} \right] = \overline{\mathbf{B}} \cdot \left(\overline{T\nabla s} + \overline{\frac{1}{\rho}\nabla \cdot \underline{\mathbf{S}}} + \sum_{k=1}^{N} \overline{\frac{\mu_k}{W_k} \nabla Y_k} \right), \tag{35}$$

where the term $\boldsymbol{\Omega} = \nabla \times \mathbf{u}$ is the vorticity. Equations (33) and (35) can be then combined to extend Doak's first version of the mean energy balance, independent of the mean momentum contribution, $\nabla \cdot (\overline{H}\,\overline{\mathbf{B}})$, that is,

$$\nabla \cdot \left[\overline{H'(\rho\mathbf{u})'} \right] = \overline{\mathbf{B}} \cdot \left[\overline{\left(\frac{\Omega}{\rho}\right)' \times (\rho \boldsymbol{u})'} \right] + \nabla \cdot \left[\overline{\underline{\mathbf{S}} \cdot \mathbf{u}} + \lambda \overline{\nabla T} \right] - \overline{\mathbf{B}} \cdot \left(\overline{T\nabla s} + \overline{\frac{1}{\rho}\nabla \cdot \underline{\mathbf{S}}} + \sum_{k=1}^{N} \overline{\frac{\mu_k}{W_k} \nabla Y_k} \right). \tag{36}$$

Equation (36) can be expressed in a more compact form by simplifying the scalar product, $\overline{\mathbf{B}} \cdot \overline{[(\Omega/\rho)' \times (\rho \boldsymbol{u})']}$, as proposed by Doak,[1] and by means of Eq. (4), as

$$\overline{\mathbf{B}} \cdot \left[\overline{(\Omega/\rho)' \times (\rho \boldsymbol{u})'} \right] = \overline{\rho \mathbf{u}} \cdot \left[\overline{(\Omega/\rho)' \times (\rho \boldsymbol{u})'} \right] = \overline{\rho \mathbf{u}} \cdot (\overline{\Omega \times \mathbf{u}}) = \overline{\mathbf{B}} \cdot (\overline{\Omega \times \mathbf{u}}), \tag{37}$$

so that Eq. (36) reduces, for multi-chemical-component flows, to

$$\nabla \cdot \left[\overline{H'(\rho\mathbf{u})'} - \left(\overline{\underline{\mathbf{S}} \cdot \mathbf{u}} + \lambda \overline{\nabla T} \right) \right] = \overline{\mathbf{B}} \cdot \left[(\overline{\Omega \times \mathbf{u}}) - \left(\overline{T\nabla s} + \overline{\frac{1}{\rho}\nabla \cdot \underline{\mathbf{S}}} + \sum_{k=1}^{N} \overline{\frac{\mu_k}{W_k} \nabla Y_k} \right) \right], \tag{38}$$

which implies, by time-averaging of Eq. (34),

$$\nabla \cdot \left[\overline{H'(\rho\mathbf{u})'} \right] = \nabla \cdot \left(\overline{\underline{\mathbf{S}} \cdot \mathbf{u}} + \lambda \overline{\nabla T} \right) + \overline{\mathbf{B}} \cdot \nabla \overline{H}. \tag{39}$$

Equation (39) relates the mean fluctuating energy fluxes due to the momentum fluctuations on the left-hand side to the mean fields on the right-hand side. Interestingly, the same result is obtained by Doak in the original MPT formulation, showing that the proposed extension for multi-chemical and reactive flow does not change the original structure of Eq. (39).

However, the balance expressed by Eq. (39) can be written in an alternative form where only fluctuations are involved. To this end, the balance of fluctuating momentum, i.e., the fluctuating part of Eq. (34), may be considered. This reads

$$\frac{\partial \mathbf{u}'}{\partial t} + (\boldsymbol{\Omega} \times \mathbf{u})' + \nabla H' - \left(T\nabla s + \frac{1}{\rho}\nabla \cdot \underline{\mathbf{S}} + \sum_{k=1}^{N} \frac{\mu_k}{W_k} \nabla Y_k \right)' = 0. \tag{40}$$

After scalar multiplication with $(\rho\mathbf{u})'$ and time-averaging, the last relation can be rearranged, by means of the results obtained above, as

$$\nabla \cdot \left[\overline{H'(\rho\mathbf{u})'} \right] = -\overline{(\rho\mathbf{u})' \cdot \left[(\Omega \times \mathbf{u})' - \left(T\nabla s + \frac{1}{\rho}\nabla \cdot \underline{\mathbf{S}} + \sum_{k=1}^{N} \frac{\mu_k}{W_k} \nabla Y_k \right)' \right]} - \overline{h' \frac{\partial \rho'}{\partial t}}, \tag{41}$$

which is the desired alternative form of Eq. (39) where only fluctuating quantities are involved. The last term on the right-hand side of Eq. (41) can be replaced by the following relation:

$$\overline{h' \frac{\partial \rho'}{\partial t}} = -\overline{(\rho T)' \frac{\partial s'}{\partial t}} + \sum_{k=1}^{N} \overline{\left(\rho \frac{\mu_k}{W_k} \right)' \frac{\partial Y_k'}{\partial t}}, \tag{42}$$

leading to a new form of the mean fluctuating energy balance due to the momentum fluctuations for multi-chemical-component and reactive flow, that is,

$$\nabla \cdot \left[\overline{H'(\rho\mathbf{u})'}\right] = -\overline{(\rho\mathbf{u})' \cdot \left[(\Omega \times \mathbf{u})' - \left(T\nabla s + \frac{1}{\rho}\nabla \cdot \underline{\mathbf{S}} + \sum_{k=1}^{N}\frac{\mu_k}{W_k}\nabla Y_k\right)'\right]} + \overline{(\rho T)'\frac{\partial s'}{\partial t}} + \sum_{k=1}^{N}\overline{\left(\rho\frac{\mu_k}{W_k}\right)'\frac{\partial Y_k'}{\partial t}}. \tag{43}$$

3. Discussion

The main results of the MPT model extension presented above are summarized by Eqs. (19), (24), and (43). For all three relations, new, clearly defined terms related to the dynamics of the mixtures are found. Compared to the original MPT, as proposed by Doak,[1] these are (i) the mixture momentum fluctuation components $\nabla\psi'_{Y_k}$, in Eq. (19); (ii) the mixture TFE components H'_{Y_k}, in Eq. (24); (iii) the terms $\overline{(\rho\mathbf{u})' \cdot (\mu_k/W_k)\nabla Y_k}$ and $\overline{(\rho(\mu_k/W_k))'(\partial Y_k'/\partial t)}$, in Eq. (43).

By inspecting the new components of momentum fluctuations [see Eqs. (8) and (9)] and TFE [see Eqs. (29)–(32)] with respect to the original vortical, acoustic, and thermal ones, one can assert that (i) the distinguishing MPT feature of having unambiguously defined and clearly separated vortical, acoustic, and thermal motions remains a fait accompli in the extended formulation as well; (ii) for each species, a new mixture component is linearly superimposed to the vortical, acoustic, and thermal ones; (iii) vortical, acoustic, and thermal components remain unchanged with respect to the original MPT formulation.

It is important to note that Eqs. (29)–(32) are obtained as a first order approximation of Eqs. (25)–(28), respectively, although neglecting the higher order fluctuating terms is not a necessary step to effectively separate the different TFE components. In fact, vortical, acoustic, thermal, and mixture TFE can be obtained just by solving—numerically or analytically (see supplementary material)—the integrals in Eqs. (25)–(28). Nevertheless, the first order relations, Eqs. (29)–(32), are useful to better compare the extended MPT formulation to Doak's original results. In this regard, it is worth noting that the thermal TFE, H'_T, reported in Eq. (31) features an additional part related to the potential ψ'_T, which was originally neglected by Doak. However, since this does not depend on the multi-chemical-component character of the flow, the difference needs to be interpreted as a kind of generalization of the original proposition.

Similar to what is observed in the case of momentum fluctuations and TFE, the new terms in Eq. (43) do not alter the features of the original model, given that these are linearly superimposed to it. This model still describes the mean fluxes of TFE due to the momentum fluctuations and can be recast, for better physical interpretation, as

$$\nabla \cdot \mathbf{J} = q_J. \tag{44}$$

Seen from the far-field, the divergence $\nabla \cdot \mathbf{J}$ approximates the radiation of acoustic power,[1] with the vector $\mathbf{J}$ defining the mean energy flux due to the momentum fluctuations, i.e.,

$$\begin{aligned}\mathbf{J} = \overline{H'(\rho\mathbf{u})'} &= \overline{H'\mathbf{B}'} + \overline{H'(-\nabla\psi'_A)} + \overline{H'(-\nabla\psi'_T)} + \sum_k^N \overline{H'(-\nabla\psi'_{Y_k})}\\ &= \overline{H'_B(\rho\mathbf{u})'} + \overline{H'_A(\rho\mathbf{u})'} + \overline{H'_T(\rho\mathbf{u})'} + \sum_k^N \overline{H'_{Y_k}(\rho\mathbf{u})'}.\end{aligned} \tag{45}$$

As shown by Eq. (45), the quantity $\mathbf{J}$ can be expanded by means of Eq. (19), when focusing on the flux due to the single components of the momentum fluctuations, or by means of Eq. (24), when focusing on the flux of a particular TFE component. Moreover, as suggested by D'Aniello *et al.*,[8] the term $\overline{H'_A(-\nabla\psi'_A)}$ can be considered as a sort of generalized acoustic intensity to quantify the acoustic energy production inside the source region, i.e., inside the combustor itself.

On the right-hand side of Eq. (44), the term q_J can be written as

$$q_J = \underbrace{-\overline{\left(\mathbf{B}' - \nabla\psi'_A - \nabla\psi'_T - \sum_k^N \nabla\psi'_{Y_k}\right)\cdot\boldsymbol{\alpha}'}}_{\text{vortical,acoustic and thermal sources due to “}\textit{mode coupling}\text{”}} \overbrace{+\overline{(\rho T)'\frac{\partial s'}{\partial t}} + \sum_k \overline{\left(\rho\frac{\mu_k}{W_k}\right)'\frac{\partial Y_k'}{\partial t}}}^{\text{“}\textit{direct}\text{”thermal sources}}. \tag{46}$$

The right-hand side of Eq. (46) describes the near-field sources of acoustic power radiated to the far-field by a multi-chemical-component and reactive flow, similar to what was originally proposed by Doak. As suggested by D'Aniello *et al.*,[8] the term $\boldsymbol{\alpha}'$ summarizes the main mechanisms of "mode coupling"—intended as scattering or conversion of each vortical, acoustic, or thermal motions into another[16]—and can be expressed, in the extended MPT framework, as

$$\boldsymbol{\alpha}' = (\Omega \times \mathbf{u})' - (T\nabla s)' - \left(\frac{1}{\rho}\nabla \cdot \underline{\mathbf{S}}\right)' - \sum_k \left(\frac{\mu_k}{W_k}\nabla Y_k\right)'. \tag{47}$$

Each of the four generation mechanisms interacts separately with vortical, acoustic, thermal, and mixture components of the momentum fluctuations. In turn, the latter can be interpreted as the "carriers" for the respective vortical, acoustic, and thermal scattering.[8] In particular, the acoustic scattering due to the mode coupling can be described by considering the term $\overline{(-\nabla\psi'_A)\cdot\boldsymbol{\alpha}'}$, which can be approximated as

$$q_A = \overline{\boldsymbol{\alpha}'\cdot\nabla\psi'_A} \approx \underbrace{\overline{(\boldsymbol{\Omega}\times\mathbf{u})'\cdot\nabla\psi'_A}}_{\text{vorticity noise source}} \overbrace{-\overline{(T\nabla s)'\cdot\nabla\psi'_A}}^{\text{entropy noise source}} \underbrace{-\sum_{k=1}^{N}\overline{\left(\frac{\mu_k}{W_k}\nabla Y_k\right)'\cdot\nabla\psi'_A}}_{\text{compositional noise source}}, \tag{48}$$

where the contribution of the viscous stresses to the acoustic production was neglected.

In Eq. (48), three sources of acoustic scattering are discerned: a first source due to the fluctuations of the Coriolis acceleration $(\boldsymbol{\Omega}\times\mathbf{u})'$; a second source due the fluctuations of temperature and entropy gradient; a third source due to the fluctuations of chemical potentials and mixture gradients. Vorticity and entropy inhomogeneities were already identified as sources of indirect acoustic scattering in an inviscid, single-component flow by Howe,[17] who also considers the stagnation enthalpy as general acoustic field. Moreover, the partition expressed by Eq. (48) suits really well the classical description of the different indirect combustion noise components (so-called vorticity,[18] entropy,[19] and compositional noise[12]) For these reasons—and more, in general due to fact that the proposed MPT model represents an *exact* model—Eq. (48) may be considered as a tool to effectively separate the sources of such noise emissions.

The last two terms on the right-hand side of Eq. (46) are defined instead as "direct" thermal sources, since they can be directly related to volumetric expansion of the fluid or, equivalently, to the heat release rate, as it will be shown in the following. As reported by Laksana *et al.*,[20] the entropy balance can be approximated, for technically relevant flames, as

$$\rho T\left(\frac{\partial s}{\partial t}+\mathbf{u}\cdot\nabla s\right) \approx -\sum_{k=1}^{N}\left(\frac{\mu_k}{W_k}\dot{\omega}_k\right) \approx \dot{q}, \tag{49}$$

where $\dot{\omega}_k$ is the production rate of the kth species and $\dot{q}$ represents the heat release rate. Indeed, averaging Eq. (49) leads for a time-stationary fluctuating flow to

$$\overline{(\rho T)'\left(\frac{\partial s}{\partial t}\right)'} \approx \bar{\dot{q}} - \overline{(\rho\mathbf{u})\cdot(T\nabla s)} \Rightarrow \overline{(\rho T)'\left(\frac{\partial s}{\partial t}\right)'} \propto \bar{\dot{q}}. \tag{50}$$

Similarly, the species conservation equation [Eq. (15)] can be used to show that, by neglecting molecular diffusion and assuming time-stationarity, the second direct thermal source of Eq. (46) reduces to

$$\sum_{k}^{N}\overline{\left(\rho\frac{\mu_k}{W_k}\right)'\frac{\partial Y'_k}{\partial t}} \approx \sum_{k=1}^{N}\overline{\left(\frac{\mu_k}{W_k}\right)\dot{\omega}_k} \Rightarrow \sum_{k=1}^{N}\overline{\left(\rho\frac{\mu_k}{W_k}\right)'\frac{\partial Y'_k}{\partial t}} \propto \bar{\dot{q}}. \tag{51}$$

Equations (50) and (51) show that both direct thermal sources are linear functions of the mean heat release rate. Hence, by considering radiation to the far-field, they could be seen as a measure for the radiated acoustic power due to the flame or, equivalently, for the source of direct combustion noise. However, similar to the first order approximation leading to Eqs. (29)–(32), the approximations in Eqs. (50) and (51) are not necessary in the explicit application of the MPT theory. They rather give a more intuitive understanding of the direct thermal sources in Eq. (46).

4. Conclusion

In this work, Doak's MPT was extended to multi-chemical-component and reactive, time-stationary fluctuating flows. Additional components related to the behavior of the mixtures were found to be superimposed on the canonical vortical, acoustic, and thermal parts of the momentum fluctuations. Similarly, new mixture-related components were linearly superimposed to vortical, acoustic, and thermal TFE components as well. These extended relations, defined by Eqs. (19) and (24), were used to develop a time-averaged model that relates the fluxes of the TFE due to the momentum fluctuations with clearly separated vortical, acoustic, and thermal sources for such fluxes.

By inspecting the newly derived mixture components, with respect to the original vortical, acoustic, and thermal ones, it was possible to assert that first, the unambiguous definition and clear separation of vortical, acoustic, and thermal motions remained a distinguishing feature of the extended MPT formulation; second, for each species a new mixture component was linearly superimposed on the original ones; and third, the original vortical, acoustic, and thermal components remained unchanged.

The time-averaged model for the mean energy fluxes due to the momentum fluctuations proposed in Eq. (43) offered a more general and comprehensive way to describe the noise generated by combustors. The radiation of the mean energy flux J, on the left-hand side of the model, was balanced by clearly separated vortical, acoustic, and thermal sources, on the right-hand side. These represented sources for the acoustic radiation to the far-field, which were recast into "mode coupling" sources—since related to the scattering of vortical, acoustic, and thermal motions into another—and "direct" thermal sources—since directly related to the mean heat release rate. In this way, indirect (including entropy, vorticity, and compositional) as well as direct combustion noise sources could be described by the framework. In this way, the proposed MPT framework recovered and generalized—at least conceptually—the classical separation between indirect (including entropy, vorticity, and compositional) and direct combustion noise sources.

Supplementary material

See the supplementary materials, Appendixes A and B, for additional details about exact analytical formulation for the TFE components and direct thermal sources of the time-averaged model.

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Author declarations

Conflict of Interest

The authors state that they have no conflicts to disclose.

Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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