PHYSICS-INFORMED FEM-BASED NEURAL NETWORKS FOR SOLVING THE STOKES AND NAVIER STOKES EQUATIONS

Franziska Griese, Philipp Knechtges, Alexander Rüttgers
German Aerospace Center (DLR)
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Why Physics Informed Neural Networks*?*

- E.g. safe control of airplanes needs surrogate modeling.

  …but people (rightfully) have reservations against AI.

- By data-driven NN approaches some natural laws are not or only poorly considered.

Conservation of		
Energy	Mass	Momentum

Why Physics Informed Neural Networks*?*

How to make AI safer and more robust*?*

- By teaching it the physics.

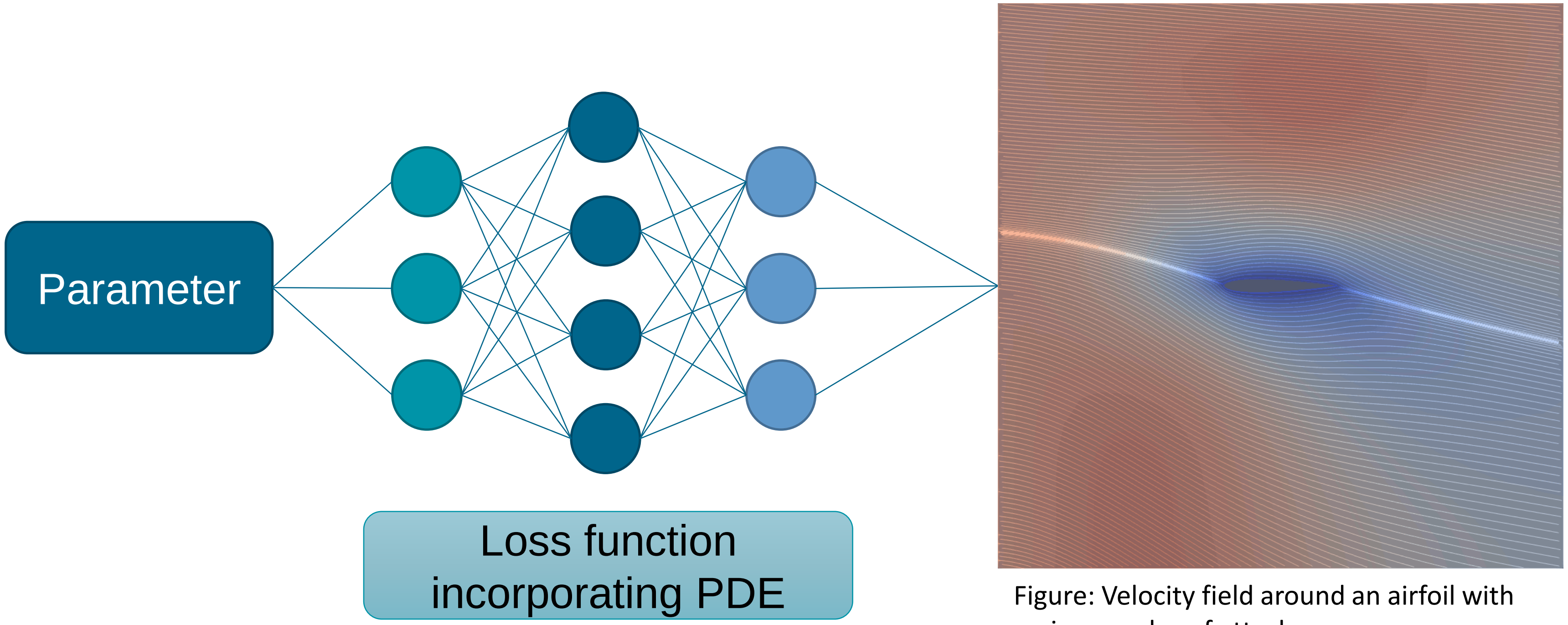


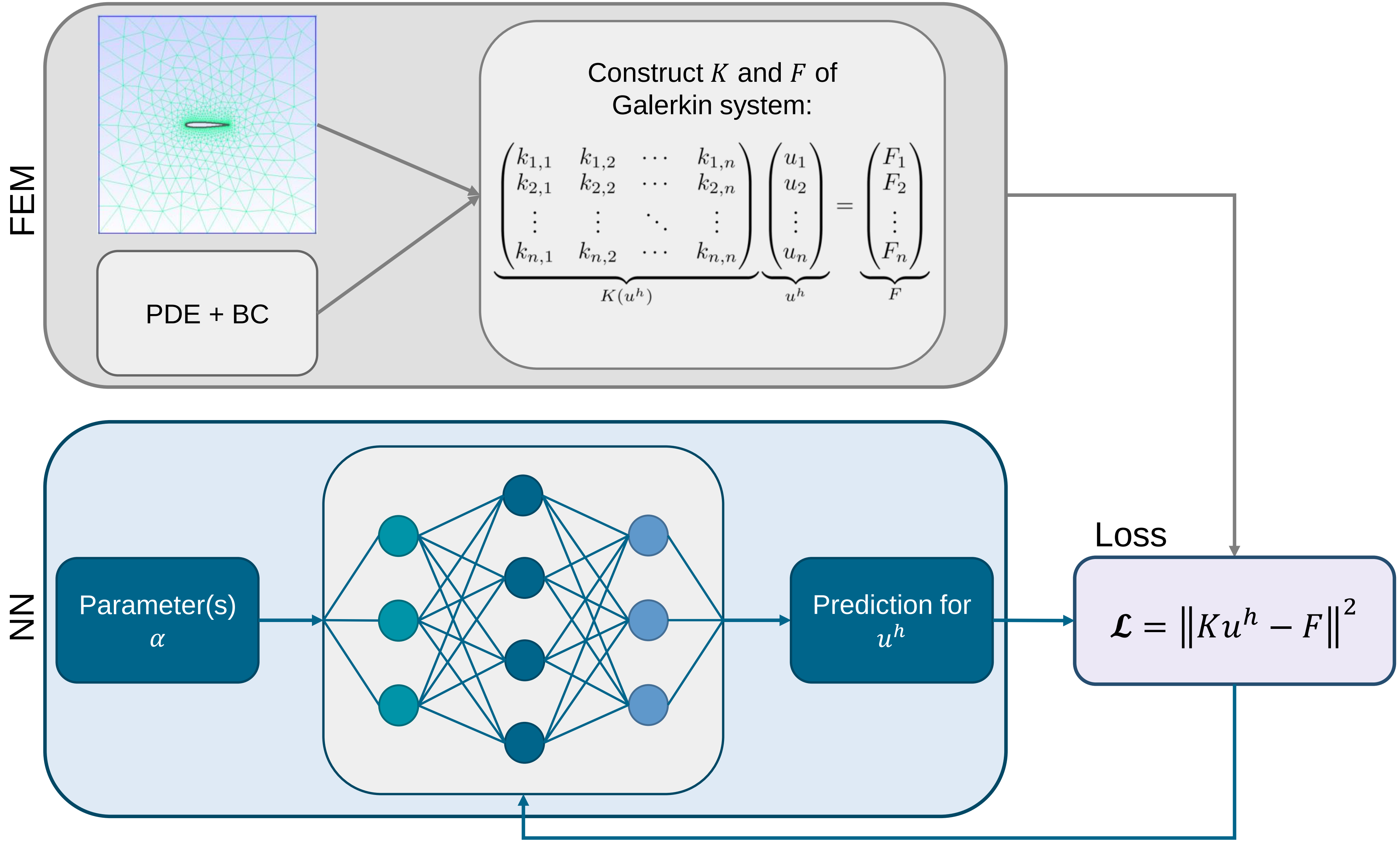
Figure: Velocity field around an airfoil with various angles of attack.

Finite Element Method based Neural Networks

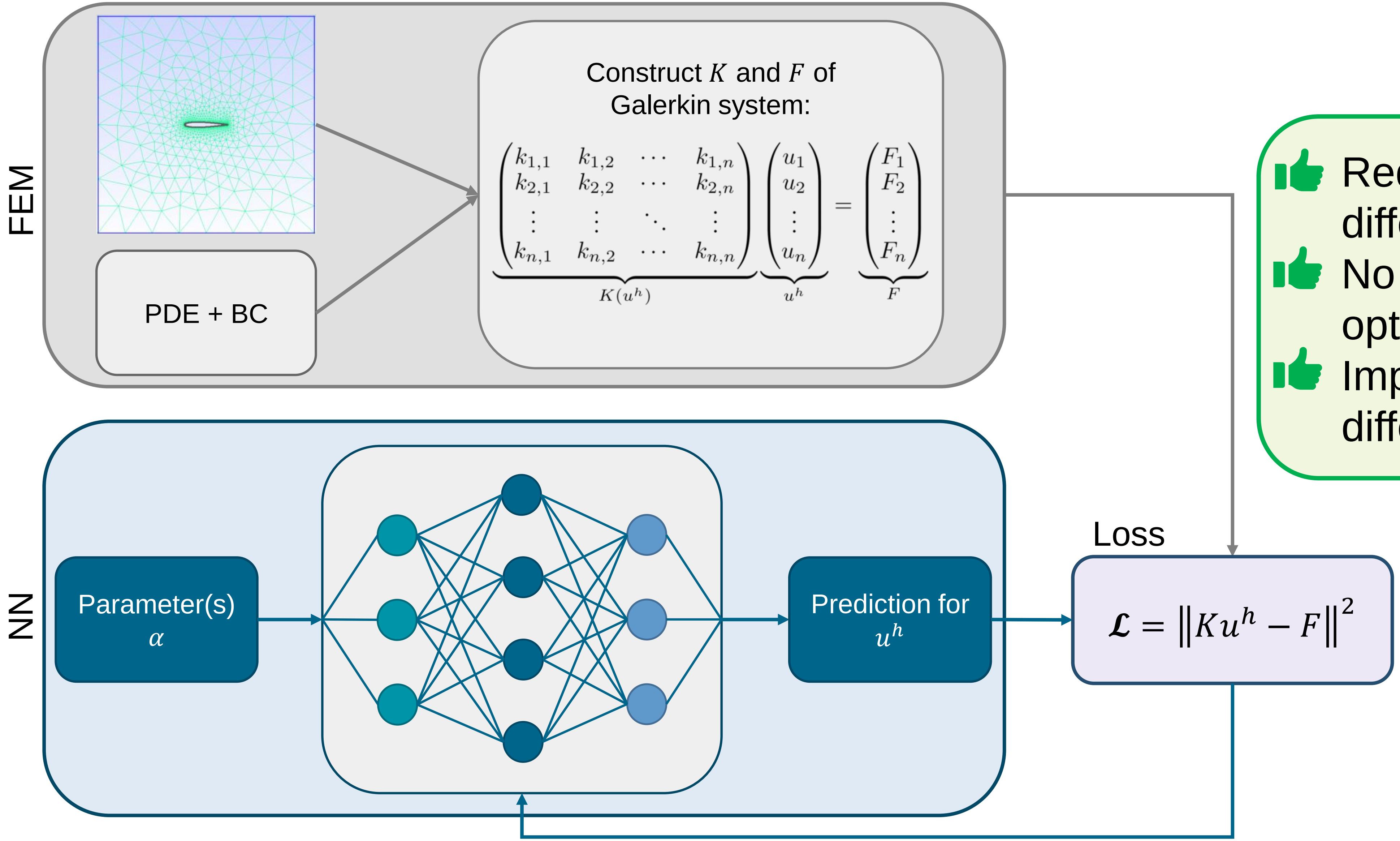
- Combination of classical finite element method and NN
- Combine strengths and compensate weaknesses of individual approaches:

FEM	NN
👎 No real-time capacity	👍 Fast prediction after training
👎 No cost amortization over multiple runs	👍 Parameterizable
👍 Sound mathematical foundation	👎 Black box model
👍 Numerical theory of errors	👎 Rudimentary convergence theory

FEM-based Neural Networks



FEM-based Neural Networks



- Reduced order of differential operators
- No multi-objective optimization
- Implemented fully differentiable

2D Stokes flow around an airfoil

- PDE:

$$\nabla \cdot u = 0,$$
$$\nabla p - \Delta u = 0$$

- with velocity u and pressure p
- BC: Dirichlet at inflow, Neumann at outflow, no-slip at airfoil

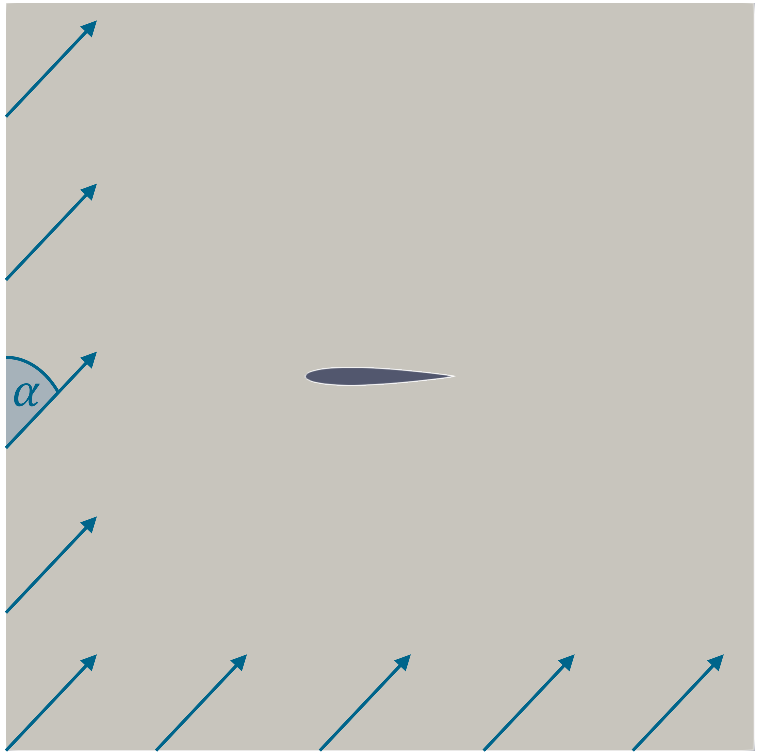
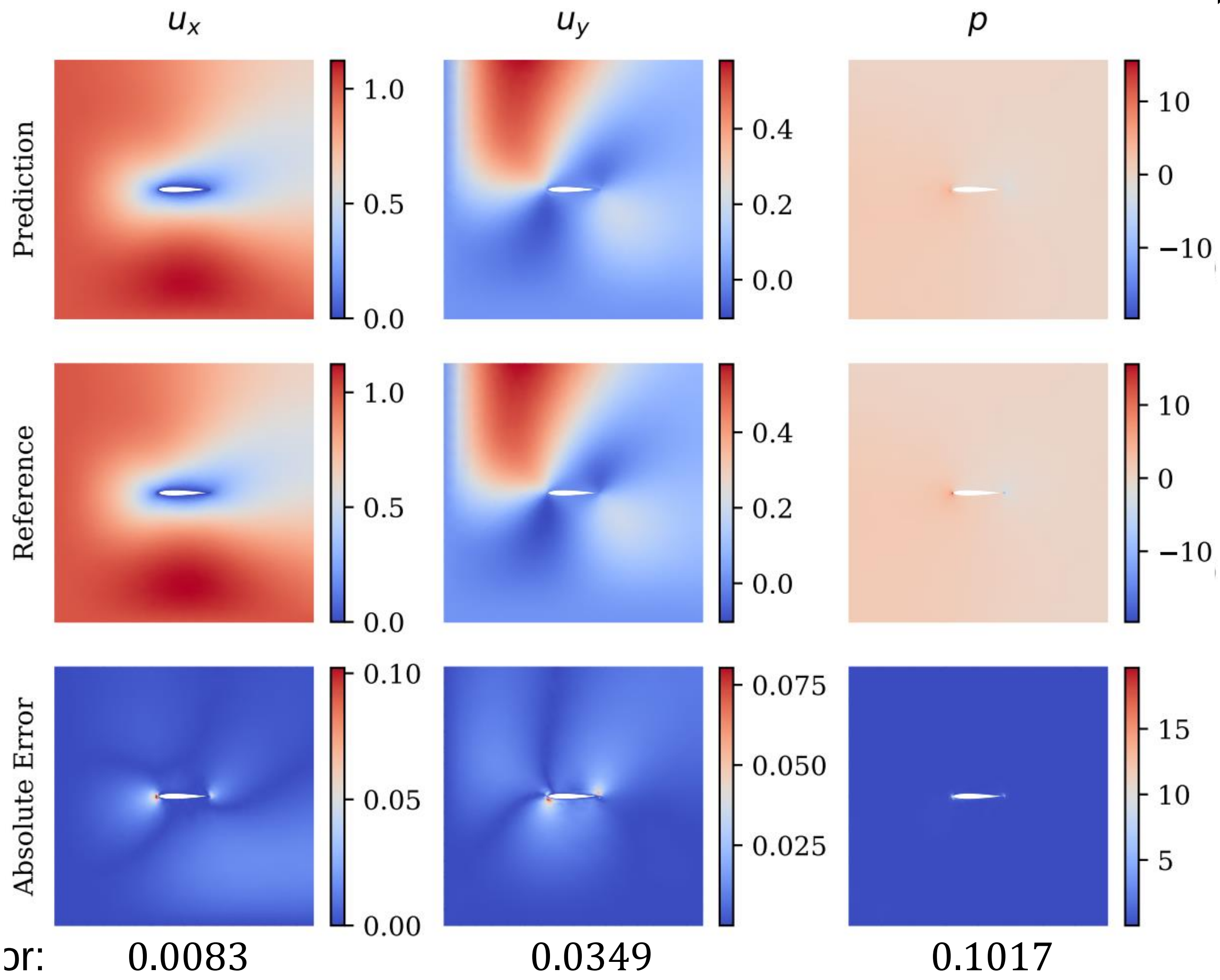


Figure: Domain around NACA 0012 airfoil with an angle of attack α.

- Saddle point problem
- Even for FEM no straight-forward problem

FEM-based NNs – Stokes

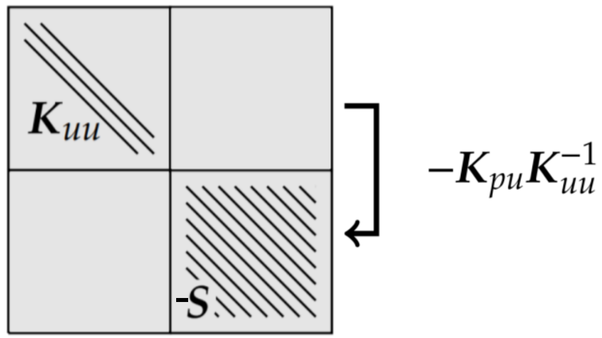
- Taylor-Hood elements to construct Galerkin system → avoid instabilities
- Training with $\alpha = 1$ and 2000 epochs of LBFGS
- CPU training time ~20 min



FEM-based NNs – Stokes Preconditioned

- Improve condition of indefinite stiffness matrix K
- Preconditioning with:
  - Schur complement
  - Cholesky decomposition $\widetilde{K} = L \cdot L^T$

$\widetilde{K} =$



$$S = -K_{up}K_{uu}^{-1}K_{pu}$$

- Training with $\alpha = 1$ and 2 epochs of LBFGS:

	u_x	u_y	p
Rel. L^∞	1.8833e-07	1.8658e-07	3.0804e-06
Rel. L^2	9.6091e-08	1.6061e-07	2.0170e-07

→ Error improvement of e-05, CPU training time < 1s

FEM-based NNs – Stokes Parameterizability & Generalizability

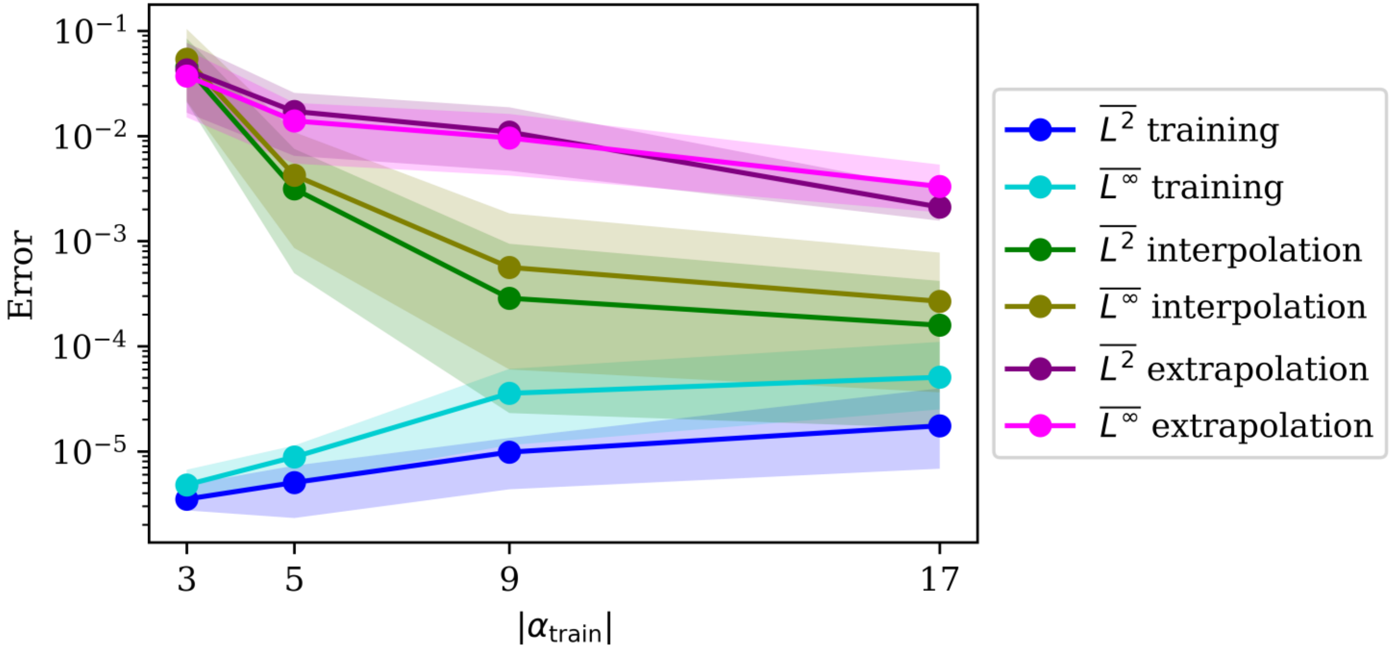
Parameterizability

- Training data $\alpha_{\text{train}} \in [1, 45]$ uniformly distributed

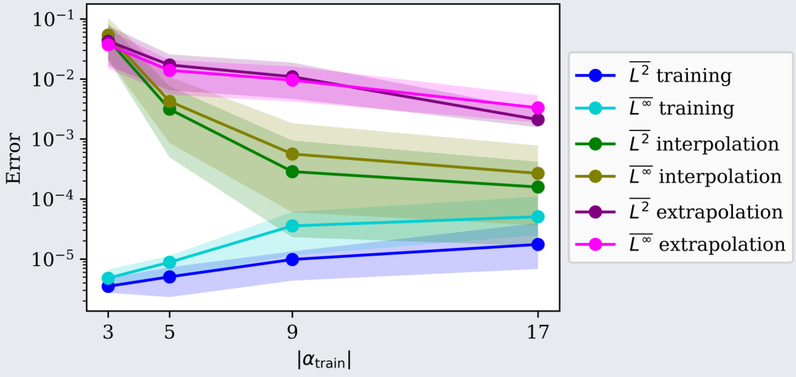
Generalizability

- Interpolation data $\alpha_{\text{in}} \in \{5, 16.5, 30, 40\}$
- Extrapolation data $\alpha_{\text{ex}} \in \{47.5, 50, 55\}$

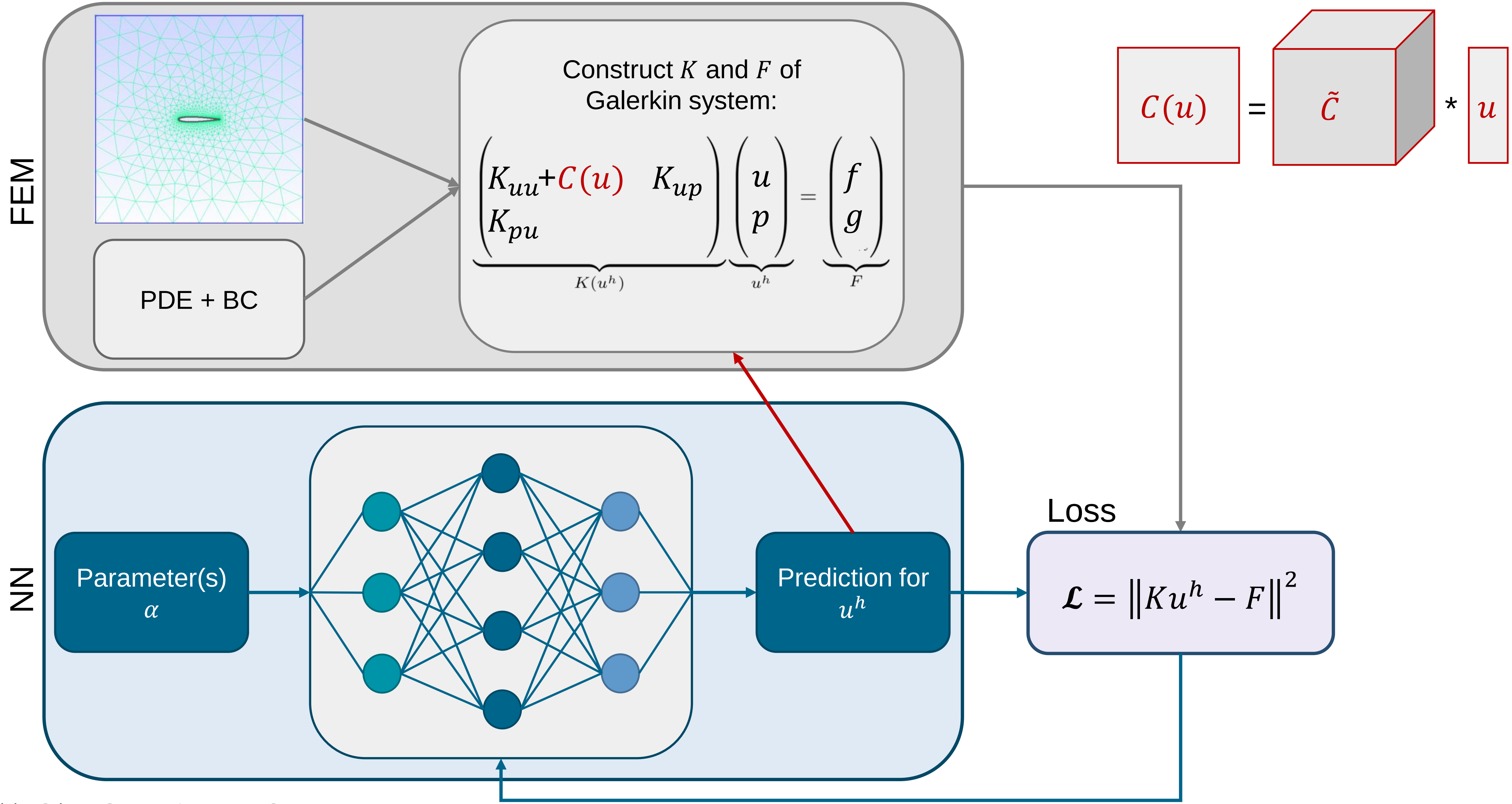
Rel. L^2 and L^∞ errors of the variable u_x



Problems

	2D Stokes	2D Navier Stokes
PDE	$\nabla p - \Delta u = 0$ $\nabla \cdot u = 0$	$-\eta\Delta u + u \cdot \nabla u + \nabla p = 0$ $\nabla \cdot u = 0$
Classification	Linear saddle point problem	Nonlinear saddle point problem
Preconditioner	Schur, Cholesky	Schur, Cholesky
Results	Rel. L^2 and L^∞ errors of the variable u_x 	?

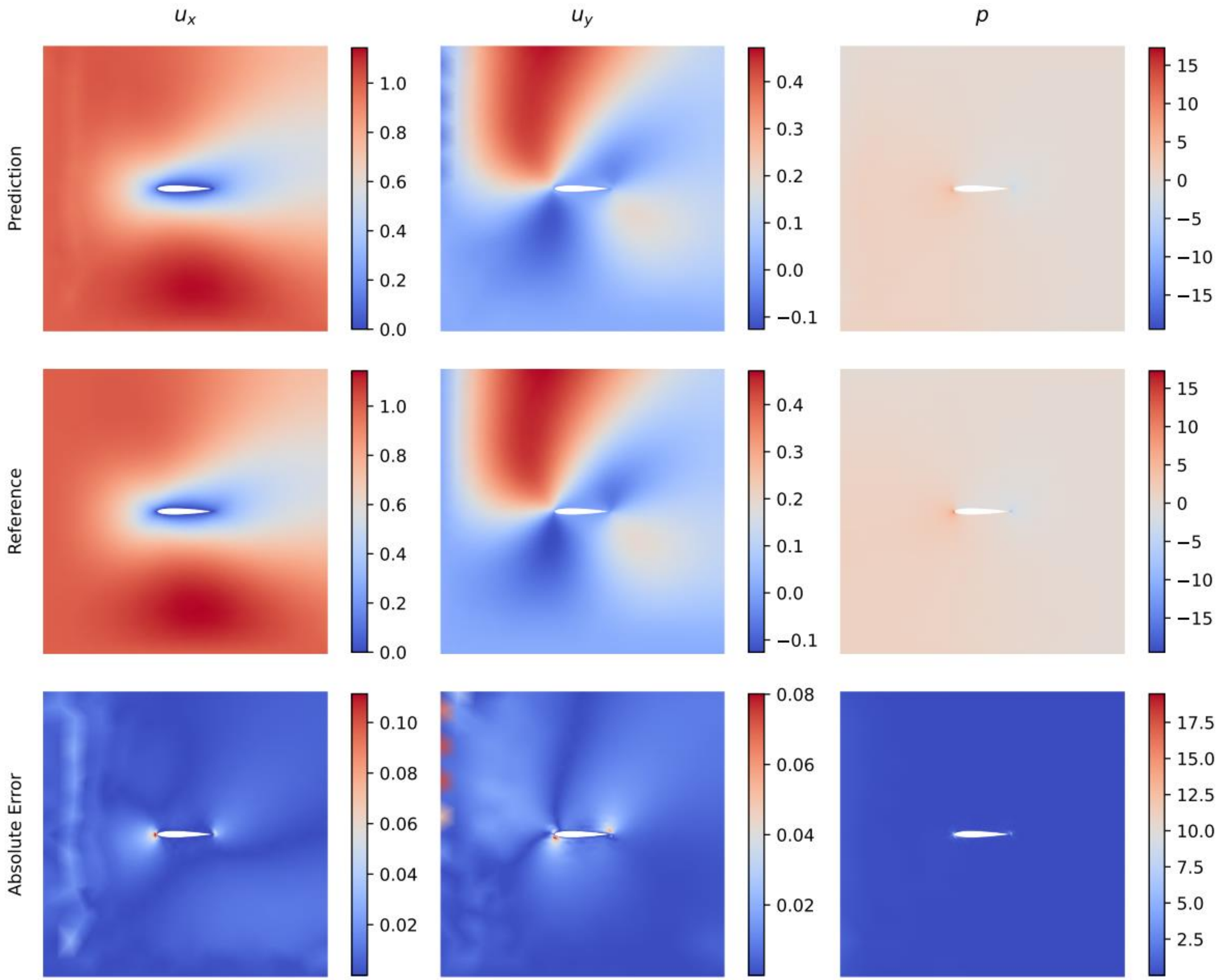
FEM-based NNs – Navier Stokes



FEM-based NNs – Navier Stokes

- Training with $\alpha = 1$, $\eta = 1$ and 2000 epochs of LBFGS

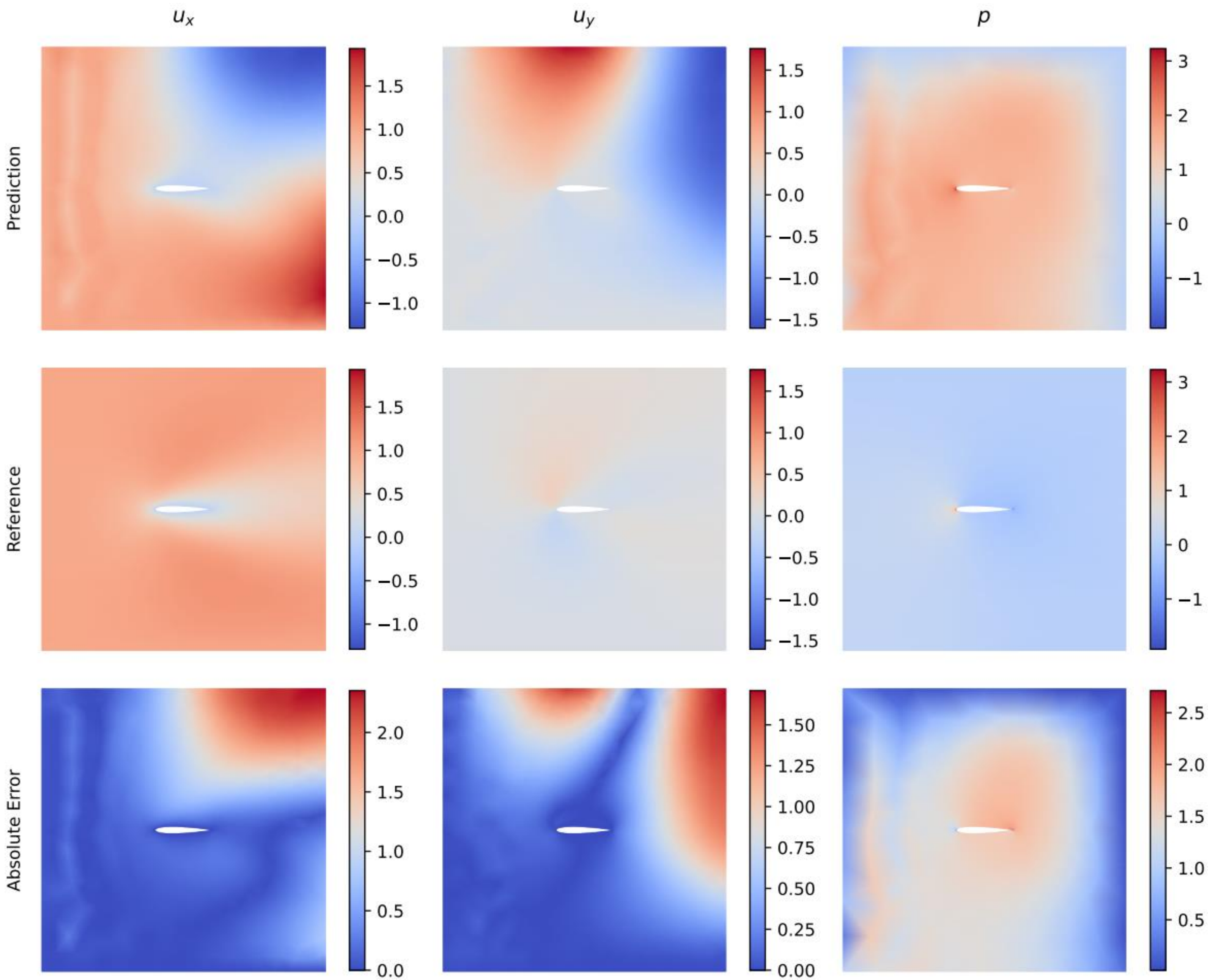
- CPU training time ~11.5 h



Rel. L^2-error: 0.0085 0.0514 0.1670

FEM-based NNs – Navier Stokes

- Training with $\alpha = 1$, $\eta = 0.1$ and 2000 epochs of LBFGS
- CPU training time ~4.3 h
- No satisfactory results



Rel. L^2-error: 0.7664 6.3577 9.3255

FEM-based NNs – Navier Stokes Preconditioned

$$K = \begin{bmatrix} K_{uu} + C_{uu}(u) & K_{up} \\ K_{pu} & \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} K_{uu} & \\ & -S \end{bmatrix}$$

$$S = -K_{up} K_{uu}^{-1} K_{pu}$$

- Training with $\alpha = 1$, $\eta = 1$ and 3 epochs of LBFGS:

	u_x	u_y	p
Rel. L^∞	3.6355e-06	1.4172e-05	1.0005e-05
Rel. L^2	1.2310e-06	1.2399e-05	7.3803e-06

- Error improvement of e-03
- CPU training time < 1 min

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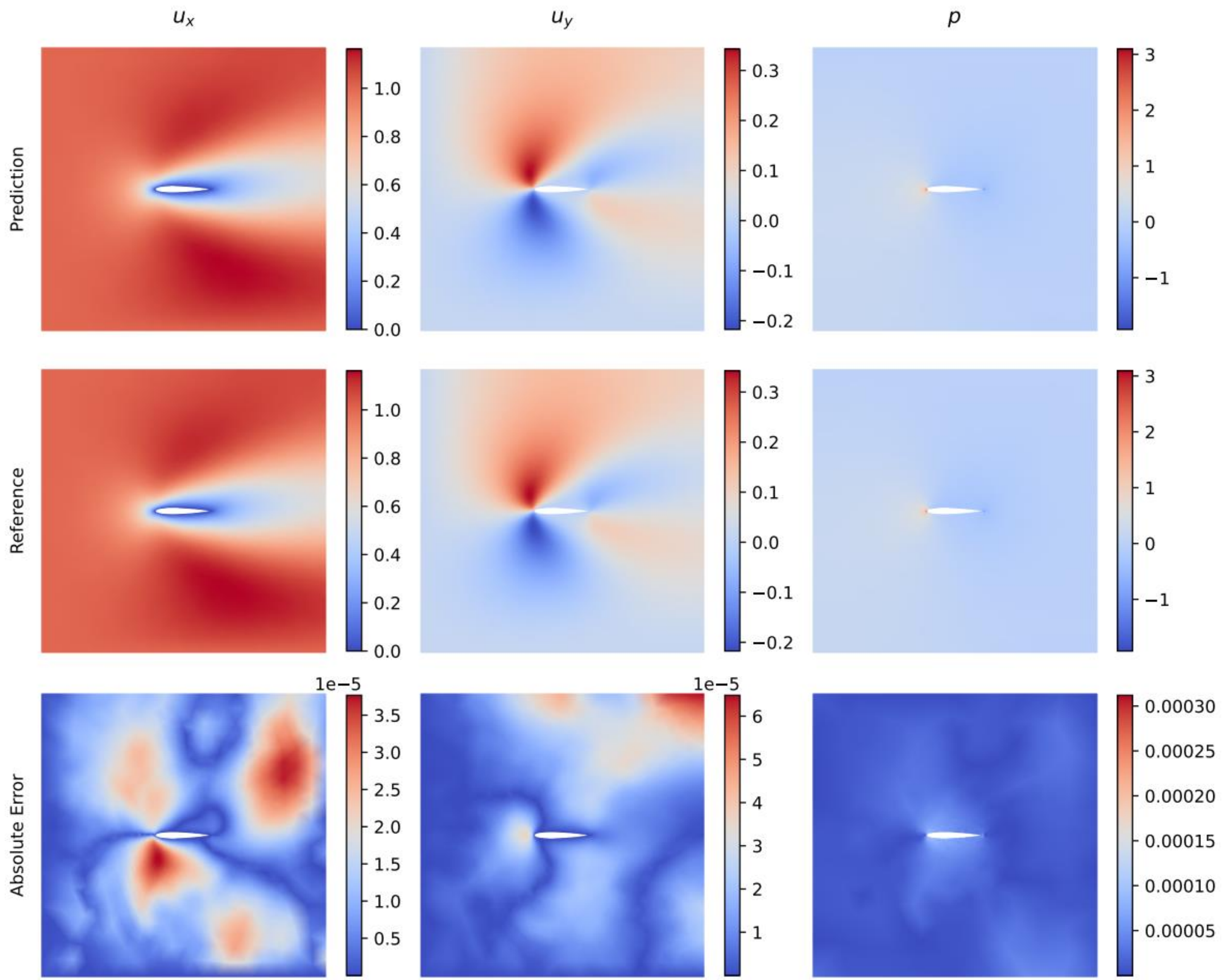
FEM-based NNs – Navier Stokes Preconditioned

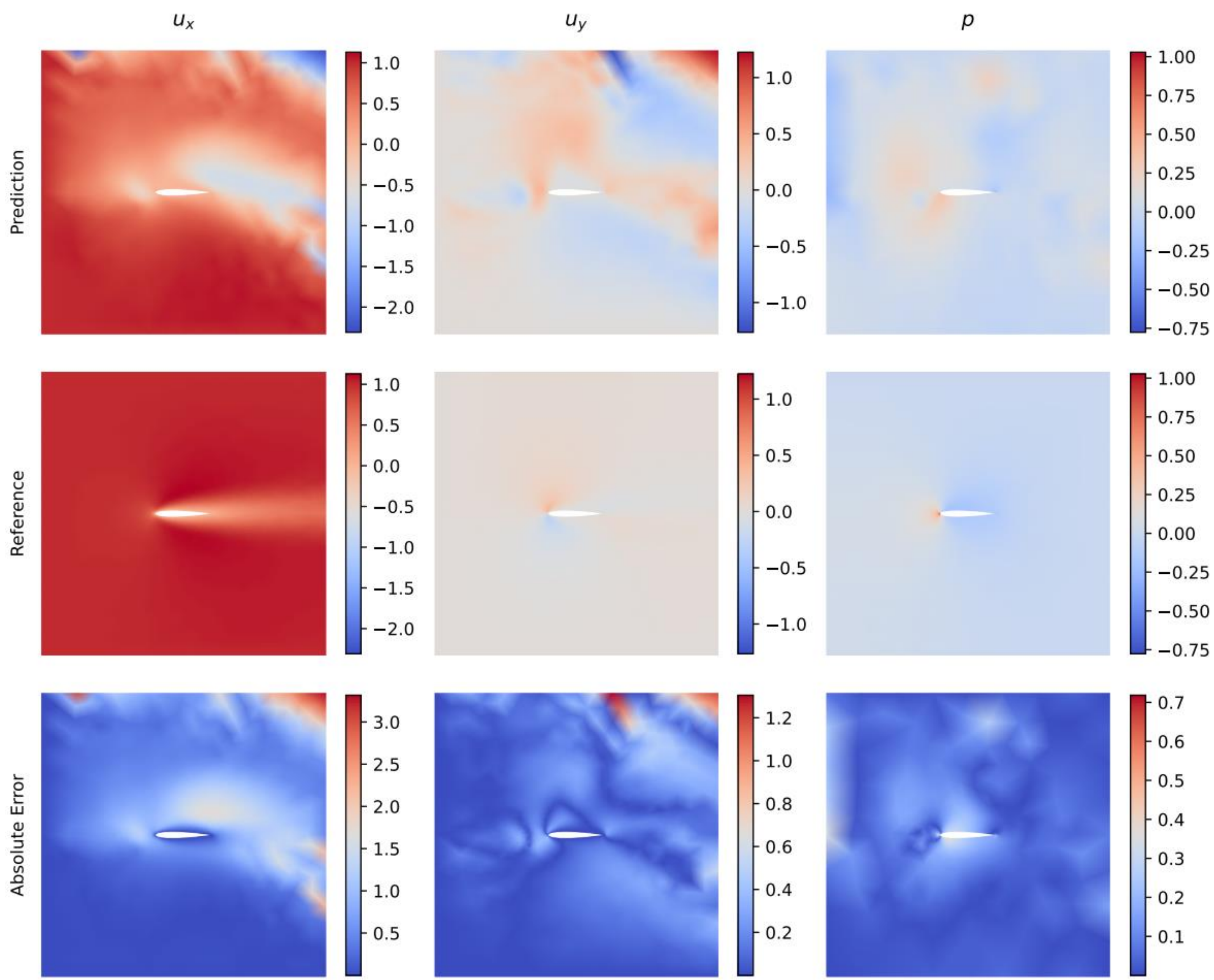
- Training with $\alpha = 1$, $\eta = 0.1$ and 12 epochs of LBFGS
- CPU training time ~11 min
- Error improvement of e-04



Rel. L^2-error: $1.4819e-05$ 0.0002 0.0001

FEM-based NNs – Navier Stokes Preconditioned

- Training with $\alpha = 1$, $\eta = 0.01$ and 100 epochs of LBFGS
- CPU training time ~1h
- No satisfactory results



Rel. L^2-error: 0.6821 0.6573 1.8404

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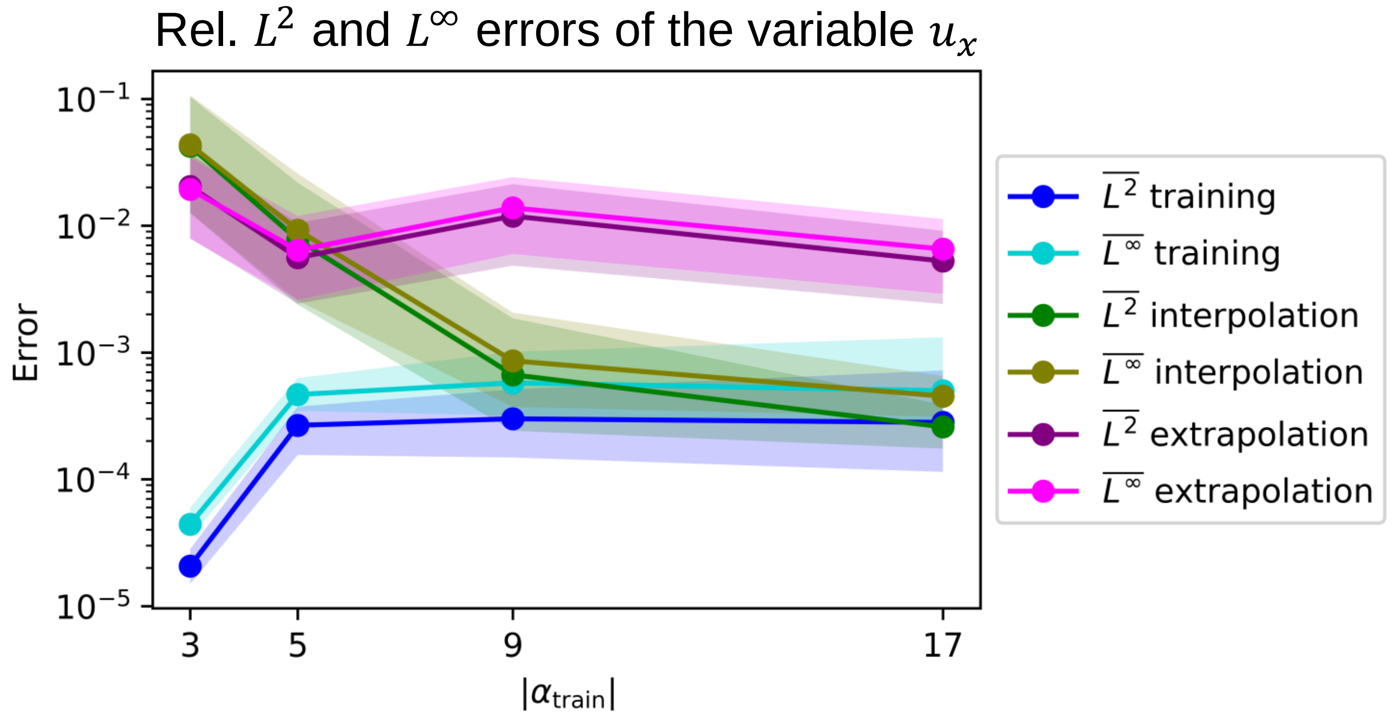
FEM-based NNs – Navier Stokes Parameterizability & Generalizability

Parameterizability

- Training data $\alpha_{\text{train}} \in [1, 45]$ uniformly distributed, $\eta = 1$

Generalizability

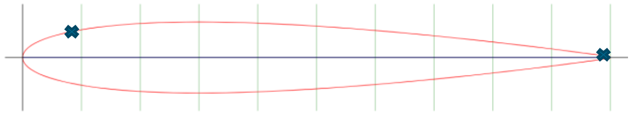
- Interpolation data $\alpha_{\text{in}} \in \{5, 16.5, 30, 40\}$
- Extrapolation data $\alpha_{\text{ex}} \in \{47.5, 50, 55\}$



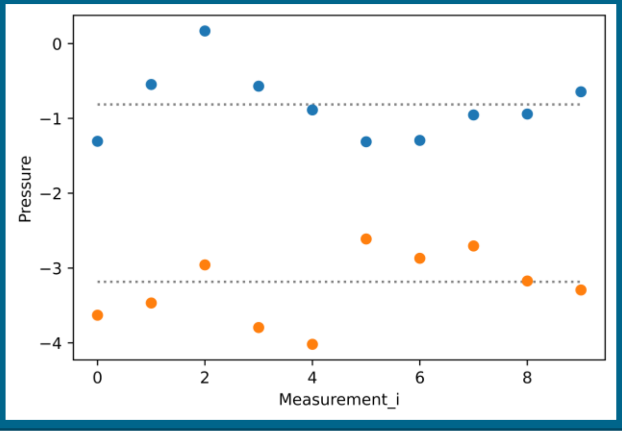
FEM-based NNs - Solving the Inverse Problem with Uncertainty Quantification

- Inverse problem:

  Measurements p at airfoil → predict angle of attack



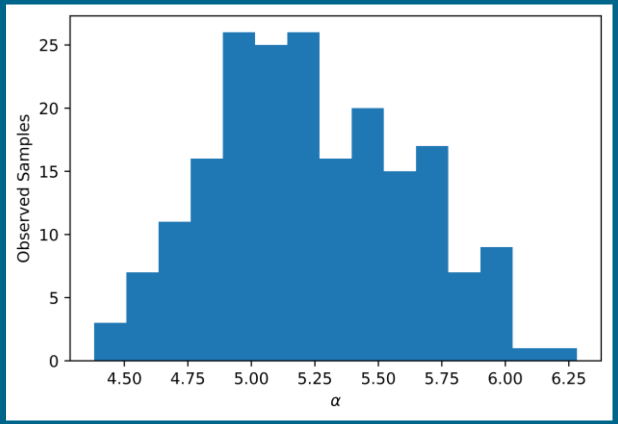
Input: trained FEM-based NN, noisy measurements



PYRO

Hamiltonian Monte Carlo method

Output: Distribution for angle of attack



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Conclusion and Outlook

Results

- Preconditioned FEM-based NNs shows the ability to parameterize and generalize well for Stokes and Navier Stokes flow with low Reynolds number
- Used fully differentiability of FEM-based NNs to solve inverse problem with UQ

Next steps for FEM-based NNs

- Try other preconditioners
- Use stabilisation methods

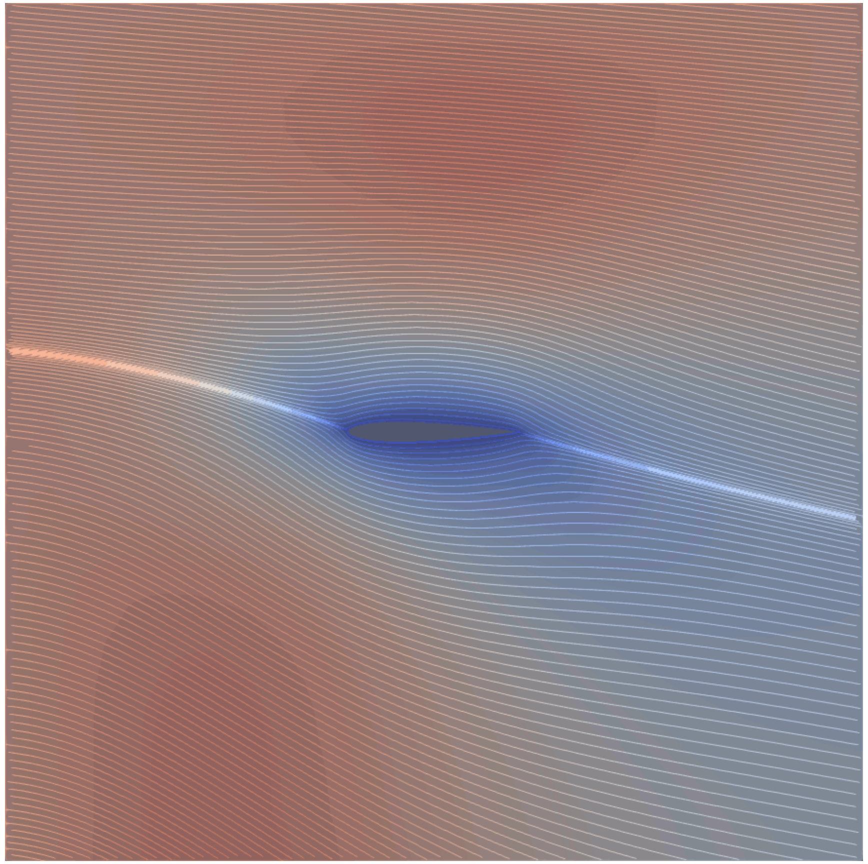

Figure: Stokes velocity field around an airfoil with various angles of attack calculated from a FEM-based NN.