PHYSICS-INFORMED FEM-BASED NEURAL NETWORKS FOR SOLVING THE NAVIER STOKES EQUATIONS

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DLR

Why Physics Informed Neural Networks*?*

- E.g. safe control of airplanes needs surrogate modeling.

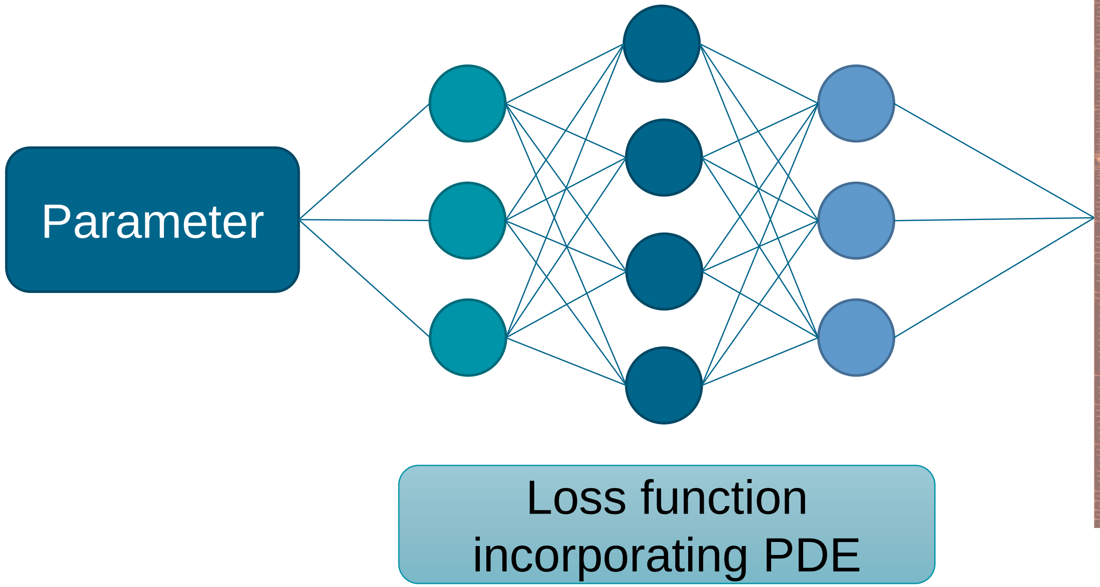
- By data-driven NN approaches some natural laws are not or only poorly considered.

Conservation of		
Energy	Mass	Momentum

Why Physics Informed Neural Networks*?*

How to make AI safer and more robust*?*

- By teaching it the physics.



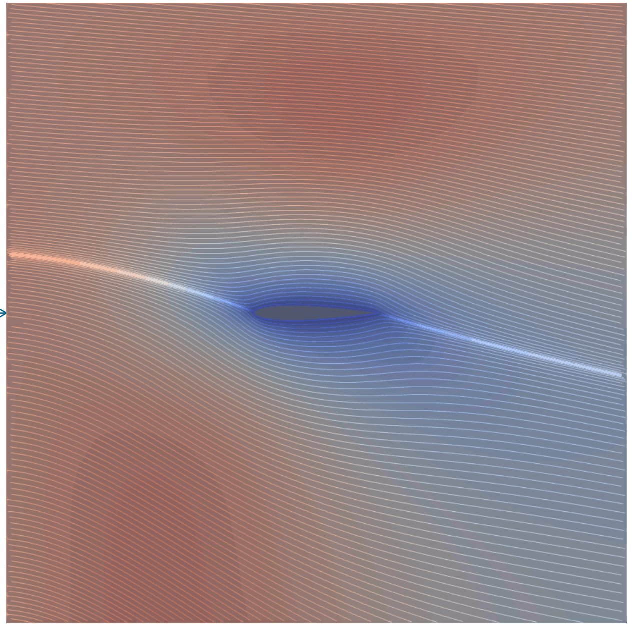


Figure: Velocity field around an airfoil with various angles of attack.

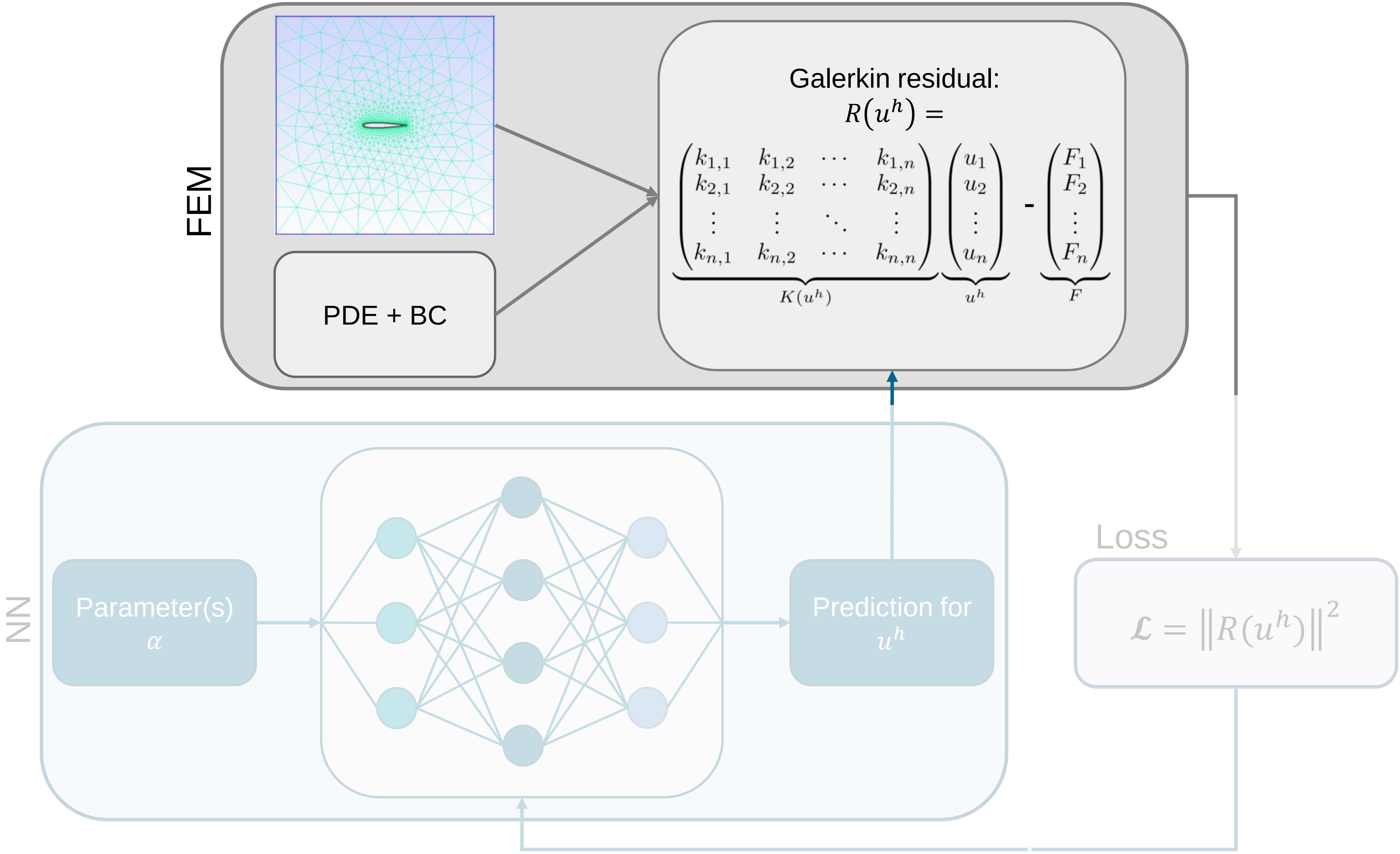
Finite Element Method based Neural Networks

- Combination of classical finite element method and NN
- Combine strengths and compensate weaknesses of individual approaches:

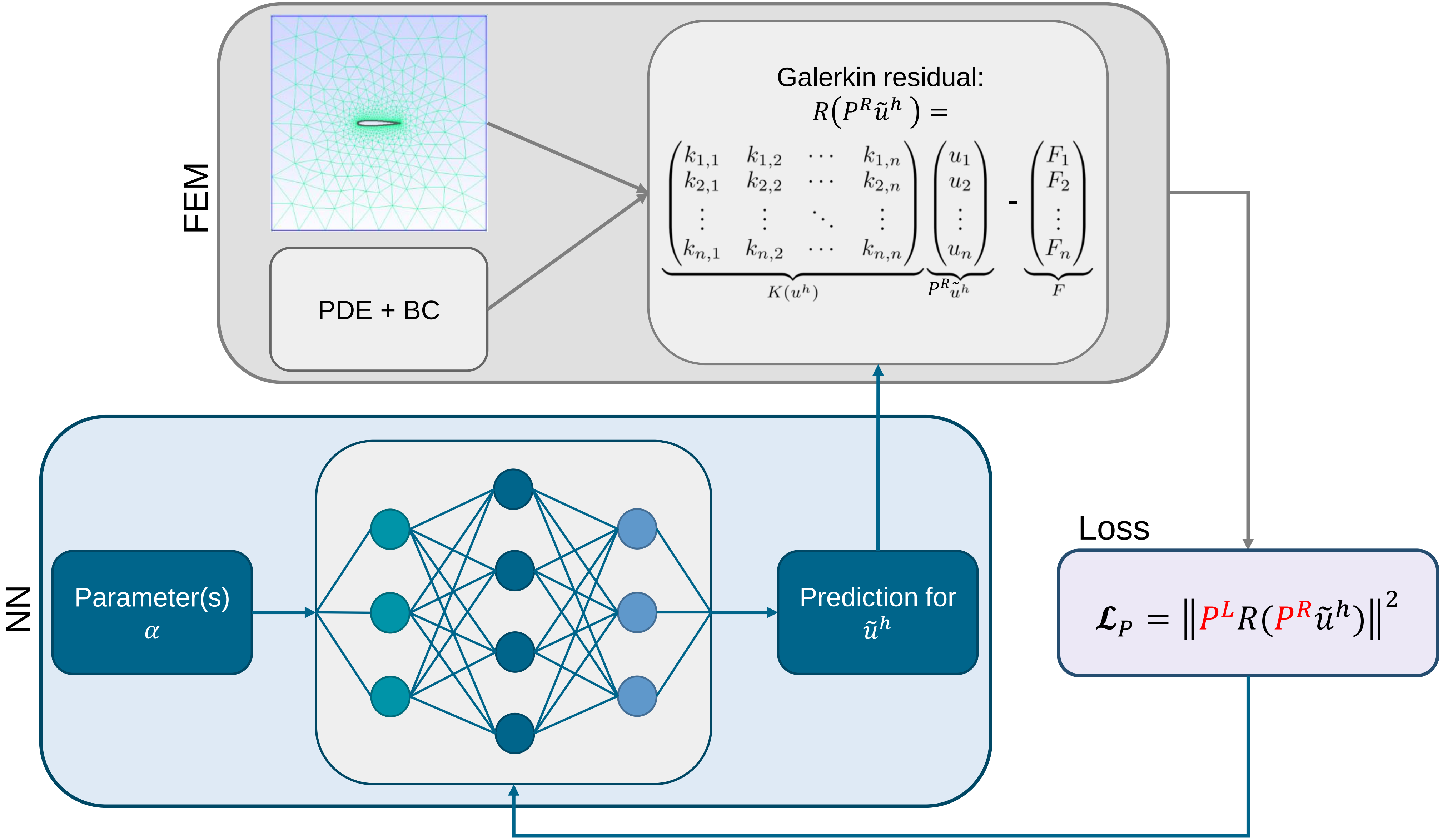
FEM	NN
No real-time capacity	Fast prediction after training
No cost amortization over multiple runs	Parameterizable
Sound mathematical foundation	Black box model
Numerical theory of errors	Rudimentary convergence theory

Implemented fully differentiable

FEM-based Neural Networks



Preconditioned FEM-based Neural Networks



Problem: Incompressible flow around an airfoil

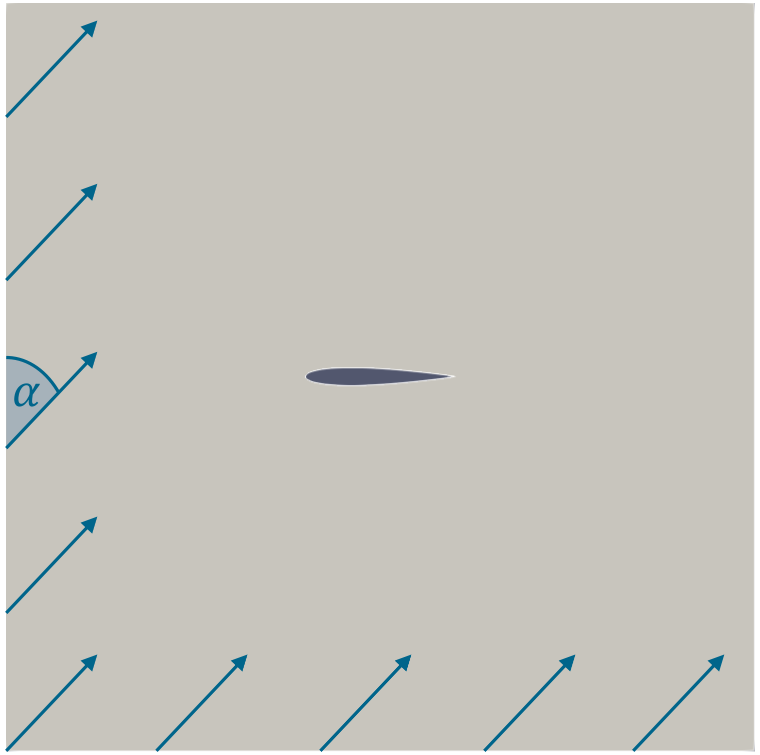


Figure: Domain around NACA 0012 airfoil with an angle of attack α.

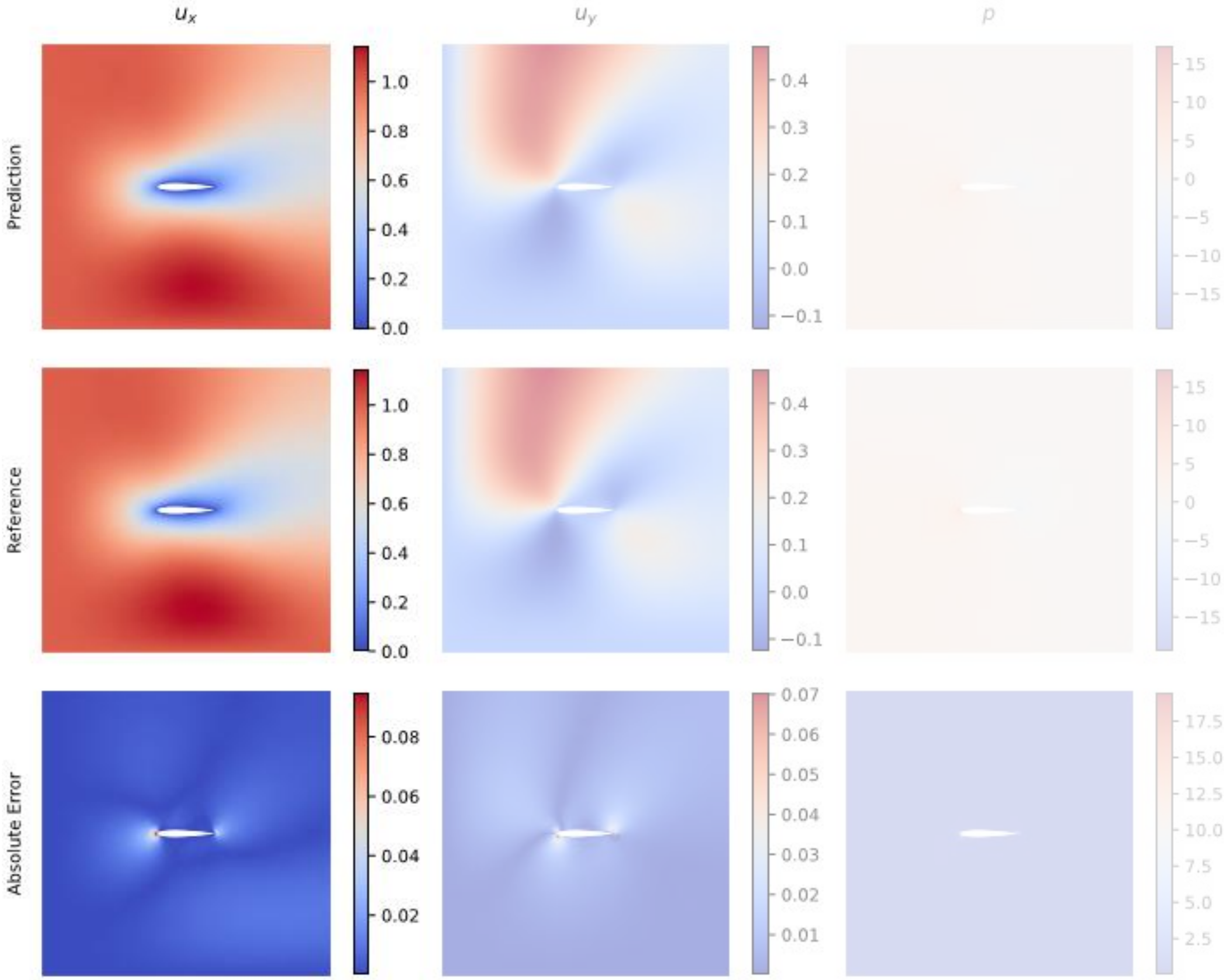
	2D Navier Stokes
PDE	$-\eta\Delta u + u \cdot \nabla \mathrm{u} + \nabla p = 0$ $\nabla \cdot u = 0$
Classification	Nonlinear saddle point problem

- BC:
  - Dirichlet at inflow
  - Neumann at outflow
  - no-slip at airfoil

FEM-based NNs – Navier Stokes

- Taylor-Hood elements to construct Galerkin system → avoid instabilities

- Training with $\alpha = 1$, $\eta = 1$ and 40000 epochs of LBFGS



Rel. L^2-error: 0.0073 0.0440 0.1558

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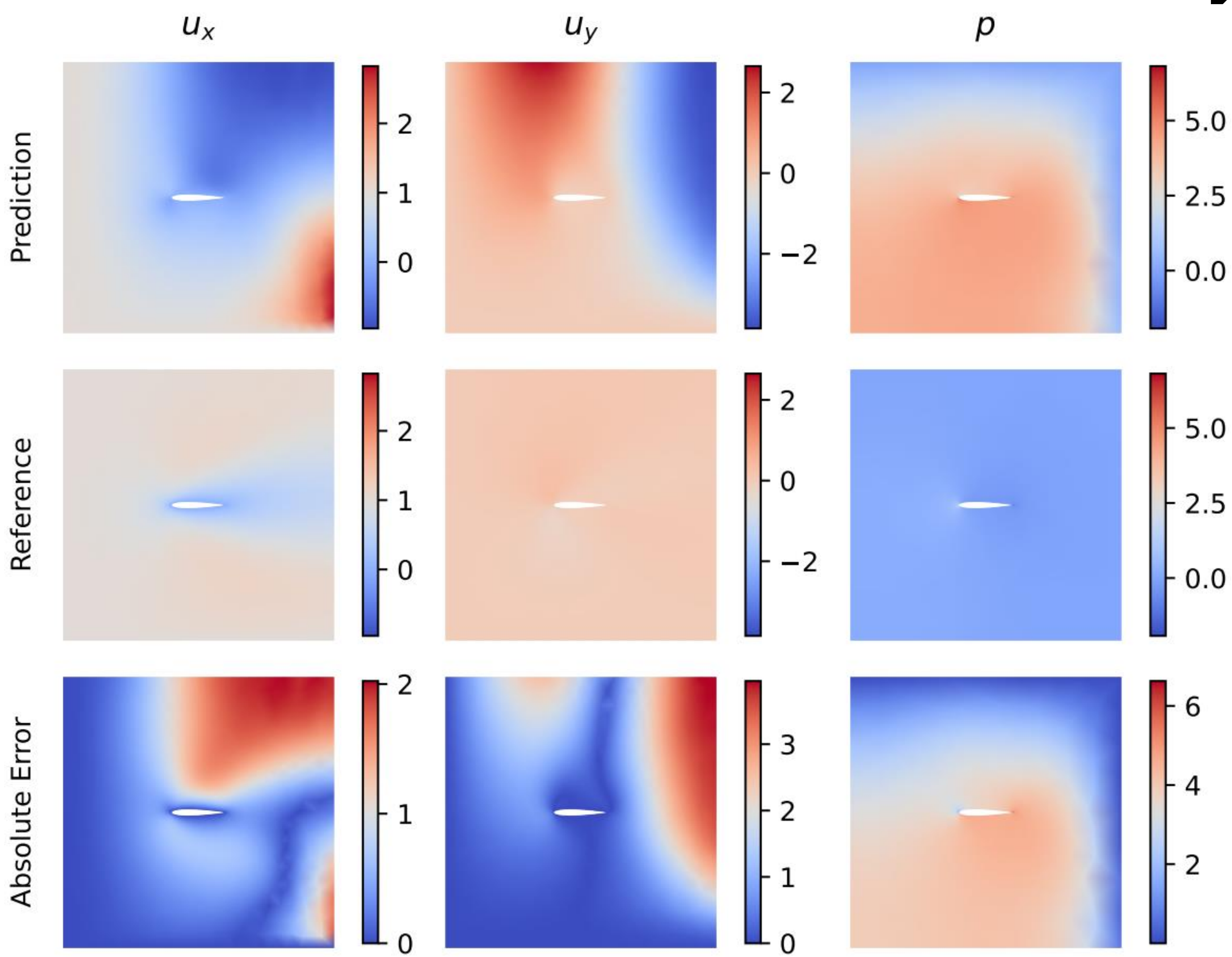
FEM-based NNs – Navier Stokes

- Training with $\alpha = 1$, $\eta = 0.1$ and 40000 epochs of LBFGS

- No satisfactory results



Rel. L^2-error: 0.8813 15.1430 24.33703

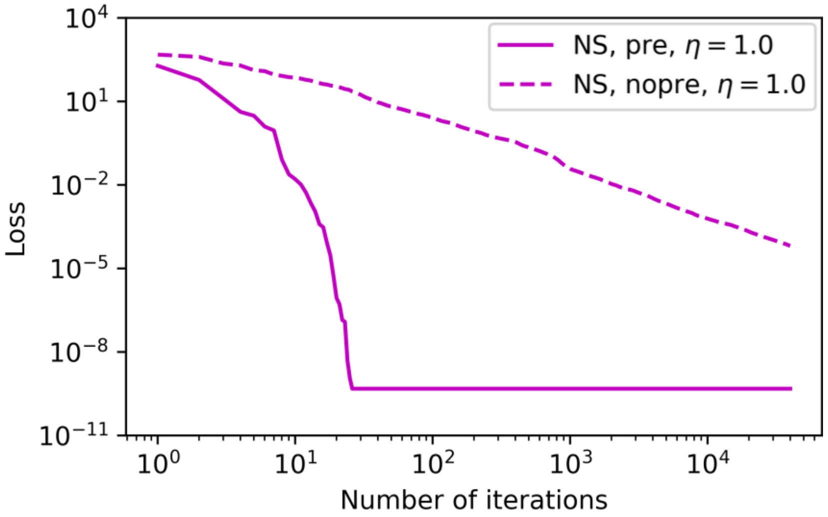
Preconditioned FEM-based NNs – Navier Stokes

$$K = \begin{bmatrix} K_{uu} + C_{uu}(u) & K_{up} \\ K_{pu} & \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} K_{uu} & \\ & -S \end{bmatrix}$$

$$S = -K_{up} K_{uu}^{-1} K_{pu}$$

- Training with $\alpha = 1$, $\eta = 1$:

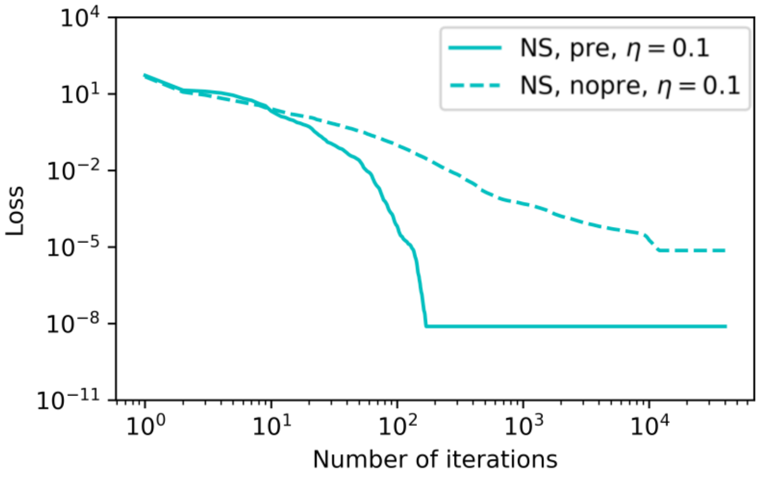


	u_x	u_y	p
Rel. L^∞	2.9603e-06	1.0582e-05	7.2077e-06
Rel. L^2	9.8632e-07	9.4800e-06	5.2756e-06

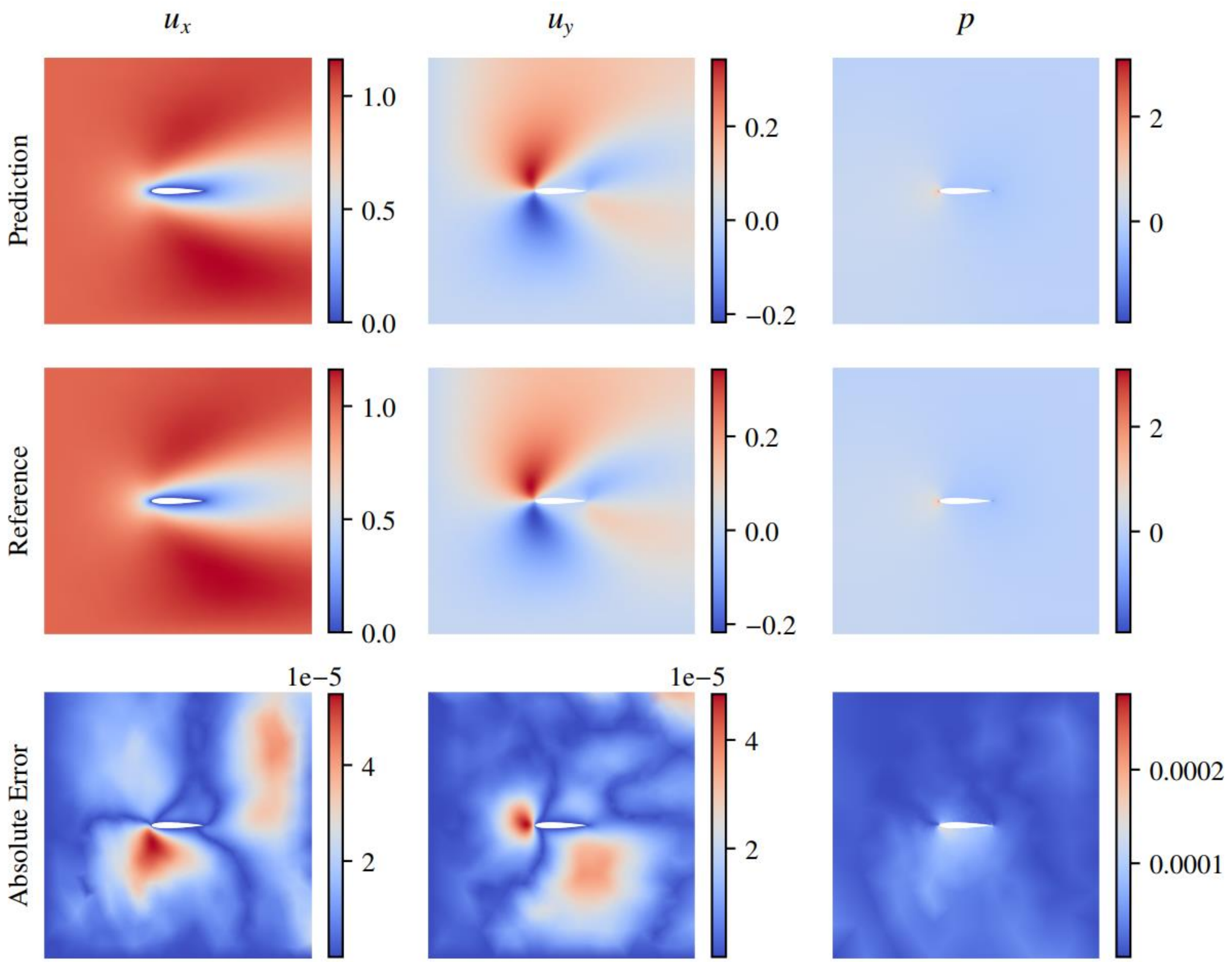
→ Error improvement of e-04

Preconditioned FEM-based NNs – Navier Stokes

- Training with $\alpha = 1$, $\eta = 0.1$



→ Satisfactory results



Rel. L^2-error: $1.7492e-05$ $1.2692e-04$ $2.0684e-04$

FEM-based NNs – Parameterizability & Generalizability

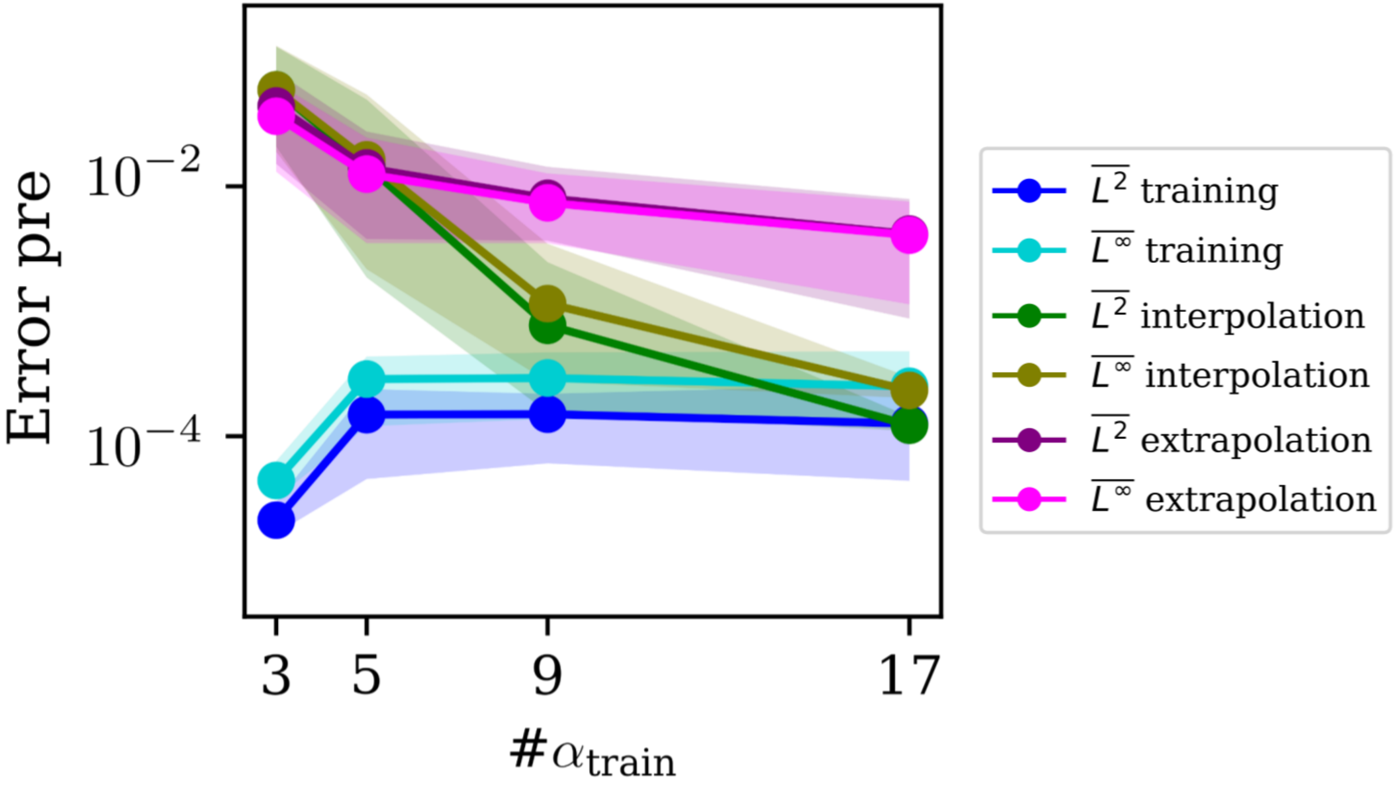
Parameterizability

- Training data $\alpha_{\text{train}} \in [1, 45]$ equidistantly distributed

Generalizability

- Interpolation data $\alpha_{\text{in}} \in \{5, 16.5, 30, 40\}$
- Extrapolation data $\alpha_{\text{ex}} \in \{47.5, 50, 55\}$

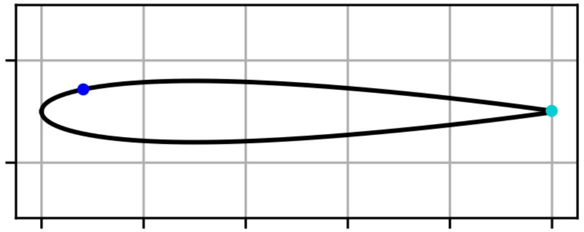
Rel. L^2 *and* L^∞ *errors of the variable* u_x



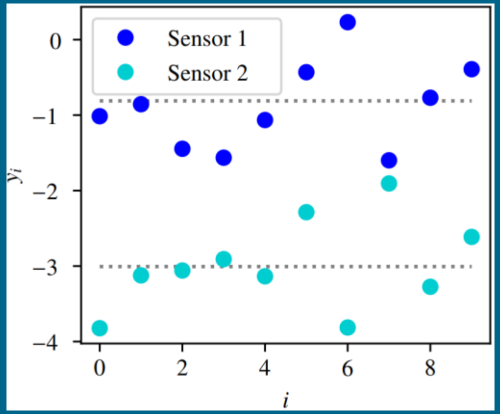
Preconditioned FEM-based NNs - Solving the Inverse Problem with Uncertainty Quantification

- Inverse problem:

Measurements p at airfoil → predict angle of attack

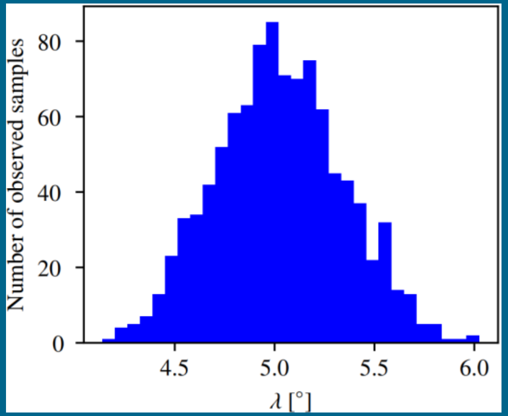


Input: trained FEM-based NN, noisy measurements



Hamiltonian Monte Carlo method

Output: Distribution for angle of attack



Conclusion and Outlook

Results

- Preconditioned FEM-based NNs shows the ability to parameterize and generalize well for Navier Stokes flow with low Reynolds number
- Used fully differentiability of FEM-based NNs to solve inverse problem with UQ

Next steps for FEM-based NNs

- Try other preconditioners
- Use stabilization methods

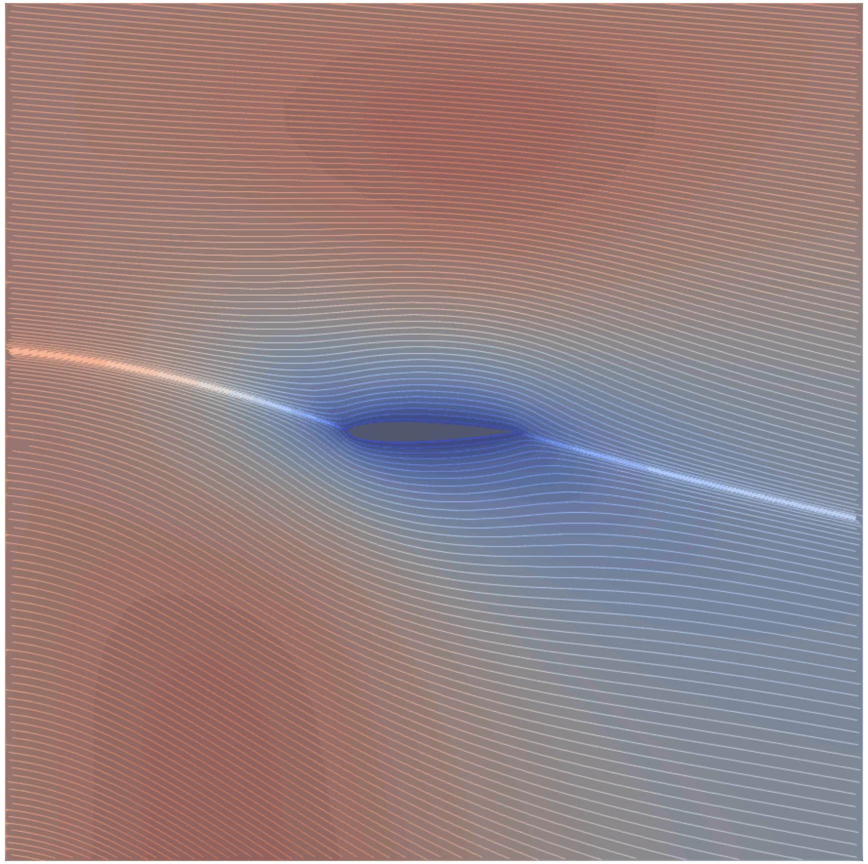


Figure: Stokes velocity field around an airfoil with various angles of attack calculated from a FEM-based NN.